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Floer Homology for Knots and SU(2)-Representations for Knot Complements and Cyclic Branched Covers

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Abstract. In this article, using 3-orbifolds singular along a knot with underlying space a homology sphere Y^3, the question of existence of non-trivial and non-abelian SU(2)-representations of the fundamental group of cyclic branched covers of Y^3 along a knot is studied. We first use Floer Homology for knots to derive an existence result of non-abelian SU(2)-representations of the fundamental group of knot complements, for knots with a non-vanishing equivariant signature. This provides information on the existence of non-trivial and non-abelian SU(2)-representations of the fundamental group of cyclic branched covers. We illustrate the method with some examples of knots in S^3.

1 Introduction

Representations of fundamental groups of low-dimensional manifolds have been studied for a long time as they are useful for the understanding of manifolds and knots in manifolds. In the last decade, representations of the fundamental group of 3-manifolds into SU(2) have been given special attention, in part because of the introduction in 3-manifold Topology of invariants such as the Casson invariant and Floer homology, where these representations play a fundamental role. If such invariants are to be useful, a minimum one can ask is that there be many 3-manifolds with non-trivial and/or non-abelian SU(2)-representations. This is a good motivation for the following problems, which appear in Kirby's updated list *Problems in Low-dimensional Topology* as Problem 3.105 (A) (see [Kir]):

Problem 1 Given a compact 3-manifold M^3 with non-trivial fundamental group $\pi_1(M)$, are there always non-trivial SU(2)-representations of $\pi_1(M)$?

Problem 2 Given a compact 3-manifold M^3 with non-abelian fundamental group $\pi_1(M)$, when are there non-abelian SU(2)-representations of $\pi_1(M)$?

Notice that the second statement is less ambitious than the first one, since as pointed by E. Klassen (see Prob. 3.105 in [Kir]), there are examples of compact 3-manifolds having a non-abelian fundamental group which admits no non-abelian representations into SU(2). Here we approach Problems 1 and 2 using 3-orbifolds and the Floer Homology for knots developed in [CS]. We exhibit a large class of 3-manifolds which have non-trivial and non-abelian SU(2)-representations of their fundamental group, in relation to the above

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problems. The article is organized as follows. After explaining some basic material in Section 2, we study, in Section 3, the existence of non-abelian SU(2)-representations of knot complements, using Floer Homology for knots. In Section 4, we apply this to the case of 3-manifolds which arise as cyclic branched covers of a homology sphere Y^3 along a knot. Section 5 provides some examples, taken from a variety of contexts in Knot Theory.

2 Basic Material

The concept of an orbifold was introduced as a generalization of the concept of a differentiable manifold, by allowing some mild singularities, which are well understood. The 3-orbifolds that will be dealt with are of the following type. They are orbifolds whose underlying space is a homology sphere Y^3, and the singular locus is a knot K in Y^3. The local groups of the 3-orbifold are trivial away from K and $\mathbb{Z}_n$, acting on $\mathbb{R}^3$ by rotations in a plane perpendicular to K, in a tubular neighbourhood, N_K, of K. Denote such 3-orbifolds by (Y^3, K, n). The 3-orbifold (Y^3, K, n) may be constructed as follows. Take $V_n(K)$ to be the n-fold cyclic branched cover of Y^3 along K. Recall how $V_n(K)$ is constructed: first consider the cyclic unbranched cover of the knot complement $Y^3 - N_K$ with automorphism group $\mathbb{Z}_n$, denoted $\tilde{X}_n$; the cyclic branched cover is then

$$V_n(K) = \tilde{X}_n \bigcup_h N_K,$$

where $h\colon \partial\tilde{X}_n \to \partial(Y^3 - N_K)$ is such that $h(\tilde{\mu}) = \mu$, for μ a meridian in $\partial(Y^3 - N_K)$ and $\tilde{\mu}$ its pre-image in $\partial\tilde{X}_n$. The construction of (Y^3, K, n) is very similar. Namely, one has

$$(Y^3, K, n) = \Big(\tilde{X}_n \bigcup_h N_K\Big)/\mathbb{Z}_n,$$

where $\mathbb{Z}_n$ acts by meridional rotations on N_K (therefore fixing the knot K) and as group of covering transformations for $\tilde{X}_n$. The 3-orbifold (Y^3, K, n) then appears as a global quotient of $V_n(K)$ by a cyclic action. The orbifold fundamental group is the group

$$\pi_1^V(Y^3, K, n) = \pi_1(Y^3 - N_K)/\langle \mu^n \rangle,$$

where "$\langle\ \rangle$" denotes the normal closure and μ is a meridian in $\pi_1(Y^3 - N_K)$. A basic notion is that of character variety of knot complements and 3-orbifolds. Recall the following definitions:

Definition 2.1 An SU(2)-representation of a group G, $\rho\colon G \to \mathrm{SU}(2)$ is said to be *non-abelian* or *irreducible* if $\rho(G)$ is not contained in a maximal torus $S^1 \subset \mathrm{SU}(2)$. Otherwise it is said to be *abelian* or *reducible*.

Definition 2.2 The space of representations of $\pi_1^V(Y^3, K, n)$ into SU(2) is called the *representation variety* of the orbifold (Y^3, K, n). The group SU(2) acts on this space via conjugation, and the quotient is called the *character variety* of (Y^3, K, n) and is denoted $\mathcal{R}(Y^3, K, n)$. Similarly, for a knot complement one has the character variety $\mathcal{R}(Y^3 - N_K)$. The subset of irreducibles is distinguished by the symbol "$*$": $\mathcal{R}_*$; while the complementary subset of reducibles is distinguished by the symbol "a": $\mathcal{R}_a$.

Remark 2.3 The topology on the representation varieties is the compact-open topology. The group SU(2) has its usual topology and the fundamental group has the discrete topology. Then the character varieties naturally inherit a topology. These spaces can be shown to carry the structure of a real algebraic variety (see [AM] for example).

It will be useful to explicitly have the relation between the character varieties of knot complements and of 3-orbifolds:

Proposition 2.4 *There is an embedding* $\Psi\colon \mathcal{R}(Y^3, K, n) \to \mathcal{R}(Y^3 - N_K)$ *and this embedding preserves irreducibility.*

Proof This is easily done using the presentation of the orbifold fundamental group $\pi_1^V(Y^3, K, n)$ given on the previous page. ∎

A word should be said about reducibles. An SU(2)-representation of $\pi_1(Y^3 - N_K)$ is reducible if and only if it factors through the abelianized group $H_1(Y^3 - N_K)$. For a knot K, $H_1(Y^3 - N_K) \simeq \mathbb{Z}$, so $\mathcal{R}_a(Y^3 - N_K)$ is the SU(2)-character variety of the group $\mathbb{Z}$. Similarly $\mathcal{R}_a(Y^3, K, n)$ is the character variety of $\mathbb{Z}_n$. Both are independent of the chosen knot.

Let us now introduce the pillow-case variety for knots in Y^3 as this will be helpful in visualizing what happens in a variety of contexts. Take the torus T^2. Then $\pi_1(T^2) = \mathbb{Z}\mu \oplus \mathbb{Z}\lambda$, where μ is a meridian and λ is a preferred longitude. Define the map $\Phi\colon \mathbb{R}^2 \to \mathcal{R}(T^2)$ by $(\alpha, \beta) = [\rho]$ where

$$\rho(\mu) = \begin{pmatrix} e^{2\pi i\alpha} & 0 \\ 0 & e^{-2\pi i\alpha} \end{pmatrix} \quad \text{and} \quad \rho(\lambda) = \begin{pmatrix} e^{2\pi i\beta} & o \\ 0 & e^{-2\pi i\beta} \end{pmatrix}.$$

One has $\Phi(\alpha, \beta) = \Phi(\alpha + m, \beta + n)$ for $m, n \in \mathbb{Z}$ and, since $\mathrm{SU}(2)/\operatorname{Ad}\mathrm{SU}(2) \simeq [-2, 2]$ via the trace function, also $\Phi(\alpha, \beta) = \Phi(-\alpha, -\beta)$. By diagonalization, it therefore follows that $\Phi(\alpha, \beta) = \Phi(\alpha', \beta')$ if and only if $(\alpha', \beta') = \pm(\alpha, \beta) + (m, n)$. Hence $\mathbb{Z}^2 \rtimes \mathbb{Z}_2$ acts on $\mathbb{R}^2$ with quotient precisely $\mathcal{R}(T^2)$. It is easy to see that $\mathcal{R}(T^2)$ has four singular points, where the stabilizer under the $\mathbb{Z}^2 \rtimes \mathbb{Z}_2$-action is not trivial. The variety $\mathcal{R}(T^2)$ is called the *pillow-case* variety. In $\mathcal{R}(T^2)$, it will be of use to consider *vertical slices* (α constant in the above description). These are

$$S_\alpha = \{\rho \in \mathcal{R}(T^2) \mid \operatorname{tr}\rho(\mu) = 2\cos(2\pi\alpha)\}.$$

For $\alpha = 0, 1/2$ these are arcs, while for $0 < \alpha < 1/2$, S_α is a circle in the pillow-case. A knot complement $Y^3 - N_K$ being a 3-manifold whose boundary is identified with T^2 in the usual way, a representation $\rho\colon \pi_1(Y^3 - N_K) \to \mathrm{SU}(2)$ restricts to $\rho_{|T^2}\colon \pi_1(T^2) \to \mathrm{SU}(2)$. This allows one to define the restriction map

$$r\colon \mathcal{R}(Y^3 - N_K) \to \mathcal{R}(T^2)$$

in the obvious way. This restriction can be important both for the study of knots and for Gauge Theory of 3-manifolds (see for example [He1] and [KK]). The variety $\mathcal{R}_a(Y^3 - N_K)$ is an arc (parameterized by $\operatorname{tr}\rho(\mu)$) and its image in the pillow-case is the lower arc, corresponding to $\rho(\lambda) = I$. Consequently, any representation whose restriction in the pillow-case is not in the lower arc is non-abelian.

There is a similar construction for 3-orbifolds (Y^3, K, n): one simply uses the embedding Ψ introduced in Proposition 2.4 to define the map:

$$r \circ \Psi\colon \mathcal{R}(Y^3, K, n) \to \mathcal{R}(T^2).$$

If one knows the restriction map r for a knot complement, it is easy to obtain the corresponding restriction map in the orbifold situation: $\rho\colon \pi_1^V(Y^3, K, n) \to \mathrm{SU}(2)$ must satisfy $\rho(\mu^n) = I$, so $\rho(\mu)$ in $\mathrm{SU}(2)$ corresponds to an n-th root of unity. Then the restriction map in the orbifold situation corresponds to vertical slices of $r\big(\mathcal{R}(Y^3 - N_K)\big)$ in $\mathcal{R}(T^2)$ by circles $S_{\frac{k}{n}}$, for $0 < k < \frac{n}{2}$ or the arcs S_0 and $S_{\frac{1}{2}}$.

Finally, we recall briefly the Alexander polynomial and equivariant signatures of a knot, as these knot invariants will play a role in here. In $Y^3 - N_K$, choose a pair (λ, μ), and consider a Seifert surface Σ. Choose a positive orientation of the normal bundle of Σ in Y^3. Let $\{x_1, \dots, x_p\}$ be a basis for $H_1(\Sigma, \mathbb{Z})$, and denote by x_j^+ the push-off of the curve x_j in the positive direction in the normal bundle of Σ. Let V_K be the matrix whose entries are $v_{ij} = \mathrm{lk}(x_i, x_j^+)$, where lk denotes the linking number of two curves in Y^3, and V_K^T its transpose.

Definition 2.5 The *Alexander polynomial* of K is the polynomial

$$\Delta_K(t) = \det(V_K^T - tV_K).$$

Definition 2.6 Let $t \in S^1$. An equivariant signature matrix is a matrix of the form $B_K(t) = (1 - t)V_K + (1 - \bar{t})V_K^T$. The signature of $B_K(t)$ is called an *equivariant signature* of K and is denoted $\sigma_t(K)$.

Remark 2.7 Notice that when $t = -1$, one gets the standard knot signature, $\sigma(K)$. To avoid cumbersome notations later on, when $t = e^{i2\pi\alpha}$, it will be useful to denote the corresponding equivariant knot signature by $\sigma_\alpha(K)$ rather than $\sigma_t(K)$.

Proposition 2.8 *The equivariant signature $\sigma_t(K)\colon S^1 \to \mathbb{Z}$ is a continuous function except possibly at the roots of the Alexander polynomial $\Delta_K(t)$.*

Proof One has that $B_K(t) = (1 - t)V_K + (1 - \bar{t})V_K^T$, hence $B_K(t) = (1 - t)(V_K - \bar{t}V_K^T)$. For $\sigma_t(K)$ to be discontinuous at t_0, there has to be a change of eigenvalue sign for the matrix $V_K - tV_K^T$, around t_0. This means that at t_0, $V_K - tV_K^T$ has a zero eigenvalue. This is only possible if $\det(V_K - \bar{t}V_K^T) = 0$. As $V_K^T - tV_K = -t(V_K - \bar{t}V_K^T)$ is the matrix whose determinant is $\Delta_K(t)$, so t_0 has to be a root of $\Delta_K(t)$. ■

3 SU(2)-Representations of Knot Complements

A first step for finding non-trivial and non-abelian representations of branched covers of Y^3 along a knot K is to obtain some information about the representations of the knot complement. As in [He2] and [HK], in this section we derive a criterion for the existence of knot complement non-abelian SU(2)-representations, expressed in terms of the equivariant knot signatures introduced in Section 2.

In [CS], a Floer Homology for knots in homology spheres was developed. The invariant generalizes ordinary Floer Homology for homology spheres. The 3-orbifolds introduced in the last section play a central role in the construction of this Floer Homology for knots, as they provide the right setting necessary to adapt ideas of Floer and Kronheimer and Mrowka to the case at hand. We recall very briefly what the construction is. The interested reader should refer to [CS] or [Co2] for more information.

Start with $\alpha \in [0, 1/2]$ a rational number, $\alpha = k/n$, such that $\Delta_K(e^{i\frac{2k}{n}2\pi}) \neq 0$. Generically, the Floer complex $\left(C_*^{(\alpha)}(Y^3, K), \partial\right)$ consists of four free abelian groups $C_i^{(\alpha)}(Y^3, K)$, $0 \leq i \leq 3$, generated by irreducible flat connections over the orbifold (Y^3, K, n) whose trace of the holonomy around a meridian is prescribed to be $2\cos(2\pi\alpha)$. The boundary operator $\partial\colon C_i^{(\alpha)}(Y^3, K) \to C_{i+1}^{(\alpha)}(Y^3, K)$ counts the number of anti-self-dual connections flowing from one irreducible flat connection to another one over the orbifold cylinder $(Y^3 \times \mathbb{R}, K \times \mathbb{R}, n)$. It is shown that $\partial^2 = 0$ and that the homology of this Floer complex is a knot invariant, denoted $\mathrm{HF}_*^{(\alpha)}(Y^3, K)$. The construction may be extended to all real numbers $\alpha \in [0, 1/2]$ such that $\Delta_K(e^{i2\alpha 2\pi}) \neq 0$.

One of the important properties of this Floer Homology for knots proved in [CS] is that it generalizes equivariant knot signatures:

Proposition 3.1 *For any knot K in a homology sphere Y^3, one has*

$$\chi\left(\mathrm{HF}_*^{(\frac{1}{4})}(Y^3, K)\right) = \frac{1}{2} \cdot \sigma(K) + 4 \cdot \lambda(Y^3),$$

where "χ" stands for the Euler characteristic of the Floer complex and $\lambda(Y^3)$ denotes the Casson invariant of Y^3. More generally, if $\Delta_K(e^{i2\pi 2\alpha}) \neq 0$, then

$$\chi\left(\mathrm{HF}_*^{(\alpha)}(Y^3, K)\right) = \frac{1}{2} \cdot \sigma_{2\alpha}(K) + 4 \cdot \lambda(Y^3).$$

We use this result to formulate our existence criterion for non-abelian SU(2)-representations of knot complements:

Theorem 3.2 *Let K be a knot in a homology sphere Y^3. If $\sigma(K) \neq 0$, there is a subinterval (α_1, α_2) in $[0, 1/2]$ such that for any $\alpha \in (\alpha_1, \alpha_2)$, there is an irreducible* SU(2)*-representation of the knot group, ρ, such that* $\operatorname{tr}\rho(\mu) = 2\cos(2\pi\alpha)$. *More generally, the result is true if some equivariant signature of K is non-vanishing.*

Proof For simplicity, suppose that $Y^3 = S^3$. The general case follows easily by taking into account the contribution of the Casson invariant. First notice that as $\Delta_K(-1)$ is always an odd integer there is a neighbourhood $(\alpha_1, \alpha_2) \subset [0, 1/2]$ about $1/4 \in [0, 1/2]$ such that the $\Delta_K(e^{i2\pi 2\alpha}) \neq 0$, for any $\alpha \in (\alpha_1, \alpha_2)$. We may then assume that on this interval, $\sigma_{2\alpha}(K) = \sigma(K)$, using Proposition 2.8. It is well-known that representations $\rho\colon \pi_1(S^3 - N_K) \to \mathrm{SU}(2)$ such that $\operatorname{tr}\rho(\mu) = 2\cos(2\pi\alpha)$ are in 1-1 correspondence with flat connections over $S^3 - N_K$ whose trace of holonomy along μ is $2\cos(2\pi\alpha)$. Therefore if for some $\alpha \in (\alpha_1, \alpha_2)$ there is no irreducible representation $\rho\colon \pi_1(S^3 - N_K) \to \mathrm{SU}(2)$ such that tr $\rho(\mu) = 2\cos(2\pi\alpha)$, it follows that $\mathrm{HF}_*^{(\alpha)}(S^3, K)$ is trivial, as it has no generators. But then by Proposition 3.1, one would obtain that $\sigma_{2\alpha}(K) = 0$, contradicting the above. This proves the first part of the theorem. The more general case is identical. ■

In fact one may prove slightly more using the following property of Floer Homology proved in [CS]:

Proposition 3.3 *When $\alpha \in [0, 1/2]$ varies, the Floer Homology $\mathrm{HF}_*^{(\alpha)}(Y^3, K)$ is constant on connected components of $[0, 1/2]$ away from the roots of $\Delta_K(t)$ which lie on S^1, the unit circle in $\mathbb{C}$.*

With this, one can be more precise about the non-abelian representations given by Theorem 3.2:

Theorem 3.4 *Let K be a knot in a homology sphere Y^3 such that $\sigma_\alpha(K) \neq 0$ for some $\alpha \in [0, 1/2]$. Then there exist an arc $\{\rho_t\} \subset \mathcal{R}_*(Y^3 - N_K)$ limiting to some reducible representation $\rho_0 \in \mathcal{R}_a(Y^3 - N_K)$, and near ρ_0, this arc may be parameterized by the trace along the meridian.*

Proof Again, suppose Y^3 is S^3 for simplicity. As $\sigma_0(K) = 0$ and $\sigma_\alpha(K) \neq 0$, by Proposition 2.8, there is an $\alpha_0 \in (0, \alpha)$ for which $e^{i2\pi\alpha_0}$ is the first root in S^1 of $\Delta_K(t)$ at which the equivariant signature changes. By Proposition 3.1 above, for $\alpha_0^{\pm}$ slightly greater or smaller than α_0, we have $\mathrm{HF}_*^{(\alpha_0^-/2)}(S^3, K)$ trivial while $\mathrm{HF}_*^{(\alpha_0^+/2)}(S^3, K)$ is non-trivial. Combining Proposition 3.3, Theorem 3.2 and the fact that $\mathcal{R}_*(S^3 - N_K)$ is a real algebraic variety gives an arc $\{\rho_t\}$ parameterized by tr $\rho_t(\mu)$. That this arc actually limits to $\rho_0 \in \mathcal{R}_a(S^3 - N_K)$ such that tr $\rho_0(\mu) = 2\cos(\alpha_0\pi)$ follows from the independence of Floer Homology with respect to perturbations compactly supported away from reducibles (see [CS, Section 3.3]). Indeed, if $\{\rho_t\}$ did not limit to ρ_0, the Floer Homology $\mathrm{HF}_*^{(\alpha_0/2)}(S^3, K)$ would be well-defined, trivial, and locally constant about $\alpha_0/2$, contradicting the fact that $\mathrm{HF}_*^{(\alpha_0^+/2)}(S^3, K)$ is non-trivial. ■

This is a slight generalization of the main result proved in [He2]; our proof being rather different, as it uses properties of Floer homology. Yet another version of this result can be found in [HK]. We conclude this section by extracting a corollary for 3-orbifolds (Y^3, K, n):

Corollary 3.5 *Let $K \hookrightarrow Y^3$ be a knot in a homology sphere which has some non-vanishing equivariant signature. Then there exists an n_0 such that for any $n \geq n_0$, the 3-orbifold (Y^3, K, n) has a fundamental group with non-abelian $\mathrm{SU}(2)$-representations.*

Proof Using Proposition 2.4, this is just Theorem 3.2. ■

4 SU(2)-Representations for Cyclic Branched Covers

We now make the transition from a 3-orbifold (Y^3, K, n) to its corresponding cyclic branched cover $V_n(K)$. First, we relate the fundamental groups:

Proposition 4.1 *There is a short exact sequence, the orbifold exact sequence:*

$$1 \longrightarrow \pi_1\big(V_n(K)\big) \longrightarrow \pi_1^V(Y^3, K, n) \longrightarrow \mathbb{Z}_n \longrightarrow 1. \tag{1}$$

Proof Consider $\tilde{X}_n$, the n-fold cyclic unbranched cover of the knot complement. By the construction of the n-fold cyclic branched cover of Y^3 along K, one has

$$\pi_1\big(V_n(K)\big) = \pi_1(\tilde{X}_n)/\langle \mu^n \rangle.$$

On the other hand, $\pi_1^V(Y^3, K, n) = \pi_1(Y^3 - N_K)/\langle \mu^n \rangle$. So using basic covering space theory, there is a homotopy exact sequence for the covering $\tilde{X}_n \to Y^3 - N_K$ which gives a commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(\tilde{X}_n) & \longrightarrow & \pi_1(Y^3 - N_K) & \longrightarrow & \mathbb{Z}_n & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & \pi_1\big(V_n(K)\big) & \longrightarrow & \pi_1^V(Y^3, K, n) & \longrightarrow & \mathbb{Z}_n & \longrightarrow & 1
\end{array}$$

yielding the exact sequence. ■

The orbifold exact sequence is the tool needed to apply to cyclic branched covers the results derived for 3-orbifolds using Floer Homology in the last section. We first consider the existence of non-trivial representations in $\mathcal{R}\big(V_n(K)\big)$, in relation to Problem 1:

Theorem 4.2 *Let K be a knot in Y^3, with $\sigma(K) \neq 0$, or more generally with some non-vanishing equivariant signature $\sigma_\alpha(K)$ for $\alpha \in [0, 1/2]$. Then there exists an n_0 such that for $n \geq n_0$, the fundamental group of $V_n(K)$ has non-trivial* SU(2)*-representations.*

Proof Let n_0 be chosen as in Corollary 3.5. Then for any $n \geq n_0$, there is at least one element $\rho \in \mathcal{R}_*(Y^3, K, n)$. Consider the restriction of ρ to $\pi_1\big(V_n(K)\big)$, subgroup of $\pi_1^V(Y^3, K, n)$. If this restriction were trivial, by the orbifold exact sequence (1), it would follow that $\rho\big(\pi_1^V(Y^3, K, n)\big)$ is an abelian subgroup of SU(2), contradicting the irreducibility of ρ. Therefore for any $n \geq n_0$, $\mathcal{R}\big(V_n(K)\big)$ contains at least one non-trivial representation. ■

The result may be extended to finite coverings of $V_n(K)$ as well:

Corollary 4.3 *For $n \geq n_1$, where n_1 is possibly greater than n_0 appearing in the statement of Theorem 4.2, any finite covering of $V_n(K)$ also has non-trivial* SU(2)*-representations of its fundamental group.*

Proof It will follow from the proof of Theorem 4.5 below, that there is an n_1 such that for $n \geq n_1$, some element $\rho \in \mathcal{R}_*(Y^3, K, n)$ has infinite image $\rho\big(\pi_1^V(Y^3, K, n)\big)$ in SU(2). Any finite cover of $V_n(K)$ has a fundamental group of finite index in $\pi_1^V(Y^3, K, n)$ and hence with non-trivial SU(2)-representations. ■

Needless to say that Theorem 4.2 and Corollary 4.3 provide a wealth of 3-manifolds with non-trivial SU(2)-representations of their fundamental group, as knots with non-zero signature are extremely common. It is interesting that a simple algebraic condition on the knot K should yield information about SU(2)-character varieties of an infinite family of 3-manifolds obtained from that knot. Our result may be seen as evidence for Problem 1.

In relation to Problem 2, a natural thing to try is to refine Theorem 4.2 to show that the non-trivial SU(2)-representations exhibited are in fact non-abelian and hence obtain that $\mathcal{R}_*\big(V_n(K)\big) \neq \varnothing$. However, Corollary 3.5 and the orbifold exact sequence are not enough for that, as it is possible that an irreducible element in $\mathcal{R}(Y^3, K, n)$, when restricted to the subgroup $\pi_1\big(V_n(K)\big)$, becomes a reducible element in $\mathcal{R}\big(V_n(K)\big)$. Some extra conditions, either on $V_n(K)$ or on K, have to be imposed, as seen below.

Theorem 4.4 *Under the same assumptions on K as in Theorem 4.2, there exists an n_0 such that for any $n \geq n_0$, if $V_n(K)$ is a homology sphere, then $\mathcal{R}_*\big(V_n(K)\big) \neq \varnothing$.*

Proof This is a consequence of Theorem 4.2 and the following fact: reducible SU(2)-representations of $\pi_1\big(V_n(K)\big)$ factor through the abelianization $H_1\big(V_n(K)\big)$. As this last group is trivial by hypothesis, the only reducible in $\mathcal{R}\big(V_n(K)\big)$ is the trivial representation, which cannot be the restriction of an element in $\mathcal{R}_*(Y^3, K, n)$, by the orbifold exact sequence (1). ■

More generally, we have:

Theorem 4.5 *Under the same assumptions on K as in Theorem 4.2, there exists an n_1 such that for any $n \geq n_1$, if $\Delta_K(e^{i\frac{k}{n}2\,pi}) \neq 0$, for $0 \leq k \leq n-1$, then $\mathcal{R}_*\big(V_n(K)\big) \neq \varnothing$.*

Proof We shall treat the case of knots in S^3 and leave the general case to the reader. We first show that for n_1 large enough, the element $\rho \in \mathcal{R}_*(S^3, K, n)$ given by Corollary 3.5 has infinite image in SU(2). For this first recall that the finite subgroups of SU(2) are the following:

(1) $\mathbb{Z}_n$ $(n \geq 1)$: cyclic
(2) D_{4n} $(n \geq 1)$: binary dihedral
(3) $\mathcal{T}^*$: binary tetrahedral (order 24)
(4) $\mathcal{O}^*$: binary octahedral (order 48)
(5) $\mathcal{I}^*$: binary icosahedral (order 120)

We have to exclude each of the possibilities. The item (1) is automatically excluded by the irreducibility of ρ. For (3)–(5), if n_1 is large enough we may suppose without loss of generality that tr $\rho(\mu) = 2\cos(2k\pi/n)$ with k and n relatively prime if $n \geq n_1$ and hence $\rho\big(\pi_1^V(Y^3, K, n)\big)$ contains an element of order n, so these cases may also be excluded. Finally, any D_{4n} lies in the group $D_\infty = S^1_A \cup S^1_B$, where, in quaternionic notation,

$$S^1_A = \{a + bi \mid a^2 + b^2 = 1\} \quad \text{and} \quad S^1_B = \{ck + dk \mid c^2 + d^2 = 1\}.$$

By Theorem 10 in [Kla], the number of representations of $\pi_1(S^3 - N_K)$ into D_∞ is equal to $\big(\Delta_K(-1) - 1\big)/2$, a finite number. It follows from Proposition 2.4 that for n_1 large enough, if $n \geq n_1$, then $\rho\big(\pi_1^V(S^3, K, n)\big)$ is not included in a binary dihedral group. If $\rho_{|\pi_1(V_n(K))}$ were reducible, then it would factor through $H_1\big(V_n(K)\big)$. But the order of $H_1\big(V_n(K)\big)$ is given as

$$|H_1\big(V_n(K)\big)| = \Big|\prod_{k=1}^{n-1} \Delta_K(e^{i\frac{k}{n}2\pi})\Big|,$$

so it is finite by our condition on $\Delta_K(t)$. The orbifold exact sequence (1) implies that $\rho\big(\pi_1^V(S^3, K, n)\big)$ is finite, a contradiction. Hence $\mathcal{R}_*\big(V_n(K)\big) \neq \varnothing$. ■

It is worth mentioning that in cases not covered by Theorem 4.5, knowledge of the pillow-case picture found in the literature can provide the same conclusion. To best understand what is going on, let us go back to Theorem 3.4, which yields an arc $\{\rho_t\}$ of irreducible elements in $\mathcal{R}(Y^3 - N_K)$, parameterized by the trace along the meridian and limiting to a reducible representation. In principle, it is possible that under the restriction map $r\colon \mathcal{R}(Y^3 - N_K) \to \mathcal{R}(T^2)$, $r(\{\rho_t\})$ be mapped entirely in the lower arc of the pillow-case, in which case for any ρ_t, one has $\rho_t(\lambda) = I$. More generally, it could also be that $\mathcal{R}_*(Y^3 - N_K)$ does not limit to the abelian arc, while $r\big(\mathcal{R}_*(Y^3 - N_K)\big)$ is contained in the lower arc of $\mathcal{R}(T^2)$. Notice however that among all the computations of the character variety of knot groups found in the literature (torus knots, twist knots, 2-bridge knots, …) there are no examples of such behavior.

By contrast, the following result is useful in relation to Problem 2 if one has some information on the image $r\big(\mathcal{R}_*(Y^3 - N_K)\big)$ in $\mathcal{R}(T^2)$, as we shall show with an example in the next section.

Theorem 4.6 *Let K be a knot such that some arc $\{\rho_t\}$ in $\mathcal{R}_*(Y^3 - N_K)$ is not mapped entirely to the lower arc of the pillow-case and which is locally parameterized by the trace along the meridian, around some $\rho_0 \in \{\rho_t\}$. Then there exists an n_0 such that for any $n \geq n_0$, $V_n(K)$ has non-abelian* SU(2)*-representations of its fundamental group.*

Proof By the local parameterization hypothesis, using Proposition 2.4, there is an n_0 such that for $n \geq n_0$, $\mathcal{R}_*(Y^3, K, n) \neq \varnothing$. Moreover, by the first hypothesis, there is at least one element $\rho \in \mathcal{R}_*(Y^3, K, n)$ satisfying $\rho(\lambda) \neq I$. By construction of (Y^3, K, n) and $V_n(K)$, notice that $\lambda \in \pi_1^V(Y^3, K, n)$ is in the subgroup $\pi_1\big(V_n(K)\big)$ and also λ is trivial when seen as an element of $H_1\big(V_n(K)\big)$. Now consider the restriction of ρ to $\pi_1\big(V_n(K)\big)$. If this restriction were reducible, it would factor through $H_1\big(V_n(K)\big)$ and as a consequence, $\rho(\lambda) = I$, contradicting the construction. ■

5 Examples

We conclude this article by giving some examples which illustrate concretely the methods developed previously. There is no doubt that the class of knots for which the results apply is much larger than what we present here, but we shall be concise here, as the purpose is mostly illustrative.

Torus Knots Let K be a torus knot of type (p,q). It is well-known that the Alexander polynomial of such a knot is given as

$$\Delta_K(t) = \frac{(1-t)(t^{pq}-1)}{(t^p-1)(t^q-1)}.$$

The equivariant signatures for torus knots have been computed by Litherland in [Li1]. They are given as $\sigma_\alpha(K) = \sigma^+ - \sigma^-$, where σ^+ and σ^- are given as follows. Let (i,j) be integer pairs such that $0 < i < p$ and $0 < j < q$. Then,

$$\sigma^+ = \text{ number of } (i,j) \text{ such that } \alpha - 1 < i/p + j/q < \alpha \pmod 2,$$

$$\sigma^- = \text{ number of } (i,j) \text{ such that } \alpha < i/p + j/q < \alpha + 1 \pmod 2.$$

By Proposition 2.8 and the formula for $\Delta_K(t)$, $\sigma_\alpha(K)$ will only change at primitive pq-th roots of unity along the unit circle and there are exactly $(p-1)(q-1)$ of these. A computation using Litherland's formula shows that for $\alpha \in \big(1/2pq, (pq-1)/2pq\big)$, $\sigma_\alpha(K)$ is non-zero. In fact, at each primitive pq-th root of unity, the jump of the function $\sigma_t(K)\colon S^1 \to \mathbb{Z}$ is of ± 2. It follows that in the case of torus knots, Theorem 3.4 can be strengthened: each abelian representation corresponding to a primitive pq-th root of unity is the limit of exactly one irreducible arc in $\mathcal{R}_*(S^3 - N_K)$. As an arc has two ends, there are $1/2\cdot(p-1)(q-1)$ arcs in $\mathcal{R}_*(S^3 - N_K)$. A different proof of this and the fact that these arcs are the totality of $\mathcal{R}_*(S^3 - N_K)$ may be found in [Kla, Theorem 1]. It will be useful for our purposes to have the image of $\mathcal{R}_*(S^3 - N_K)$ in the pillow-case. Under the covering $\Phi\colon \mathbb{R}^2 \to \mathcal{R}(T^2)$, this is given by the images under Φ of lines of slope $-1/pq$ and initial points $(1/2pq, 0)$, $(2/2pq, 0), \ldots$ for first coordinate values between 0 and 1/4. Given this, we can give a sharp version of Theorems 4.2 and 4.5:

Proposition 5.1 *Let K be a non-trivial torus knot of type (p,q). Then for any $n \geq 3$ there are non-trivial* SU(2)*-representations of $\pi_1\big(V_n(K)\big)$. In fact, except possibly for values of n such that $3 \leq n \leq pq$ for which $(n,pq) \neq 1$, there are non-abelian* SU(2)*-representations of $\pi_1\big(V_n(K)\big)$.*

Proof As K is non-trivial, the pair (p,q) satisfies

$$1/2pq \leq 1/12 \leq 5/12 \leq (pq-1)/2pq.$$

For any $n \geq 3$ there a k such that $0 \leq k \leq n/2$ and $1/2pq \leq k/n \leq (pq-1)/2pq$. By the discussion above, $\sigma_{2k/n}(K) \neq 0$, so that $\mathcal{R}_*(S^3, K, n) \neq \varnothing$ and hence $\mathcal{R}\big(V_n(K)\big)$ has non-trivial elements.

For the irreducibility, it is simpler to use the argument given in the proof of Theorem 4.6. In the case where $(n,pq) = 1$, the representations $\rho \in \mathcal{R}_*(S^3, k, n)$ will satisfy $\rho(\lambda) \neq I$, by the description above of the pillow-case image. Hence, the restriction to $\pi_1\big(V_n(K)\big)$ will be an element in $\mathcal{R}_*\big(V_n(K)\big)$, as requested. ■

Notice that in this case the cyclic branched coverings are simply Brieskorn spheres $\Sigma(p,q,n)$, whose SU(2)-character variety had already been computed in [FS], for example.

Algebraic Knots Let us turn our attention to knots which arise as links of singularities, usually referred to as *algebraic* knots. Recall that a knot $K \hookrightarrow S^3$ is said to be algebraic if there is an irreducible complex polynomial $f\colon \mathbb{C}^2 \to \mathbb{C}$ having a singularity at 0 and such that $K = f^{-1}(0) \cap S^3_\epsilon$, where S^3_ϵ is a small 3-sphere about the origin in $\mathbb{C}^2$. The simplest example is the case of a torus knot of type (p, q) which may been seen as the link of singularity of $f(x, y) = x^p + y^q$. The class of algebraic knots has been studied quite extensively, and a lot is known about them (see for example [Le]). It is well-known that algebraic knots may be realized as a particular kind of iterated torus knots. We recall that an iterated torus knot is simply a satellite of a torus knot obtained by using another torus knot in the process. Given an algebraic knot K which is the link of $f\colon \mathbb{C}^2 \to \mathbb{C}$, by taking a Puiseux expansion, one has that K is an iterated torus knot, say of type $\big((p_1, q_1), \dots, (p_r, q_r)\big)$, satisfying the conditions $p_i > 0$ and $q_{i+1} > p_i p_{i+1} q_i$. And conversely, an iterated torus knot of type $\big((p_1, q_1), \dots, (p_r, q_r)\big)$ is algebraic if $p_i > 0$ and $q_{i+1} > p_i p_{i+1} q_i$ (we refer the reader to [EN] for details on this correspondence). Notice that, in particular, if K is algebraic then $p_i > 0$ and $q_i > 0$ for all i's.

Now, Litherland gave in [Li2] a general formula for the computation of equivariant knot signatures of satellite knots in terms of the the equivariant signatures of the constituent knots. We briefly describe the procedure in the case of iterated torus knots. Let $K_1 = (p_1, q_1)$ be a torus knot. Using another torus knot $K_2 = (p_2, q_2)$, consider the iterated torus knot of type $\big((p_1, q_1), (p_2, q_2)\big)$: K^{12}. Then by Theorem 1 in [Li2],

$$\sigma_\alpha(K^{12}) = \sigma_{\alpha p_1}(K_2) + \sigma_\alpha(K_1).$$

For a general iterated torus knot $K^{1\dots r}$ of type $\big((p_1, q_1), \dots, (p_r, q_r)\big)$, its equivariant signature $\sigma_\alpha(K^{1\dots r})$ will then be computed inductively as

$$\sigma_\alpha(K^{1\dots r}) = \sigma_{\alpha p_1 p_2 \dots p_{r-1}}(K_r) + \sigma_{\alpha p_1 p_2 \dots p_{r-2}}(K_{r-1}) + \cdots + \sigma_\alpha(K_1).$$

The crucial thing to observe in this formula is that an iterated torus knot, K, which is algebraic, satisfies $p_i, q_i > 0$ for $1 \le i \le r$. Consequently, all the equivariant signatures appearing in the equation above are non-negative. In particular, in light of what was done in the case of torus knots, one has $\sigma_{2\alpha}(K) \neq 0$ for $\alpha \in [1/12, 5/12]$. Therefore we may generalize the result obtained for torus knots:

Proposition 5.2 *Let K be a non-trivial algebraic knot of type $\big((p_1, q_1), \dots, (p_r, q_r)\big)$. Then for any $n \ge 3$ there are non-trivial* SU(2)*-representations of $\pi_1\big(V_n(K)\big)$. In fact, except possibly for values of n such that $3 \le n \le p_i q_i$ for which $(n, p_i q_i) \neq 1$ for $1 \le i \le r$, there are non-abelian* SU(2)*-representations of $\pi_1\big(V_n(K)\big)$.*

Proof The proof of the first statement in the case of torus knots generalizes readily here. As for the second statement, it simply depends on the observation that one may construct irreducible SU(2)-representations for an iterated torus knot from irreducible SU(2)-representations of the constituent torus knots. We explain briefly the case with two knots. Let K be an iteration of K_1 by K_2, both torus knots. K_1 lies on the boundary of a solid torus V_1, so we may slightly push it inside. The knot K_2 is the core of the solid torus V_2.

Let $h\colon V_1 \to V_2$ be the iteration homeomorphism. By construction, we can decompose $S^3 - N_K$ as

$$S^3 - N_K = (S^3 - N_{K_2}) \cup_{\partial V_2} (V_2 - N_K) \tag{2}$$

where, of course, $V_2 - N_K \cong V_1 - N_{K_1}$. Notice that that $\pi_1(V_1 - N_{K_1})$ has an extra generator than $\pi_1(S^3 - N_{K_1})$, given by the core of the solid torus V_1. We then have $\mathcal{R}_*(S^3 - N_{K_1}) \subset \mathcal{R}_*(V_1 - N_{K_1})$ as those representations which send that extra generator to I in SU(2).

We wish to obtain elements in $\mathcal{R}_*(S^3 - N_K)$ using Equation (2) by gluing elements of $\mathcal{R}_*(V_1 - N_{K_1})$ to abelian representations of $\pi_1(S^3 - N_{K_2})$. Now an abelian representation $\rho\colon \pi_1(S^3 - N_{K_2}) \to \mathrm{SU}(2)$ sends the longitude λ_{K_2} to I, so it follows that the representations in $\mathcal{R}_*(V_1 - N_{K_1})$ which can be glued to such abelian ones are precisely those in $\mathcal{R}_*(S^3 - N_{K_1})$. The result now follows from Proposition 5.1. ■

Notice that one could also have glued non-abelian elements in $\mathcal{R}(S^3 - N_{K_2})$ to abelian elements in $\mathcal{R}(V_2 - N_{K_2})$, as done in [CL, Theorem 1] in the case of $\mathrm{SL}(2,\mathbb{C})$-representations. In fact both gluing points-of-view are hinted at by combining Proposition 3.1 and Litherland's formula for iterated torus knots mentioned previously. A similar result may be proved for iterated torus knots which are not algebraic, and more generally for other satellite knots.

Figure eight knot Let K be the figure eight knot. This knot has Alexander polynomial $\Delta_K(t) = 3 - t^{-1} + t$, a polynomial with no roots along the unit circle $S^1 \subset \mathbb{C}$. It follows that on $[0,1]$, one has $\sigma_\alpha(K) \equiv 0$. This means that the eventual existence of non-trivial and non-abelian SU(2)-representations of $\pi_1\big(V_n(K)\big)$ cannot be obtained by our main results Theorems 4.2, 4.4 and 4.5, for which it is essential to have non-vanishing equivariant signatures. But explicit knowledge of the pillow-case restriction enables one to apply Theorem 4.6 instead. A similar argument can be applied to other twist knots and 2-bridge knots, using work in [Kla] and [Bur].

The space $\mathcal{R}_*(S^3 - N_K)$ was computed in [Kla], and simply consists of a circle. The image in the pillow-case of this circle is given explicitly in [KK, Proposition 5.4]. The important point for our purpose is that for any $\alpha \in [1/6, 1/3]$, there are representations $\rho_\alpha\colon \pi_1(S^3 - N_K) \to \mathrm{SU}(2)$ such that $\mathrm{tr}\,\rho(\mu) = 2\cos(2\pi\alpha)$. This readily gives a sharpened version of Theorem 4.6:

Proposition 5.3 *For K the figure eight knot, given any $n \geq 3$, there are non-trivial* SU(2)*-representations of $\pi_1\big(V_n(K)\big)$. In fact, $\pi_1\big(V_n(K)\big)$ has non-abelian* SU(2)*-representations for any $n \geq 3$, except possibly for $n = 4$.*

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