

ALMOST-PERFECT MODULES

PINAR AYDOĞDU and A. ÇİĞDEM ÖZCAN

Department of Mathematics, Hacettepe University,
06800 Beytepe Ankara, Turkey
e-mails: paydogdu@hacettepe.edu.tr, ozcan@hacettepe.edu.tr

Abstract. We call a module M *almost perfect* if every M -generated flat module is M -projective. Any perfect module is almost perfect. We characterize almost-perfect modules and investigate some of their properties. It is proved that a ring R is a left almost-perfect ring if and only if every finitely generated left R -module is almost perfect. R is left perfect if and only if every (projective) left R -module is almost perfect.

2000 *Mathematics Subject Classification.* Primary 16A51, secondary 16A50, 16D40.

1. Introduction. Throughout this paper, R denotes an associative ring with unit and all modules are unitary left R -modules. The notation \ll will be used for small submodules of modules. We refer the reader to [3, 7, 11] for the definitions used but not defined in the paper.

Amini et al. [2] call a ring R *left almost perfect* (A -perfect) if every flat left R -module is R -projective. In this paper, we are motivated to study a module theoretic version of almost-perfect rings. We see that any perfect module is almost perfect, and any projective almost-perfect module satisfying $(*)$ is semi-perfect (the definitions are given in the text). We notice that the class of non-zero almost-perfect abelian groups coincide with the class of non-zero torsion abelian groups. Some basic properties of the class of almost-perfect modules are also investigated. We obtain some necessary and sufficient conditions for a module to be almost perfect, and a ring to be left almost-perfect or left perfect in terms of almost-perfect modules. In the final part of this paper, we consider the endomorphism ring of almost-perfect modules.

2. Results. DEFINITION 1. A module M is called *almost perfect* (A -perfect)¹ if every M -generated flat module is M -projective.

By definitions, R is a left A -perfect ring if and only if ${}_R R$ is an A -perfect module.

EXAMPLE 2. It is obvious that if M is a semi-simple module, then it is A -perfect. Moreover, an A -perfect module over a (von Neumann) regular ring is semi-simple. Indeed, let M be an A -perfect module over a regular ring and N a submodule of M . Since the factor module M/N is M -generated flat, it is M -projective. It follows that N is a direct summand of M . Thus, M is semi-simple.

EXAMPLE 3. Torsion modules over an integral domain are A -perfect.

Dedicated to Professor Patrick F. Smith on his 65th birthday

¹See Remark 26

Proof. Let R be an integral domain, M a torsion R -module and K an M -generated flat R -module. Then K is torsion-free and there exists an epimorphism $g : M^{(\Lambda)} \rightarrow K$ for an index set Λ . Since $M^{(\Lambda)}$ is torsion, we have that $\text{Im} g \subseteq T(K) = 0$, where $T(K)$ is the torsion submodule of K . Hence, $K = 0$ and so K is M -projective. \square

The set of rational numbers \mathbb{Q} is not A -perfect as a \mathbb{Z} -module because $\mathbb{Q}_{\mathbb{Z}}$ is flat \mathbb{Q} -generated but not \mathbb{Q} -projective.

Note that A -perfect flat modules are quasi-projective.

Recall some definitions: An epimorphism $f : P \rightarrow M$ is called a *projective cover* of the module M in case P is a projective module and kernel of f is a small submodule. An epimorphism $f : F \rightarrow M$ with F flat is called a *flat cover* of the module M if, for each homomorphism $g : H \rightarrow M$ with H flat, there exists a homomorphism $h : H \rightarrow F$ such that $fh = g$ and every endomorphism k of F with $fk = f$ is an automorphism of F . Due to [4], every module has a flat cover.

Semi-perfect and perfect modules are defined by Mares [8] as a generalization of Bass' notion of semi-perfect and perfect rings. Perfect modules are studied by a few authors, for example, Cunningham-Rutter [5], Varadarajan [9] and Wisbauer [11]. A module M is called *semi-perfect* if every factor module of M has a projective cover. It is known that M is semi-perfect if and only if every finitely M -generated module has a projective cover. It is also obvious that if M is semi-perfect, then every finitely M -generated flat module is projective. A module M is called *perfect* if any direct sum of copies of M are semi-perfect.

It can be easily seen that projective covers of M -generated modules are M -generated for a projective module M . But flat covers of M -generated modules need not be M -generated for any module M (see Example 7). We don't know whether flat covers of M -generated modules are M -generated or not for a projective module M .

In this paper, a module M is said to satisfy $(*)$ if flat covers of M -generated modules are M -generated. Note that any free module, in particular, any ring satisfies $(*)$.

The following well-known lemma will be used in this paper (see [2, Lemma 3.6]).

LEMMA 4. *Let $f : F \rightarrow M$ be a flat cover of the module M . If F is projective, then $f : F \rightarrow M$ is a projective cover of M .*

The following result may be known but we don't have a reference. We give a proof for completeness' sake.

PROPOSITION 5. *Let M be a module. Consider the following statements:*

- (1) M is perfect.
- (2) Every M -generated module has a projective cover.
- (3) Every M -generated flat module is projective.
- (4) Flat covers of M -generated modules are projective.

Then (4) \Rightarrow (1) \Leftrightarrow (2) \Rightarrow (3); (3) \Rightarrow (4) if M satisfies $()$.*

Proof. The implication (4) \Rightarrow (1) follows from the fact that if a flat cover of a module is projective, then it is a projective cover of the module by Lemma 4. The equivalency (1) \Leftrightarrow (2) is obvious. The implication (2) \Rightarrow (3) follows from the fact that any flat module which has a projective cover is projective. For (3) \Rightarrow (4), suppose that M satisfies $(*)$. Then the flat cover of any M -generated module is projective by hypothesis. \square

We conclude from Proposition 5 that the following implication holds for modules.

$$\text{perfect} \Rightarrow A\text{-perfect.}$$

The following theorem characterizes A -perfect modules.

THEOREM 6. *Let M be a module. Consider the following statements:*

(1) *M is semi-perfect and flat covers of finitely M -generated modules are finitely M -generated.*

(2) *Finitely M -generated flat modules are projective and flat covers of finitely M -generated modules are finitely M -generated.*

(3) *Flat covers of finitely M -generated modules are projective.*

(4) *Flat covers of M -cyclic modules are projective.*

(5) *Finitely M -generated flat modules are M -projective and flat covers of finitely M -generated modules are finitely M -generated.*

(6) *Flat covers of finitely M -generated modules are M -projective.*

(7) *Flat covers of M -cyclic modules are M -projective.*

(8) *Every flat module is M -projective.*

(9) *M is A -perfect.*

Then (1) \Leftrightarrow (2) \Rightarrow (3) \Leftrightarrow (4) \Rightarrow (6) \Leftrightarrow (7) \Leftrightarrow (8) \Rightarrow (9); (5) \Rightarrow (6); (3) \Rightarrow (2) and (4) \Rightarrow (5) if M is flat; (9) \Rightarrow (8) if M satisfies (); (6) \Rightarrow (4) if M is projective.*

Proof. (1) \Rightarrow (2) Let N be a finitely M -generated flat module. Then there exists an epimorphism $M^n \rightarrow N$ for some positive integer n . Since M is semi-perfect, M^n is semi-perfect ([7, 11.3.4]) and so N has a projective cover. Let the projective module be P and the epimorphism $f : P \rightarrow N$ with $\text{Ker } f \ll P$. Since $P/\text{Ker } f \cong N$ is flat, $\text{Ker } f = 0$ [7, 10.5.3]. Hence, $P \cong N$ is projective.

(2) \Rightarrow (1) and (2) \Rightarrow (3) \Rightarrow (4) are obvious.

(4) \Rightarrow (3) Let X be a finitely M -generated module. Then flat covers of X -cyclic modules are projective by [1, Corollary 3.4 and Proposition 3.2]. Hence, flat cover of X is projective.

(4) \Rightarrow (6) \Rightarrow (7) are obvious.

(7) \Rightarrow (8) Let N be a flat module, $g : N \rightarrow M/K$ a homomorphism and $f : F \rightarrow M/K$ a flat cover of M/K . Since N is flat and f is a flat cover, there exists a homomorphism $h : N \rightarrow F$ such that $fh = g$. By assumption, F is M -projective. So there exists a homomorphism $k : F \rightarrow M$ such that $\pi k = f$, where $\pi : M \rightarrow M/K$ is the canonical epimorphism. Define $\alpha = kh$. Then $\pi\alpha = g$, and so N is M -projective. So (8) holds.

(8) \Rightarrow (6) and (8) \Rightarrow (9) are obvious.

(9) \Rightarrow (8) Assume that M satisfies (*). Let F be a flat cover of an M -cyclic module. By (*), F is M -generated. By hypothesis, F is M -projective. Hence (7), and so (8) holds.

(3) \Rightarrow (1) Assume that M is flat. By hypothesis and Lemma 4, every finitely M -generated module has a projective cover which is equivalent to the fact that M is semi-perfect. Now, let X be a finitely M -generated module. Then there exists an epimorphism $f : M^n \rightarrow X$ for some positive integer n . Let $g : F \rightarrow X$ be a flat cover of X . By assumption, F is projective and so g is also a projective cover of X . M^n being flat implies that there exists a homomorphism $h : M^n \rightarrow F$ such that $gh = f$. Then $F = \text{Im } h + \text{Ker } g$. Since $\text{Ker } g \ll F$, we have $F = \text{Im } h$. Hence, F is finitely M -generated.

(6) \Rightarrow (4) By [1, Proposition 3.2].

Consequently, the statements above are all equivalent if M is projective and satisfies (*). □

We obtain the following implication for modules by Theorem 6:

$$\text{projective } A\text{-perfect with } (*) \Rightarrow \text{semi-perfect.}$$

The following example shows that $(*)$ does not hold in general.

EXAMPLE 7. Let $R = \mathbb{Z}$ and the \mathbb{Z} -module $M = \mathbb{Z}/(p)$ for a prime p . The flat cover of M is the ring of p -adic integers which is not (finitely) M -generated. Hence M does not satisfy $(*)$. Moreover, since M is simple, it is A -perfect but not semi-perfect.

The projectivity condition on M in Theorem 6 ($9 \Rightarrow 4$) can not be removed and even replaced by flatness:

EXAMPLE 8. Let R be a regular ring and M a semi-simple left R -module which is not projective. We claim that M satisfies $(*)$ and is A -perfect flat but is not semi-perfect.

Since R is regular, every left R -module is flat and so M satisfies $(*)$. Since M is semi-simple, it is A -perfect. If M has a projective cover, $f : P \rightarrow M$, then $P/\ker f \cong M$ is flat. Since $\ker f \ll P$, $\ker f = 0$ (see [7, 10.5.3]). This gives that $P \cong M$ is projective, which is a contradiction. It follows that M is not semi-perfect.

To be specific, we can take the ring $R = \{(x_1, \dots, x_n, x, x, \dots) \mid x_i, x \in \mathbb{Z}_2, i = 1, \dots, n\}$. Then R is regular and $M := R/\bigoplus_{i=1}^{\infty} F_i$ is simple singular (so it is not projective) R -module, where $F_i = \mathbb{Z}_2, i = 1, 2, \dots$

PROPOSITION 9. *Let M be a flat module. If flat covers of M -generated modules are projective, then M satisfies $(*)$.*

Proof. Let X be an M -generated module and $f : F \rightarrow X$ be a flat cover of X . By hypothesis, F is projective and then by Lemma 4, f is a projective cover of X . Let g be the epimorphism $M^{(\Lambda)} \rightarrow X$ for some index set Λ . Since $M^{(\Lambda)}$ is flat, there exists a homomorphism $h : M^{(\Lambda)} \rightarrow F$ such that $fh = g$. Since $\ker f \ll F$, h is an epimorphism. So F is M -generated. □

Recall that an ideal I of a ring R is called *left t -nilpotent* if, for any sequence a_1, a_2, \dots in I , there exists an n such that $a_1 a_2 \dots a_n = 0$. A module M is called a *progenerator* if M is a finitely generated projective generator.

Mares [8, Theorem 7.6] prove that if M is a progenerator, then M is perfect if and only if M is semi-perfect and the Jacobson radical $J(R)$ is left t -nilpotent. After Mares, in [5, Theorem 1], it is proved that a projective module M is perfect if and only if M is semi-perfect and $J(\text{Tr}(M))$ is left t -nilpotent, where $\text{Tr}(M)$ is the trace ideal $\sum\{f(M) \mid f \in \text{Hom}_R(M, R)\}$ of M . This gives the following result via Theorem 6.

THEOREM 10. *If M is a projective module which satisfies $(*)$, then the following are equivalent.*

- (1) M is perfect.
- (2) M is A -perfect and $J(\text{Tr}(M))$ is left t -nilpotent.

If M is a generator, then the trace ideal of M is R .

COROLLARY 11. *If M is a projective generator, then the following are equivalent.*

- (1) M is perfect.
- (2) M is A -perfect and $J(R)$ is left t -nilpotent.

PROPOSITION 12. *The class of A -perfect modules is closed under factor modules.*

Proof. Let N be a submodule of an A -perfect module M and K an M/N -generated flat module. Then K is M -generated flat and by assumption, it is M -projective. Hence, K is M/N -projective. Thus, M/N is A -perfect. \square

We know from [6] that an abelian group is quasi-projective if and only if it is free or a torsion group such that every p -component A_p is a direct sum of cyclic groups of the same order p^n . If G is a non-zero A -perfect flat (= torsion-free) abelian group, then it is quasi-projective and hence it is free. But this leads to a contradiction because \mathbb{Z} is not an A -perfect \mathbb{Z} -module. As a consequence we obtain the result below:

PROPOSITION 13. *A non-zero abelian group G is torsion if and only if it is A -perfect.*

Proof. The necessity follows from Example 3. For the sufficiency, let G be A -perfect and consider the torsion subgroup $T(G)$ of G . If $T(G) \neq G$, then $G/T(G)$ is a non-zero torsion-free A -perfect abelian group by Proposition 12, but this is impossible. Thus, $G = T(G)$. \square

It can be easily seen that a principal ideal domain R is A -perfect if and only if there exists a finitely generated torsion-free A -perfect R -module.

The class of A -perfect modules need not be closed under direct sums.

EXAMPLE 14. If R is a left A -perfect ring which is not left perfect (see [2] for such a ring), then $R^{(\mathbb{N})}$ is not A -perfect as a left R -module.

Proof. Since ${}_R R^{(\mathbb{N})}$ is free, it is a generator for left R -modules, and so it satisfies (*). If ${}_R R^{(\mathbb{N})}$ was A -perfect, then it would be semi-perfect by Theorem 6. Thus, R would be left perfect by [11, 43.9], which is a contradiction. \square

PROPOSITION 15. *Let $M = \bigoplus_{i=1}^n M_i$ be a module. Suppose that $\bigoplus_{k=1}^{i-1} M_k$ is M_i -generated and M_i is $\bigoplus_{k=1}^{i-1} M_k$ -generated for each $i = 2, \dots, n$. Then each M_i is A -perfect if and only if M is A -perfect.*

Proof. The sufficiency is clear by Proposition 12. For the necessity it is enough to prove the statement for $n = 2$. The rest of the proof follows from induction. Let M_1 and M_2 be A -perfect and suppose that M_1 is M_2 -generated and M_2 is M_1 -generated. If K is an $M_1 \oplus M_2$ -generated flat module, then K is both M_1 - and M_2 -generated by hypothesis. Hence, K is both M_1 - and M_2 -projective which implies that K is $M_1 \oplus M_2$ -projective. \square

COROLLARY 16. *A module M is A -perfect if and only if M^n is A -perfect for any positive integer n .*

PROPOSITION 17. *If M_1 is an A -perfect generator and M_2 is semi-simple, then $M_1 \oplus M_2$ is A -perfect.*

Proof. Let X be an $M_1 \oplus M_2$ -generated flat module. Since M_1 is a generator, X is M_1 -generated. By hypothesis, it is M_1 -projective. X is also M_2 -projective because M_2 is semi-simple. Hence, X is $M_1 \oplus M_2$ -projective and thus $M_1 \oplus M_2$ is A -perfect. \square

The next two theorems characterize left A -perfect and left perfect rings in terms of A -perfect modules, respectively.

THEOREM 18. *The following are equivalent for a ring R .*

- (1) R is a left A -perfect ring.
- (2) Every finitely generated left R -module is A -perfect.
- (3) Every finitely generated projective left R -module is A -perfect.

Proof. The implications (2) \Rightarrow (3) \Rightarrow (1) are obvious. For (1) \Rightarrow (2), let M be a finitely generated R -module and F an M -generated flat R -module. Then consider the epimorphism $g : R^n \rightarrow M$ for some n and the canonical epimorphism $\pi : M \rightarrow M/N$ for any submodule N of M . Since F is R -projective, there exists $h : F \rightarrow R^n$ such that $\pi gh = f$, for any homomorphism $f : F \rightarrow M/N$. Define $h' = gh$. Then we obtain that $\pi h' = f$ which means that F is M -projective. \square

Note that a ring R is left perfect if and only if every left R -module is semi-perfect, if and only if every projective left R -module is semi-perfect (see [11, 42.3; 43.9]).

THEOREM 19. *The following are equivalent for a ring R .*

- (1) R is left perfect.
- (2) Every left R -module is A -perfect.
- (3) Every projective left R -module is A -perfect.
- (4) Every free left R -module is A -perfect.
- (5) ${}_R R^{(\mathbb{N})}$ is A -perfect.

Proof. (1) \Rightarrow (2) is obvious because every flat left module is projective over a left perfect ring. The implications (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5) are obvious. For (5) \Rightarrow (1), ${}_R R^{(\mathbb{N})}$ is semi-perfect by Theorem 6 and hence R is left perfect by [11, 43.9]. \square

In [2], it is proved that the polynomial ring $R[x]$, in one indeterminate x , is not an (left or right) A -perfect ring for any ring R . However, by Theorem 19, we see that $R[x]$ is A -perfect as a left R -module if R is left perfect.

THEOREM 20. *Let ${}_R M$ be a progenerator and $S = \text{End}_R(M)$. The following are equivalent.*

- (1) ${}_R M$ is A -perfect.
- (2) S is left A -perfect.
- (3) R is left A -perfect.

Proof. (1) \Rightarrow (2) We will use the notation \otimes instead of \otimes_S in this proof. Let X be a flat left S -module. We claim that X is S -projective, that is,

$$\text{Hom}_S(X, S) \longrightarrow \text{Hom}_S(X, S/I) \longrightarrow 0$$

is exact for any exact sequence $S \longrightarrow S/I \longrightarrow 0$, where I is a left ideal of S . Since M is an R - S -bimodule and ${}_R M$ is flat, $M \otimes X$ is a flat left R -module, so it is M -projective by hypothesis. Note that $M \cong M \otimes S$ as an R -module. So $M \otimes X$ is $M \otimes S$ -projective. This gives the following exact sequences, where vertical maps are isomorphisms by

[3, Propositions 20.6 and 20.10] and this completes the proof.

$$\begin{array}{ccccccc}
 \text{Hom}_R(M \otimes X, M \otimes S) & \longrightarrow & \text{Hom}_R(M \otimes X, M \otimes S/I) & \longrightarrow & 0 \\
 \downarrow & & \downarrow & & \\
 \text{Hom}_S(X, \text{Hom}_R(M, M \otimes S)) & \longrightarrow & \text{Hom}_S(X, \text{Hom}_R(M, M \otimes S/I)) & \longrightarrow & 0 \\
 \downarrow & & \downarrow & & \\
 \text{Hom}_S(X, S) & \longrightarrow & \text{Hom}_S(X, S/I) & \longrightarrow & 0
 \end{array}$$

(2) \Rightarrow (3) Since ${}_R M$ is a progenerator, R is Morita equivalent to S (see [3, Corollary 22.5]). By [2, Proposition 3.4], R is left A -perfect.

(3) \Rightarrow (1) Since ${}_R M$ is finitely generated, ${}_R M$ is A -perfect by Theorem 18. □

COROLLARY 21. *Let $e^2 = e \in R$ such that $ReR = R$. Then Re is an A -perfect left R -module if and only if $\text{End}_R(Re) \cong eRe$ is a left A -perfect ring, if and only if R is a left A -perfect ring.*

Proof. $\text{Tr}(Re) = ReR = R$ and so ${}_R Re$ is a progenerator. So the proof follows from Theorem 20. □

If ${}_R M$ is a progenerator, then M_S is a progenerator, where $S = \text{End}_R(M)$ and $R \cong \text{End}_S(M_S)$ (see [11, 18.8]). Then by Theorem 20, M_S is A -perfect if and only if S is right A -perfect, if and only if R is right A -perfect. Note that the notion of A -perfect rings is not left–right symmetric [2, Example 3.3].

In Theorem 20, (1) $\not\Rightarrow$ (2) and (3) if M is not a generator:

EXAMPLE 22. Let K be a field and I an infinite index set. Let $R = \prod_{i \in I} K_i$ such that for each $i \in I$, $K_i = K$. Then $M := \bigoplus_{i \in I} K_i$ is a non-finitely generated projective R -module which is not a generator. $\text{End}_R(M) \cong R$ is not A -perfect since R is not semi-perfect. But M is A -perfect since it is semi-simple.

In Theorem 20, (3) $\not\Rightarrow$ (1) and (2) if M is not finitely generated:

EXAMPLE 23. Consider an A -perfect ring R that is not left perfect. Let ${}_R M = R^{(\mathbb{N})}$. Then M is a non-finitely generated projective generator. ${}_R M$ and $\text{End}({}_R M)$ are not A -perfect by Example 14 and [11, 43.9].

In Theorem 20, (2) $\not\Rightarrow$ (1):

EXAMPLE 24. As we mentioned before, the abelian group \mathbb{Q} is not A -perfect. On the other hand, since $\text{End}(\mathbb{Q}_{\mathbb{Z}}) \cong \mathbb{Q}_{\mathbb{Q}}$, $\text{End}(\mathbb{Q}_{\mathbb{Z}})$ is an A -perfect ring.

In Theorem 20, (2) $\not\Rightarrow$ (3) if M is not a generator:

EXAMPLE 25. Let R be a ring with a simple projective module M and not right A -perfect (e.g. any ring with non-zero projective socle which is not semi-perfect). Then $\text{End}(M)$ is a division ring and so a right A -perfect ring. But M is not a generator.

REMARK 26. After the submission of our paper, the paper [1] is appeared and Amini–Amini–Ershad call any module M almost-perfect if flat covers of M -cyclic

modules are projective. This is the condition (4) in Theorem 6 and so almost-perfect in the sense of [1] implies almost-perfect in our sense. But the converse need not be true. For example, any semi-simple module is almost-perfect in our sense but need not be almost-perfect in the sense of [1]. We should also note that eRe is left A -perfect if and only if R is left A -perfect for any non-zero idempotent e in R by [1, Proposition 2.24] (cf. Corollary 21).

ACKNOWLEDGEMENTS. The authors are supported by the project of Hacettepe University of number 08 G 702 001. The first author thanks The Scientific Technological Research Council of Turkey (TUBITAK) for the grant.

REFERENCES

1. B. Amini, A. Amini and M. Ershad, Almost-perfect rings and modules, *Comm. Algebra* **37** (2009), 4227–4240.
2. A. Amini, M. Ershad and H. Sharif, Rings over which flat covers of finitely generated modules are projective, *Commun. Algebra* **36** (8) (2008), 2862–2871.
3. F. W. Anderson and K. R. Fuller, *Rings and categories of modules* (Spring-Verlag, New York, 1974).
4. L. Bican, R. El Bashir and E. Enochs, All modules have flat covers, *Bull. Lond. Math. Soc.* **33** (2001), 385–390.
5. R. S. Cunningham and E. A. Rutter, Jr., Perfect modules, *Math. Z.* **140** (1974), 105–110.
6. L. Fuchs and K. M. Rangaswamy, Quasi-projective abelian groups, *Bull. Soc. Math. France* **98** (1970), 5–8.
7. F. Kasch, *Modules and rings*, *London Mathematical Society Monographs*, 17 (Academic Press, London, New York, 1982).
8. E. Mares, Semiperfect modules, *Math. Z.* **82** (1963), 347–360.
9. K. Varadarajan, Perfect modules, *Acta Math. Hungar.* **78** (1–2) (1998), 1–9.
10. R. Ware, Endomorphism rings of projective modules, *Trans. Amer. Math. Soc.* **155** (1971), 233–256.
11. R. Wisbauer, *Foundations of module and ring theory* (Gordon and Breach, Reading, PA, 1991).