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FILTER ADJUNCTION OF SPACES AND COMPACTIFICATIONS

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1. Introduction. The problem of describing the T_1 compactifications of a given T_1 space arises quite naturally in many contexts, and has been approached from a number of directions. One characteristic of all approaches has been the exclusive consideration of strict topological extensions. There are obvious advantages to this approach. Points of the compactification may be distinguished by their trace filters, and the topology of the compactification is readily described in a natural manner. Moreover every T_2 compactification is strict, so the method loses no generality in this most important special case. However, in the study of T_1 compactifications many important ones are not strict, and are in fact of an entirely opposite nature. The present paper examines a special and in a sense prototypical class of such non-strict compactifications.

The principal method through which the present results are obtained is the method of filter adjunction, through which from a pair (X, Y) of spaces and a filter λ on X a space $T(X, Y, \lambda)$ is obtained which contains X as a dense open subspace with remainder Y. The process is functorial in nature and frequently preserves topological properties of the spaces and maps to which it is applied. It is a simple generalization of the usual construction of one point extensions.

The main result of this paper is a description of the possible outgrowths of a space in its T_1 compactifications. The result is very simple; the class of outgrowths of an infinite discrete space in T_1 compactifications is the class of nonempty compact T_1 spaces, and the class of outgrowths of a non-discrete T_1 space is the class of all nonempty spaces. Thus outgrowth classes do not distinguish spaces.

Several results concerning the embedding of categories of spaces into categories of compact spaces are also established.

The term *space* shall mean a nonempty topological space, and the term *map* shall mean a continuous function between spaces.

2. The adjunction construction. All constructions in this paper begin with the disjoint union bifunctor in the category of sets and functions. To establish notation, given a pair (X, Y) of sets the object $T(X, Y)$ is their disjoint union with inclusion $G(X, Y)$ of X into $T(X, Y)$ and inclusion $H(X, Y)$ of Y into $T(X, Y)$. Given a pair (f, g) of functions $f : X \to W$ and $g : Y \to Z$ the function $T(f, g) : T(X, Y) \to T(W, Z)$ is defined by the relations $G(W, Z) \cdot f = T(f, g) \cdot G(X, Y)$ and $H(W, Z) \cdot g = T(f, g) \cdot H(X, Y)$.

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Now let X and Y be spaces, with a filter λ on X. Define a topology on the set $T(X, Y)$ by declaring V to be open if $G(X, Y)^{\leftarrow}[V]$ is open in X, $H(X, Y)^{\leftarrow}[V]$ is open in Y, and $G(X, Y)^{\leftarrow}[V] \in \lambda$ whenever $H(X, Y)^{\leftarrow}[V] \neq \emptyset$. Write $T(X, Y, \lambda)$ for the resulting space.

There are various important filters on X. In particular the improper filter θ of all open subsets of X, the filter λ_{kc} of complements of closed compact subsets, the filter λ_{fc} of complements of closed finite subsets, and the smallest filter $\lambda_X = \{X\}$.

2.1. *Let X, Y be spaces with an open filter λ on X.*

a) *$G(X, Y)$ is an embedding of X as open subspace of $T(X, Y, \lambda)$.*

b) *$H(X, Y)$ is an embedding of Y as closed subspace of $T(X, Y, \lambda)$.*

c) *$T(X, Y, \lambda)$ is the sum of X and Y if and only if $\lambda = \theta$.*

d) *$G(X, Y)[X]$ is dense in $T(X, Y, \lambda)$ if and only if $\lambda \neq \theta$.*

2.2. *Let X, Y be spaces with an open filter λ on X.*

a) *$T(X, Y, \lambda)$ is T_0 if and only if X and Y are T_0.*

b) *$T(X, Y, \lambda)$ is T_1 if and only if X and Y are T_1 and $\lambda_{fc} \subset \lambda$.*

c) *$T(X, Y, \lambda)$ is T_2 if and only if X is T_2, Y has exactly one point, and λ has no cluster point in X.*

d) *$T(X, Y, \lambda)$ is compact if and only if Y is compact and $\lambda \subset \lambda_{kc}$.*

e) *$T(X, Y, \lambda)$ is connected if and only if X is the only closed member of λ.*

One particularly simple type of extension is useful in attaching non-compact spaces as outgrowths in compactifications.

Given a space X and $x \in X$, set $X_x = X - x$, and let λ_x be the trace on X_x of the neighborhood filter of $x \in X$.

2.3. *For any space X and $x \in X$, and for any space Y and $y \in Y$, the subspace $G(X_x, Y)[X_x] \cup H(X_x, Y)(y)$ of $T(X_x, Y, \lambda_x)$ is homeomorphic to X.*

2.4. *Let X, Y, W, Z be spaces with open filters λ on X and ω on W, and functions $f : X \rightarrow W$ and $g : Y \rightarrow Z$.*

a) *$T(f, g)$ is one-one if and only if f and g are one-one.*

b) *$T(f, g)$ is onto if and only if f and g are onto.*

c) *$T(f, g)$ is a map if and only if f and g are maps and $f^{\leftarrow}(\omega) \subset \lambda$.*

d) *$T(f, g)$ is an embedding if and only if f and g are embeddings and $\lambda = f^{\leftarrow}(\omega)$.*

e) *$T(f, g)$ is open if and only if f and g are open and $f(\lambda) \subset \omega$.*

f) *$T(f, g)$ is closed if and only if f and g are closed, $\lambda \subset f^{\leftarrow}(\omega)$, and either $f^{\leftarrow}(\omega) = \theta$ or g is onto.*

Proof. The proofs of all the assertions are similar. The proof of f) will be given as a sample. Write $T = T(X, Y, \lambda)$, $S = T(W, Z, \omega)$, $G = G(X, Y)$, $H = H(X, Y)$, $K = G(W, Z)$, and $L = H(W, Z)$.

Suppose $T(f, g)$ is a closed function. Let $A \subset X$ be closed; then $G(A) \cup H[Y] \subset T$ is closed, so $K[f[A]] \cup L[g[Y]]$ is closed in S, and thus $f[A]$ is closed in W. Similarly if $B \subset Y$ is closed then $G[X] \cup H[B]$ is closed in T, so

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$K[f[X]] \cup L[g[B]]$ is closed in S, and thus $g[B]$ is closed in Y. This establishes that f and g are closed.

Now if $V \in \lambda$ then $G[X - V] \subset T$ is closed, so $K[f[X - V]] \subset S$ is closed, which means $U = W - f[X - V] \in \omega$. Since $f^{\leftarrow}[U] \subset V$ it follows that $V \in f^{\leftarrow}(\omega)$. Thus $\lambda \subset f^{\leftarrow}(\omega)$. Also since $G[X] \cup H[Y]$ is closed then $K[f[X)) \cup L[g[Y]]$ is closed, which means either $g[Y] = Z$, or $f[X] \cap V = \emptyset$ for some $V \in \omega$, that is, $f^{\leftarrow}(\omega) = \theta$.

Conversely, suppose f and g are closed, $\lambda \subset f^{\leftarrow}(\omega)$, and $T(f, g)$ is not a closed function. Then $f^{\leftarrow}(\omega) \neq \theta$, and $g[Y] \neq Z$. To see these last two statements, note first that since $T(f, g)$ is not closed there are closed sets $A \subset X$ and $B \subset Y$ such that $G[A] \cup H[B]$ is closed and $K[f[A]] \cup L[g[B]]$ is not closed. Since $f[A]$ and $g[B]$ are closed then $g[B] \neq Z$, and $f[A] \cap V \neq \emptyset$ for all $V \in \omega$. This gives $A \cap f^{\leftarrow}[V] \neq \emptyset$ for all $V \in \omega$, so $f^{\leftarrow}(\omega) \neq \theta$. Also since $\lambda \subset f^{\leftarrow}(\omega)$ then $A \cap W \neq \emptyset$ for all $W \in \lambda$. Since $G[A] \cup H[B]$ is closed then $B = Y$, and thus $g[Y] \neq Z$.

2.5. *Remark.* The construction of $T(X, Y, \lambda)$ may also be performed as a combination of a generalized one-point extension, as in [**1**, 6, § 5], and a map attachment construction, as in [**2**, VI, § 6]. Specifically, let A be the set Y with discrete topology, and let j be the identity map from A to Y. Construct Z as a space whose ground set is the disjoint union of X and A, with X as open subspace and A as closed subspace, in which the trace filter on X of each point of A is the filter λ. Then $T(X, Y, \lambda)$ is homeomorphic to the attachment space $Z \cup_j Y$.

The extensions $T(X, Y, \lambda_X)$ and $T(X, Y, \lambda_{fc})$ have special minimal properties.

2.6. *$T(X, Y, \lambda_X)$ has the smallest topology for which $G(X, Y)$ is an open embedding and $H(X, Y)$ is an embedding.*

2.7. *If X and Y are T_1 and X is infinite then $T(X, Y, \lambda_{fc})$ has the smallest T_1 topology for which $G(X, Y)$ is an open embedding and $H(X, Y)$ is an embedding.*

3. Compactifications. A *compactification* of the infinite space X is a pair (Z, k) in which Z is a compact space and k is a dense embedding of X into Z; the *outgrowth* is the subspace $Z_k = Z - k[X]$. A *map* from the compactification (Z, k) to the compactification (Y, h) is a map m from Z to Y such that $mk = h$ and $m[Z_k] \subset Y_h$. The compactifications of X form a category $\mathscr{C}_X$, preordered by the relation $(Z, k) > (Y, h)$ if there is a map m with $mk = h$.

The following results show that the order structure of the category of compactifications of any infinite space is as complex as the order structure of the category $\mathscr{K}$ of compact spaces, in which $Z > Y$ if there is a map from Z to Y.

Given $Z \in \mathscr{K}$, set $K(Z) = (T(X, Z, \lambda_{fc}), G(X, Z))$, and if m is a map from Z to Y set $K(m) = T(\mathrm{id}_X, m)$.

3.1. *The function K is a full embedding functor from the category λ to the category $\mathscr{C}_X$.*

Proof. By 2.1, 2.2, and 2.4 K is a functor from $\mathscr{K}$ to $\mathscr{C}_X$. It is clearly one-one on objects and maps. If p is a map from the compactification $K(Z)$ of X to the compactification $K(Y)$ then since p preserves outgrowths there is a map m from Z to Y such that $p = K(m)$.

The class of maps considered in the preorder $Z > Y$ may be restricted, as for example to closed surjections, and the compactifications may be correspondingly restricted.

In the remainder of this paper we shall consider only T_1 spaces and compactifications.

3.2. *An infinite space contains every compact space in its outgrowth class.*

3.3. *A discrete space is open in every extension.*

3.4. *The outgrowth class of an infinite discrete space is the class of all compact spaces.*

3.5. *The outgrowth class of a non-discrete space is the class of all spaces.*

Proof. Let X be a non-discrete space, with a non-isolated point x. Let Y be any space, with $y \in Y$. Set $R = T(Y_y, x, \lambda_{fc})$ and $S = T(X_x, R, \lambda_x)$. Let ν be the collection of all open subsets V of S such that $G(X_x, R)^{\leftarrow}[V] \in \lambda_{kc}$ and $G(Y_y, x)^{\leftarrow}[H(X_x, R)^{\leftarrow}[V]] \in \lambda_y$. It is easy to see that $\lambda_{fc} \subset \nu \subset \lambda_{kc}$. It follows from 2.1 b) and d) that $Z = T(S, y, \nu)$ is a compact (T_1) space. The subspace $G(S, y)[G(X_x, R)[X_x] \cup H(X_x, R)[H(Y_y, x)(x)]]$ of Z is homeomorphic to X and its complement is homeomorphic to Y. Thus X has a compactification with outgrowth Y.

3.6. *A non-discrete space has a compactification in which it is not open.*

Proof. It is simple to see that X is not open in the compactification constructed in 3.5.

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