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Some Estimates for Generalized Commutators of Rough Fractional Maximal and Integral Operators on Generalized Weighted Morrey Spaces

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Abstract. In this paper, we establish BMO estimates for generalized commutators of rough fractional maximal and integral operators on generalized weighted Morrey spaces, respectively.

1 Introduction and Main Results

The classical Morrey spaces $M_{p,\lambda}$ were introduced by Morrey in [14] to study the local behavior of solutions of second order elliptic partial differential equations (PDEs). In recent years there has been an explosion of interest in the study of the boundedness of operators on Morrey-type spaces. It has been obtained that many properties of solutions to PDEs are concerned with the boundedness of some operators on Morrey-type spaces. In fact, better inclusion between Morrey and Hölder spaces allows one to obtain higher regularity of the solutions to different elliptic and parabolic boundary problems; see [2,6,16,17] for details. Moreover, a variety of Morrey spaces are defined in the process of study. Mizuhara [13] introduced the generalized Morrey spaces $M_{p,\varphi}$; Komori and Shirai [11] defined the weighted Morrey spaces $L_{p,\kappa}(w)$, and Guliyev [8] and Karaman [10] gave a concept of generalized weighted Morrey spaces $M_{p,\varphi}(w)$ that could be viewed as extension of both $M_{p,\varphi}$ and $L_{p,\kappa}(w)$. The boundedness of some operators such as Hardy–Littlewood maximal operator, the fractional integral operator as well as the fractional maximal operator and the Calderón–Zygmund singular integral operator on these Morrey spaces can be seen in [8,10,11,13].

Let us consider the following generalized commutator of rough fractional integral operators:

$$T^A_{\Omega,\alpha}f(x)=\int_{\mathbb{R}^n}\frac{\Omega(x-y)}{|x-y|^{n-\alpha+m-1}}R_m(A;x,y)f(y)dy \qquad 0<\alpha<n$$

and the corresponding generalized commutator of rough fractional maximal operators:

$$M^A_{\Omega,\alpha}f(x)=\sup_{r>0}\frac{1}{r^{n-\alpha+m-1}}\int_{|x-y|<r}\Big|\Omega(x-y)R_m(A;x,y)f(y)\Big|dy \qquad 0<\alpha<n,$$

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where $\Omega \in L_s(S^{n-1})(s>1)$ is homogeneous of degree zero in $\mathbb{R}^n$, $m \in \mathbb{N}$, A is a function defined on $\mathbb{R}^n$, and $R_m(A;x,y)$ denotes the m-th order Taylor series remainder of A at x about y, that is,

$$R_m(A;x,y) = A(x) - \sum_{|\gamma|<m} \frac{1}{\gamma!} D^\gamma A(y)(x-y)^\gamma,$$

$\gamma = (\gamma_1,\ldots,\gamma_n)$, each $\gamma_i (i = 1,\ldots,n)$ is a nonnegative integer, $|\gamma| = \sum_{i=1}^n \gamma_i$, $\gamma! = \gamma_1! \cdots \gamma_n!$, $x^\gamma = x_1^{\gamma_1} \cdots x_n^{\gamma_n}$, and $D^\gamma = \frac{\partial^{|\gamma|}}{\partial^{\gamma_1} x_1 \cdots \partial^{\gamma_n} x_n}$.

For $m = 1$, $T^A_{\Omega,\alpha}$ and $M^A_{\Omega,\alpha}$ are obviously the commutator operators,

$$\begin{aligned}[A, T_{\Omega,\alpha}]f(x) &= A(x)T_{\Omega,\alpha}f(x) - T_{\Omega,\alpha}(Af)(x)\\ &= \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n-\alpha}} \big(A(x) - A(y)\big) f(y)dy\end{aligned}$$

and

$$\begin{aligned}[A, M_{\Omega,\alpha}]f(x) &= A(x)M_{\Omega,\alpha}f(x) - M_{\Omega,\alpha}(Af)(x)\\ &\le \sup_{r>0} \frac{1}{r^{n-\alpha}} \int_{|x-y|<r} |\Omega(x-y)||A(x) - A(y)||f(y)|dy,\end{aligned}$$

where rough fractional integral operator $T_{\Omega,\alpha}$ and rough fractional maximal operator $M_{\Omega,\alpha}$ are defined by

$$T_{\Omega,\alpha}f(x) = \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n-\alpha}} f(y)dy \qquad 0 < \alpha < n$$

and

$$M_{\Omega,\alpha}f(x) = \sup_{r>0} \frac{1}{r^{n-\alpha}} \int_{|x-y|<r} |\Omega(x-y)||f(y)t|dy \qquad 0 < \alpha < n.$$

The weighted (L_p, L_q)-boundedness and weak boundedness of the operators $T_{\Omega,\alpha}$ and $M_{\Omega,\alpha}$ were given in [4] and [5], respectively. On the other hand, if $m \ge 2$, then $T^A_{\Omega,\alpha}$ and $M^A_{\Omega,\alpha}$ are nontrivial generalization of the above commutators, respectively. The weighted (L_p, L_q)-boundedness of the operators $T^A_{\Omega,\alpha}$ and $M^A_{\Omega,\alpha}$ have been given by Wu and Yang in [19], where they proved the following result.

Theorem 1.1 *Suppose that* $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, Ω *is homogeneous of degree zero with* $\Omega \in L_s(S^{n-1})(s>1)$. *Moreover,* $|\gamma| = m-1$, $m \ge 2$, *and* $D^\gamma A \in BMO(\mathbb{R}^n)$. *If* $s' < p$, $w(x)^{s'} \in A(\frac{p}{s'}, \frac{q}{s'})$, *then there exists a constant C, independent of A and f, such that*

$$\|T^A_{\Omega,\alpha}f\|_{L_q(w^q,\mathbb{R}^n)} \le C \sum_{|\gamma|=m-1} \|D^\gamma A\|_* \|f\|_{L_p(w^p,\mathbb{R}^n)},$$

$$\|M^A_{\Omega,\alpha}f\|_{L_q(w^q,\mathbb{R}^n)} \le C \sum_{|\gamma|=m-1} \|D^\gamma A\|_* \|f\|_{L_p(w^p,\mathbb{R}^n)}.$$

Here and in the sequel, p' always denotes the conjugate index of any $p > 1$; that is, $\frac{1}{p} + \frac{1}{p'} = 1$, and C stands for a constant that is independent of the main parameters, but it may vary from line to line.

Let $B = B(x_0, r_B)$ denote the ball with the center x_0 and radius r_B. For a given measurable set E, we also denote the Lebesgue measure of E by $|E|$. Let X be a measurable set in $\mathbb{R}^n$. For any given $X \subseteq \mathbb{R}^n$ and $0 < p < \infty$, denote by $L_p(X)$ the spaces of all functions f satisfying

$$\|f\|_{L_p(X)} = \Big(\int_X |f(x)|^p dx \Big)^{\frac{1}{p}} < \infty.$$

We recall the definition of classical Morrey spaces $M_{p,\lambda}$ as

$$M_{p,\lambda}(\mathbb{R}^n) = \Big\{ f : \|f\|_{M_{p,\lambda}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n, r>0} r^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,r))} < \infty \Big\},$$

where $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$, $0 \le \lambda \le n$, and $1 \le p < \infty$.

Note that $M_{p,0} = L_p(\mathbb{R}^n)$ and $M_{p,n} = L_\infty(\mathbb{R}^n)$. If $\lambda < 0$ or $\lambda > n$, then $M_{p,\lambda} = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.

We also denote by $WM_{p,\lambda} \equiv WM_{p,\lambda}(\mathbb{R}^n)$ the weak Morrey space of all functions $f \in WL_p^{\mathrm{loc}}(\mathbb{R}^n)$ for which

$$\|f\|_{WM_{p,\lambda}} \equiv \|f\|_{WM_{p,\lambda}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n,\ r>0} r^{-\frac{\lambda}{p}} \|f\|_{WL_p(B(x,r))} < \infty,$$

where $WL_p(B(x,r))$ denotes the weak L_p-space of measurable functions f for which

$$\begin{aligned}\|f\|_{WL_p(B(x,r))} &\equiv \|f\chi_{B(x,r)}\|_{WL_p(\mathbb{R}^n)} \\ &= \sup_{t>0} t\big|\{y \in B(x,r) : |f(y)| > t\}\big|^{1/p} \\ &= \sup_{0<t\le|B(x,r)|} t^{1/p}(f\chi_{B(x,r)})^*(t) < \infty,\end{aligned}$$

where g^* denotes the non-increasing rearrangement of a function g.

Throughout the paper we assume that $x \in \mathbb{R}^n$ and $r > 0$ and also that $B(x,r)$ denotes the open ball centered at x of radius r, $B^C(x,r)$ denotes its complement, and $|B(x,r)|$ is the Lebesgue measure of the ball $B(x,r)$ and $|B(x,r)| = v_n r^n$, where $v_n = |B(0,1)|$. It is known that $M_{p,\lambda}(\mathbb{R}^n)$ is an extension of $L_p(\mathbb{R}^n)$ in the sense that $M_{p,0} = L_p(\mathbb{R}^n)$.

On the other hand, Mizuhara [13] has given generalized Morrey spaces $M_{p,\varphi}$ considering $\varphi(r)$ instead of r^λ in the above definition of the Morrey space. Later, we have defined the generalized Morrey spaces $M_{p,\varphi}$ with normalized norm as follows.

Definition 1.2 (Generalized Morrey space) Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and $1 \le p < \infty$. We denote by $M_{p,\varphi} \equiv M_{p,\varphi}(\mathbb{R}^n)$ the generalized Morrey space, the space of all functions $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$ with finite quasinorm

$$\|f\|_{M_{p,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} |B(x,r)|^{-\frac{1}{p}} \|f\|_{L_p(B(x,r))}.$$

Also by $WM_{p,\varphi} \equiv WM_{p,\varphi}(\mathbb{R}^n)$ we denote the weak generalized Morrey space of all functions $f \in WL_p^{\mathrm{loc}}(\mathbb{R}^n)$ for which

$$\|f\|_{WM_{p,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} |B(x,r)|^{-\frac{1}{p}} \|f\|_{WL_p(B(x,r))} < \infty.$$

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According to this definition, we recover the Morrey space $M_{p,\lambda}$ and weak Morrey space $WM_{p,\lambda}$ under the choice $\varphi(x,r) = r^{\frac{\lambda-n}{p}}$:

$$M_{p,\lambda} = M_{p,\varphi}\,|_{\varphi(x,r)=r^{\frac{\lambda-n}{p}}}, \quad WM_{p,\lambda} = WM_{p,\varphi}\,|_{\varphi(x,r)=r^{\frac{\lambda-n}{p}}}.$$

During the last decades various classical operators, such as maximal, singular, and potential operators have been widely investigated in classical and generalized Morrey spaces.

Komori and Shirai [11] introduced a version of the weighted Morrey space $L_{p,\kappa}(w)$, which is a natural generalization of the weighted Lebesgue space $L_p(w)$, and investigated the boundedness of classical operators in harmonic analysis.

Definition 1.3 (Weighted Morrey space) Let $1 \le p < \infty$, $0 < \kappa < 1$ and let w be a weight function. We denote by $L_{p,\kappa}(w) \equiv L_{p,\kappa}(\mathbb{R}^n, w)$ the weighted Morrey space of all classes of locally integrable functions f with the norm

$$\|f\|_{L_{p,\kappa}(w)} = \sup_{x\in\mathbb{R}^n, r>0} w(B(x,r))^{-\frac{\kappa}{p}} \|f\|_{L_{p,w}(B(x,r))} < \infty.$$

Furthermore, by $WL_{p,\kappa}(w) \equiv WL_{p,\kappa}(\mathbb{R}^n, w)$ we denote the weak weighted Morrey space of all classes of locally integrable functions f with the norm

$$\|f\|_{WL_{p,\kappa}(w)} = \sup_{x\in\mathbb{R}^n, r>0} w(B(x,r))^{-\frac{\kappa}{p}} \|f\|_{WL_{p,w}(B(x,r))} < \infty.$$

Remark 1.4 Alternatively, we could define the weighted Morrey spaces with cubes instead of balls. Hence, we shall use these two definitions of weighted Morrey spaces appropriate to calculation.

Remark 1.5 (i) If $w \equiv 1$ and $\kappa = \lambda/n$ with $0 \le \lambda \le n$, then $L_{p,\lambda/n}(1) = M_{p,\lambda}(\mathbb{R}^n)$ is the classical Morrey space.

(ii) If $\kappa = 0$, then $L_{p,0}(w) = L_p(w)$ is the weighted Lebesgue space.

On the other hand, the generalized weighted Morrey spaces $M_{p,\varphi}(w)$ were introduced by Guliyev [8] and Karaman [10] as follows.

Definition 1.6 (Generalized weighted Morrey space) Let $1 \le p < \infty$, $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and let w be non-negative measurable function on $\mathbb{R}^n$. We denote by $M_{p,\varphi}(w) \equiv M_{p,\varphi}(\mathbb{R}^n, w)$ the generalized weighted Morrey space, the space of all classes of functions $f \in L^{\text{loc}}_{p,w}(\mathbb{R}^n)$ with finite norm

$$\|f\|_{M_{p,\varphi}(w)} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} w(B(x,r))^{-\frac{1}{p}} \|f\|_{L_{p,w}(B(x,r))},$$

where $L_{p,w}(B(x,r))$ denotes the weighted $L_{p,w}$-space of measurable functions f for which

$$\|f\|_{L_{p,w}(B(x,r))} \equiv \|f\chi_{B(x,r)}\|_{L_{p,w}(\mathbb{R}^n)} = \Big(\int_{B(x,r)} |f(y)|^p w(y)\,dy\Big)^{\frac{1}{p}}.$$

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Furthermore, by $WM_{p,\varphi}(w) \equiv WM_{p,\varphi}(\mathbb{R}^n, w)$ we denote the weak generalized weighted Morrey space of all classes of functions $f \in WL^{\rm loc}_{p,w}(\mathbb{R}^n)$ for which

$$\|f\|_{WM_{p,\varphi}(w)} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} w(B(x,r))^{-\frac{1}{p}} \|f\|_{WL_{p,w}(B(x,r))} < \infty,$$

where $WL_{p,w}(B(x,r))$ denotes the weighted weak $WL_{p,w}$-space of measurable functions f for which

$$\|f\|_{WL_{p,w}(B(x,r))} \equiv \|f\chi_{B(x,r)}\|_{WL_{p,w}(\mathbb{R}^n)} = \sup_{t>0} tw\Big(\{y \in B(x,r): |f(y)| > t\}\Big)^{\frac{1}{p}} < \infty.$$

Remark 1.7 (i) If $w \equiv 1$, then $M_{p,\varphi}(1) = M_{p,\varphi}$ is the generalized Morrey space.

(ii) If $\varphi(x,r) \equiv w(B(x,r))^{\frac{\kappa-1}{p}}$, $0 < \kappa < 1$, then $M_{p,\varphi}(w) = L_{p,\kappa}(w)$ is the weighted Morrey space.

(iii) If $\varphi(x,r) \equiv \nu(B(x,r))^{\frac{\kappa}{p}} w(B(x,r))^{-\frac{1}{p}}$, $0 < \kappa < 1$, then $M_{p,\varphi}(w) = L_{p,\kappa}(\nu, w)$ is the two weighted Morrey space.

(iv) If $w \equiv 1$ and $\varphi(x,r) = r^{\frac{\lambda-n}{p}}$ with $0 \le \lambda \le n$, then $M_{p,\varphi}(1) = M_{p,\lambda}$ is the classical Morrey space and $WM_{p,\varphi}(1) = WM_{p,\lambda}$ is the weak Morrey space.

(v) If $\varphi(x,r) \equiv w(B(x,r))^{-\frac{1}{p}}$, then $M_{p,\varphi}(w) = L_p(w)$ is the weighted Lebesgue space.

The aim of this paper is to investigate the boundedness of generalized commutators of rough fractional maximal and integral operators on generalized weighted Morrey spaces, respectively. Our main results can be formulated as follows.

Theorem 1.8 *Suppose that $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, Ω is homogeneous of degree zero with $\Omega \in L_s(S^{n-1})(s > 1)$. Moreover, let A be a function defined on $\mathbb{R}^n$, $|\gamma| = m-1$, $m \ge 2$ and $D^\gamma A \in BMO(\mathbb{R}^n)$. If $s' < p$, $w(x)^{s'} \in A(\frac{p}{s'}, \frac{q}{s'})$, then there exists a constant C, independent of A and f, such that*

$$\text{(1.1)} \quad \|T^A_{\Omega,\alpha} f\|_{L_q(w^q, B(x_0,r))} \le C \sum_{|\gamma|=m-1} \|D^\gamma A\|_* \Big(w^q\big(B(x_0,r)\big)\Big)^{\frac{1}{q}} \times \int_{2r}^{\infty} (1 + \ln\frac{t}{r}) \|f\|_{L_p(w^p, B(x_0,t))} \Big(w^q\big(B(x_0,t)\big)\Big)^{-\frac{1}{q}} \frac{1}{t} dt.$$

Theorem 1.9 *Let $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, Ω be homogeneous of degree zero with $\Omega \in L_s(S^{n-1})(s > 1)$. Suppose that $s' < p$, $w(x)^{s'} \in A(\frac{p}{s'}, \frac{q}{s'})$ and the pair (φ_1, φ_2) satisfies the condition*

$$\text{(1.2)} \qquad \int_r^\infty (1 + \ln\frac{t}{r}) \frac{\operatorname{essinf}_{t<\tau<\infty} \varphi_1(x,\tau)(w^p(B(x,\tau)))^{\frac{1}{p}}}{(w^q(B(x,t)))^{\frac{1}{q}}} \frac{dt}{t} \le C_0 \varphi_2(x,r),$$

where C_0 does not depend on x and r. If $D^\gamma A \in BMO(\mathbb{R}^n)(|\gamma| = m-1, m \ge 2)$, then there is a constant $C > 0$, independent of A and f, such that

$$\text{(1.3)} \qquad \|T^A_{\Omega,\alpha} f\|_{M_{q,\varphi_2}(w^q,\mathbb{R}^n)} \le C \sum_{|\gamma|=m-1} \|D^\gamma A\|_* \|f\|_{M_{p,\varphi_1}(w^p,\mathbb{R}^n)},$$

(1.4) $$\|M^A_{\Omega,\alpha}f\|_{M_{q,\varphi_2}(w^q,\mathbb{R}^n)} \le C \sum_{|\gamma|=m-1} \|D^\gamma A\|_* \|f\|_{M_{p,\varphi_1}(w^p,\mathbb{R}^n)}.$$

Corollary 1.10 *Let $0<\alpha<n$, $1<p<\frac{n}{\alpha}$, $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$, Ω be homogeneous of degree zero with $\Omega\in L_s(S^{n-1})(s>1)$. Also let $s'<p$, $w(x)^{s'}\in A(\frac{p}{s'},\frac{q}{s'})$, and $0<\kappa<\frac{p}{q}$. If $D^\gamma A\in BMO(\mathbb{R}^n)(|\gamma|=m-1, m\ge 2)$, then there is a constant $C>0$, independent of A and f, such that*

$$\|T^A_{\Omega,\alpha}f\|_{L_{q,\frac{\kappa q}{p}}(w^q,\mathbb{R}^n)} \le C \sum_{|\gamma|=m-1} \|D^\gamma A\|_* \|f\|_{L_{p,\kappa}(w^p,w^q,\mathbb{R}^n)},$$

$$\|M^A_{\Omega,\alpha}f\|_{L_{q,\frac{\kappa q}{p}}(w^q,\mathbb{R}^n)} \le C \sum_{|\gamma|=m-1} \|D^\gamma A\|_* \|f\|_{L_{p,\kappa}(w^p,w^q,\mathbb{R}^n)}.$$

In the case of $w=1$ from Theorem 1.9, we get the following new result.

Corollary 1.11 (see [1]) *Let $0<\alpha<n$, $1<p<\frac{n}{\alpha}$, $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$, Ω be homogeneous of degree zero with $\Omega\in L_s(S^{n-1})(s>1)$. Suppose that $s'<p$ and the pair (φ_1,φ_2) satisfies the condition*

$$\int_r^\infty \Big(1+\ln\frac{t}{r}\Big)\frac{\operatorname{essinf}_{t<\tau<\infty}\varphi_1(x,\tau)\tau^{\frac{n}{p}}}{t^{\frac{n}{q}}}\frac{dt}{t} \le C_0\varphi_2(x,r),$$

where C_0 does not depend on x and r. If $D^\gamma A\in BMO(\mathbb{R}^n)(|\gamma|=m-1, m\ge 2)$, then there is a constant $C>0$, independent of A and f, such that

$$\|T^A_{\Omega,\alpha}f\|_{M_{q,\varphi_2}(\mathbb{R}^n)} \le C \sum_{|\gamma|=m-1} \|D^\gamma A\|_* \|f\|_{M_{p,\varphi_1}(\mathbb{R}^n)},$$

$$\|M^A_{\Omega,\alpha}f\|_{M_{q,\varphi_2}(\mathbb{R}^n)} \le C \sum_{|\gamma|=m-1} \|D^\gamma A\|_* \|f\|_{M_{p,\varphi_1}(\mathbb{R}^n)}.$$

2 Some Preliminaries and Basic Lemmas

We begin with some properties of $A_p(\mathbb{R}^n)$ weights that play a great role in the proofs of our main results.

A weight function is a locally integrable function on $\mathbb{R}^n$ that takes values in $(0,\infty)$ almost everywhere. For a weight function w and a measurable set E, we define $w(E)=\int_E w(x)dx$, the Lebesgue measure of E by $|E|$ and the characteristic function of E by χ_E. Given a weight function w, we say that w satisfies the doubling condition if there exists a constant $D>0$ such that for any ball B, we have $w(2B)\le Dw(B)$. When w satisfies this condition, we denote $w\in\Delta_2$, for short.

If w is a weight function, we denote by $L_p(w)\equiv L_p(\mathbb{R}^n,w)$ the weighted Lebesgue space defined by the norm

$$\|f\|_{L_{p,w}} = \Big(\int_{\mathbb{R}^n}|f(x)|^p w(x)dx\Big)^{\frac{1}{p}} < \infty, \quad \text{when } 1\le p<\infty$$

and by $\|f\|_{L_{\infty,w}} = \operatorname{esssup}_{x\in\mathbb{R}^n}|f(x)|w(x)$ when $p=\infty$.

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We denote by $WL_p(w)$ the weighted weak space consisting of all measurable functions f such that

$$\|f\|_{WL_p(w)} = \sup_{t>0} tw\Big(\{x\in\mathbb{R}^n : |f(x)| > t\Big)^{\frac{1}{p}} < \infty.$$

We recall that a weight function w is in the Muckenhoupt's class $A_p(\mathbb{R}^n)$, $1 < p < \infty$, if

$$\begin{aligned}(2.1)\qquad [w]_{A_p} &:= \sup_B [w]_{A_p(B)}\\ &= \sup_B\Big(\frac{1}{|B|}\int_B w(x)dx\Big)\Big(\frac{1}{|B|}\int_B w(x)^{1-p'}dx\Big)^{p-1} < \infty,\end{aligned}$$

where the supremum is taken with respect to all the balls B and $\frac{1}{p} + \frac{1}{p'} = 1$. The expression $[w]_{A_p}$ is called the characteristic constant of w. Note that, by Hölder's inequality, for all balls B we have

$$[w]_{A_p}^{1/p} \geq [w]_{A_p(B)}^{1/p} = |B|^{-1}\|w\|_{L_1(B)}^{1/p}\|w^{-1/p}\|_{L_{p'}(B)} \geq 1.$$

For $p = 1$, the class $A_1(\mathbb{R}^n)$ is defined by

$$(2.2)\qquad \frac{1}{|B|}\int_B w(x)dx \leq C \inf_{x\in B} w(x)$$

for every ball $B \subset \mathbb{R}^n$. Thus, we have the condition $Mw(x) \leq Cw(x)$ with $[w]_{A_1} = \sup_{x\in\mathbb{R}^n}\frac{Mw(x)}{w(x)}$, and also for $p = \infty$, we define

$$A_\infty = \bigcup_{1\leq p<\infty} A_p, \quad [w]_{A_\infty} = \inf_{1\leq p<\infty}[w]_{A_p}, \quad \text{and} \quad [w]_{A_\infty} \leq [w]_{A_p}.$$

One knows that $A_p \subset A_q$ if $1 \leq p < q < \infty$, and that $w \in A_p$ for some $1 < p < q$ if $w \in A_q$ with $q > 1$, and also $[w]_{A_p} \leq [w]_{Aq}$.

By (2.1), we have

$$\Big(w^{-\frac{p'}{p}}(B)\Big)^{\frac{1}{p'}} = \|w^{-\frac{1}{p}}\|_{L_{p'}(B)} \leq C|B|w(B)^{-\frac{1}{p}}$$

for $1 < p < \infty$. Note that

$$(2.3)\qquad \Big(\operatorname*{essinf}_{x\in E} f(x)\Big)^{-1} = \operatorname*{esssup}_{x\in E}\frac{1}{f(x)}$$

is true for any real-valued nonnegative function f and is measurable on E (see [18, p. 143]), and by (2.2) we get

$$\|w^{-1}\|_{L_\infty(B)} = \operatorname*{esssup}_{x\in B}\frac{1}{w(x)} = \frac{1}{\operatorname{essinf}_{x\in B} w(x)} \leq C|B|w(B)^{-1}.$$

We also need another weight class $A(p,q)$ introduced by Muckenhoupt and Wheeden in [15] to study weighted boundedness of fractional integral operators.

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A weight function w belongs to the Muckenhoupt–Wheeden class $A(p,q)$ [15] for $1 < p < q < \infty$ if

$$[w]_{A(p,q)} := \sup_B [w]_{A(p,q)(B)} = \sup_B \Big(\frac{1}{|B|} \int_B w(x)^q dx \Big)^{\frac{1}{q}} \Big(\frac{1}{|B|} \int_B w(x)^{-p'} dx \Big)^{\frac{1}{p'}} < \infty, \tag{2.4}$$

where the supremum is taken with respect to all the balls B. Note that, by Hölder's inequality, for all balls B we have

$$[w]_{A(p,q)} \geq [w]_{A(p,q)(B)} = |B|^{\frac{1}{p} - \frac{1}{q} - 1} \|w\|_{L_q(B)} \|w^{-1}\|_{L_{p'}(B)} \geq 1.$$

Moreover, if $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ with $1 < p < \frac{n}{\alpha}$ and $0 < \alpha < n$, then it is easy to deduce that

$$w(x) \in A(p,q) \iff w(x)^q \in A_{\frac{q(n-\alpha)}{n}} \iff w(x)^q \in A_{1+\frac{q}{p'}}.$$

For $p = 1$, w is in $A_{1,q}$ with $1 < q < \infty$ if

$$[w]_{A(1,q)} := \sup_B [w]_{A(1,q)(B)} = \sup_B \Big(\frac{1}{|B|} \int_B w(x)^q dx \Big)^{\frac{1}{q}} \Big(\operatorname*{esssup}_{x \in B} \frac{1}{w(x)} \Big) < \infty,$$

where the supremum is taken with respect to all the balls B. Thus, we get

$$\Big(\frac{1}{|B|} \int_B w(x)^q dx \Big)^{\frac{1}{q}} \leq C \inf_{x \in B} w(x)$$

for every ball $B \subset \mathbb{R}^n$.

By (2.4), we have

$$\Big(\int_B w(x)^q dx \Big)^{\frac{1}{q}} \Big(\int_B w(x)^{-p'} dx \Big)^{\frac{1}{p'}} \leq C |B|^{\frac{1}{q} + \frac{1}{p'}}. \tag{2.5}$$

We summarize some properties about Muckenhoupt–Wheeden class $A(p,q)$; see [7, 15].

Lemma 2.1 *Given $1 \leq p \leq q < \infty$, the following statements hold:*

(i) $w(x) \in A(p,q) \iff w(x)^q \in A_{1+\frac{q}{p'}}$;

(ii) $w(x) \in A(p,q) \iff w(x)^{-p'} \in A_{1+\frac{p'}{q}}$;

(iii) *if $p_1 < p_2$ and $q_1 < q_2$, then $A(p_1,q_1) \subset A(p_2,q_2)$.*

Let us recall the definition and some properties of $BMO(\mathbb{R}^n)$. A locally integrable function b is said to be in BMO if

$$\|b\|_* = \sup_{x \in \mathbb{R}^n, r > 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |b(y) - b_{B(x,r)}| dy < \infty,$$

where

$$b_{B(x,r)} = \frac{1}{|B(x,r)|} \int_{B(x,r)} b(y) dy.$$

Define

$$BMO(\mathbb{R}^n) = \{b \in L_1^{\mathrm{loc}}(\mathbb{R}^n) \ : \ \|b\|_* < \infty\}.$$

If one regards two functions whose difference is a constant as one (modulo constants), then the space $BMO(\mathbb{R}^n)$ is a Banach space with respect to norm $\|\cdot\|_*$.

An early work about $BMO(\mathbb{R}^n)$ space can be attributed to John and Nirenberg [9]. For $1 < p < \infty$, there is a close relation between $BMO(\mathbb{R}^n)$ and A_p weights:

$$BMO(\mathbb{R}^n) = \left\{\alpha \log w : w \in A_p, \alpha \geq 0\right\}.$$

Lemma 2.2 (John–Nirenberg inequality; see [9]) *There are constants C_1, $C_2 > 0$, such that for all $b \in BMO(\mathbb{R}^n)$ and $\beta > 0$,*

$$|\{x \in B : |b(x) - b_B| > \beta\}| \leq C_1 |B| e^{-C_2\beta/\|b\|_*}, \quad \forall B \subset \mathbb{R}^n.$$

By Lemma 2.2, it is easy to get the following.

Lemma 2.3 *Let $w \in A_\infty$ and $b \in BMO(\mathbb{R}^n)$. Then for any $p \geq 1$, we have*

$$\Big(\frac{1}{w(B)} \int_B |b(y) - b_B|^p w(y) dy\Big)^{\frac{1}{p}} \leq C\|b\|_*.$$

Lemma 2.4 (see [12]) *Let b be a function in $BMO(\mathbb{R}^n)$. Also let $1 \leq p < \infty$, $x \in \mathbb{R}^n$, and $r_1, r_2 > 0$. Then*

$$\Big(\frac{1}{|B(x,r_1)|} \int_{B(x,r_1)} |b(y) - b_{B(x,r_2)}|^p dy\Big)^{\frac{1}{p}} \leq C\Big(1 + \Big|\ln \frac{r_1}{r_2}\Big|\Big) \|b\|_*,$$

where $C > 0$ is independent of b, x, r_1, and r_2.

By Lemmas 2.3 and 2.4, it is easily to prove the following result.

Lemma 2.5 *Let $w \in A_\infty$ and $b \in BMO(\mathbb{R}^n)$. Let also $1 \leq p < \infty$, $x \in \mathbb{R}^n$, and $r_1, r_2 > 0$. Then*

$$\Big(\frac{1}{w(B(x,r_1))} \int_{B(x,r_1)} |b(y) - b_{B(x,r_2)}|^p w(y) dy\Big)^{\frac{1}{p}} \leq C\Big(1 + \Big|\ln \frac{r_1}{r_2}\Big|\Big) \|b\|_*, \tag{2.6}$$

where $C > 0$ is independent of b, w, x, r_1, and r_2.

At the end of this section, we list some known results about $R_m(A; x, y)$.

Lemma 2.6 (see [3]) *Let A be a function on $\mathbb{R}^n$ and $D^\gamma A \in L_q^{\mathrm{loc}}(\mathbb{R}^n)$ for $|\gamma| = m$ and some $q > n$. Then*

$$|R_m(A; x, y)| \leq C|x - y|^m \sum_{|\gamma|=m} \Big(\frac{1}{|\widetilde{Q}(x,y)|} \int_{\widetilde{Q}(x,y)} |D^\gamma A(z)|^q dz\Big)^{\frac{1}{q}},$$

where $\widetilde{Q}(x, y)$ is the cube centered at x with edges parallel to the axes and having diameter $5\sqrt{n}|x - y|$.

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Lemma 2.7 (see [3]) *For fixed $x \in \mathbb{R}^n$, let*

$$\overline{A}(x) = A(x) - \sum_{|\gamma|=m-1} \frac{1}{\gamma!} (D^\gamma A)_{B(x,5\sqrt{n}|x-y|)} x^\gamma.$$

Then $R_m(A;x,y) = R_m(\overline{A};x,y)$.

Lemma 2.8 (see [1]) *Let $x \in B(x_0,r)$, $y \in B(x_0,2^{j+1}r) \setminus B(x_0,2^j r)$. Then*

$$|R_m(A;x,y)| \le C|x-y|^{m-1}\Big(j \sum_{|\gamma|=m-1} \|D^\gamma A\|_* + \sum_{|\gamma|=m-1} |D^\gamma A(y) - (D^\gamma A)_{B(x_0,r)}|\Big).$$

3 Proofs of the Main Results

Proof of Theorem 1.8 We write as $f = f_1 + f_2$, where $f_1(y) = f(y)\chi_{B(x_0,2r)}(y)$, $\chi_{B(x_0,2r)}$ denotes the characteristic function of $B(x_0,2r)$. Then

$$\|T^A_{\Omega,\alpha} f\|_{L_q(w^q,B(x_0,r))} \le \|T^A_{\Omega,\alpha} f_1\|_{L_q(w^q,B(x_0,r))} + \|T^A_{\Omega,\alpha} f_2\|_{L_q(w^q,B(x_0,r))}.$$

Since $f_1 \in L_p(w^p,\mathbb{R}^n)$, by the boundedness of $T^A_{\Omega,\alpha}$ from $L_p(w^p,\mathbb{R}^n)$ to $L_q(w^q,\mathbb{R}^n)$ (see Theorem 1.1), we get

$$\begin{aligned}\|T^A_{\Omega,\alpha} f_1\|_{L_q(w^q,B(x_0,r))} &\le \|T^A_{\Omega,\alpha} f_1\|_{L_q(w^q,\mathbb{R}^n)} \le C \sum_{|\gamma|=m-1} \|D^\gamma A\|_* \|f_1\|_{L_p(w^p,\mathbb{R}^n)} \\ &= C \sum_{|\gamma|=m-1} \|D^\gamma A\|_* \|f\|_{L_p(w^p,B(x_0,2r))}.\end{aligned}$$

Since $1 < p < q$ and $\frac{s'p}{p'(p-s')} \ge 1$, then by Hölder's inequality

$$\begin{aligned} 1 &\le \Big(\frac{1}{|B|}\int_B w(y)^p dy\Big)^{\frac{1}{p}} \Big(\frac{1}{|B|}\int_B w(y)^{-p'} dy\Big)^{\frac{1}{p'}} \\ &\le \Big(\frac{1}{|B|}\int_B w(y)^q dy\Big)^{\frac{1}{q}} \Big(\frac{1}{|B|}\int_B w(y)^{-\frac{s'p}{(p-s')}} dy\Big)^{\frac{(p-s')}{s'p}}. \end{aligned}$$

This means

$$r^{\frac{n}{s'}-\alpha} \le \big(w^q(B(x_0,r))\big)^{\frac{1}{q}} \|w^{-1}\|_{L_{\frac{s'p}{(p-s')}}(B(x_0,r))}.$$

Thus,

$$\begin{aligned} &\|f\|_{L_p(w^p,B(x_0,2r))} \\ &\quad \le C r^{\frac{n}{s'}-\alpha} \|f\|_{L_p(w^p,B(x_0,2r))} \int_{2r}^\infty \frac{dt}{t^{\frac{n}{s'}-\alpha+1}} \\ &\quad \le C\big(w^q(B(x_0,r))\big)^{\frac{1}{q}} \|w^{-1}\|_{L_{\frac{s'p}{(p-s')}}(B(x_0,r))} \int_{2r}^\infty \|f\|_{L_p(w^p,B(x_0,t))} \frac{dt}{t^{\frac{n}{s'}-\alpha+1}} \\ &\quad \le C\big(w^q(B(x_0,r))\big)^{\frac{1}{q}} \int_{2r}^\infty \|f\|_{L_p(w^p,B(x_0,t))} \|w^{-1}\|_{L_{\frac{s'p}{(p-s')}}(B(x_0,t))} \frac{dt}{t^{\frac{n}{s'}-\alpha+1}}. \end{aligned}$$

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Since $w(x)^{s'} \in A(\frac{p}{s'}, \frac{q}{s'})$, by (2.5), we get

(3.1) $$\big(w^q(B(x_0,t))\big)^{\frac{1}{q}}\|w^{-1}\|_{L_{\frac{s'p}{(p-s')}}(B(x_0,t))} \le Ct^{\frac{n}{s'}-\alpha}$$

holds for all $t > 0$. Thus,

$$\|T^A_{\Omega,\alpha}f_1\|_{L_q(w^q,B(x_0,r))} \le C\sum_{|\gamma|=m-1}\|D^\gamma A\|_*\big(w^q(B(x_0,r))\big)^{\frac{1}{q}} \times \int_{2r}^{\infty}(1+\ln\frac{t}{r})\|f\|_{L_p\left(w^p,B(x_0,t)\right)}\big(w^q(B(x_0,t))\big)^{-\frac{1}{q}}\frac{1}{t}dt.$$

Let $\Delta_i = B(x_0, 2^{j+1}r) \setminus B(x_0, 2^j r)$ and $x \in B(x_0,r)$. By Lemma 2.8, we get

$$\begin{aligned}
&|T^A_{\Omega,\alpha}f_2(x)|\\
&\le \Big|\int_{(B(x_0,2r))^C}\frac{\Omega(x-y)}{|x-y|^{n-\alpha+m-1}}R_m(A;x,y)f(y)dy\Big|\\
&\le \sum_{j=1}^{\infty}\int_{\Delta_i}\frac{|\Omega(x-y)f(y)|}{|x-y|^{n-\alpha}}\Big(j\sum_{|\gamma|=m-1}\|D^\gamma A\|_* + \sum_{|\gamma|=m-1}|D^\gamma A(y)-(D^\gamma A)_{B(x_0,r)}|\Big)dy\\
&\le C\sum_{|\gamma|=m-1}\|D^\gamma A\|_*\sum_{j=1}^{\infty}j\int_{\Delta_i}\frac{|\Omega(x-y)f(y)|}{|x-y|^{n-\alpha}}dy\\
&\quad + C\sum_{|\gamma|=m-1}\sum_{j=1}^{\infty}\int_{\Delta_i}\frac{|\Omega(x-y)f(y)|}{|x-y|^{n-\alpha}}|D^\gamma A(y)-(D^\gamma A)_{B(x_0,r)}|dy\\
&= I_1 + I_2.
\end{aligned}$$

By Hölder's inequality, we have

(3.2) $$\int_{\Delta_i}\frac{|\Omega(x-y)f(y)|}{|x-y|^{n-\alpha}}dy \le \Big(\int_{\Delta_i}|\Omega(x-y)|^s dy\Big)^{\frac{1}{s}}\Big(\int_{\Delta_i}\frac{|f(y)|^{s'}}{|x-y|^{(n-\alpha)s'}}dy\Big)^{\frac{1}{s'}}.$$

When $x \in B(x_0,s)$ and $y \in \Delta_i$, then by a direct calculation, we can see that $2^{j-1}r \le |y-x| < 2^{j+1}r$. Hence,

(3.3) $$\Big(\int_{\Delta_i}|\Omega(x-y)|^s dy\Big)^{\frac{1}{s}} \le C\|\Omega\|_{L_s(S^{n-1})}|B(x_0,2^{j+1}r)|^{\frac{1}{s}}.$$

It is clear that $x \in B(x_0,r)$, $y \in B(x_0,2r)^C$ implies $\frac{1}{2}|x_0-y| \le |x-y| \le \frac{3}{2}|x_0-y|$. Consequently,

$$\Big(\int_{\Delta_i}\frac{|f(y)|^{s'}}{|x-y|^{(n-\alpha)s'}}dy\Big)^{\frac{1}{s'}} \le \frac{1}{|B(x_0,2^{j+1}r)|^{1-\frac{\alpha}{n}}}\Big(\int_{B(x_0,2^{j+1}r)}|f(y)|^{s'}dy\Big)^{\frac{1}{s'}}.$$

Then

$$I_1 \le C\sum_{|\gamma|=m-1}\|D^\gamma A\|_*\sum_{j=1}^{\infty}j(2^{j+1}r)^{\alpha-\frac{n}{s'}}\Big(\int_{B(x_0,2^{j+1}r)}|f(y)|^{s'}dy\Big)^{\frac{1}{s'}}.$$

Since $s' < p$, it follows from Hölder's inequality that

$$\Big(\int_{B(x_0,2^{j+1}r)}|f(y)|^{s'}dy\Big)^{\frac{1}{s'}} \le C\|f\|_{L_p(w^p,B(x_0,2^{j+1}r))}\|w^{-1}\|_{L_{\frac{s'p}{(p-s')}}(B(x_0,2^{j+1}r))}.$$

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Then

$$\sum_{j=1}^{\infty} j(2^{j+1}r)^{\alpha-\frac{n}{s'}}\Big(\int_{B(x_0,2^{j+1}r)}|f(y)|^{s'}dy\Big)^{\frac{1}{s'}}$$

$$\le C\sum_{j=1}^{\infty}\Big(1+\ln\frac{2^{j+1}r}{r}\Big)(2^{j+1}r)^{\alpha-\frac{n}{s'}}\|f\|_{L_p(w^p,B(x_0,2^{j+1}r))}\|w^{-1}\|_{L_{\frac{s'p}{(p-s')}}(B(x_0,2^{j+1}r))}$$

$$\le C\sum_{j=1}^{\infty}\int_{2^{j+1}r}^{2^{j+2}r}\Big(1+\ln\frac{t}{r}\Big)\|f\|_{L_p(w^p,B(x_0,t))}\|w^{-1}\|_{L_{\frac{s'p}{(p-s')}}(B(x_0,t))}\frac{dt}{t^{\frac{n}{s'}-\alpha+1}}$$

$$\le C\int_{2r}^{\infty}\Big(1+\ln\frac{t}{r}\Big)\|f\|_{L_p(w^p,B(x_0,t))}\|w^{-1}\|_{L_{\frac{s'p}{(p-s')}}(B(x_0,t))}\frac{dt}{t^{\frac{n}{s'}-\alpha+1}}.$$

By (3.1), we know

$$\|w^{-1}\|_{L_{\frac{s'p}{(p-s')}}(B(x_0,t))}\le Ct^{\frac{n}{s'}-\alpha}\big(w^q(B(x_0,t))\big)^{-\frac{1}{q}}. \tag{3.4}$$

Then

$$I_1\le C\sum_{|\gamma|=m-1}\|D^\gamma A\|_*\int_{2r}^{\infty}\Big(1+\ln\frac{t}{r}\Big)\|f\|_{L_p\left(w^p,B(x_0,t)\right)}\big(w^q(B(x_0,t))\big)^{-\frac{1}{q}}\frac{1}{t}dt.$$

On the other hand, by Hölder's inequality, (3.2) and (3.3) we have

$$\int_{\Delta_i}\frac{|\Omega(x-y)f(y)|}{|x-y|^{n-\alpha}}|D^\gamma A(y)-(D^\gamma A)_{B(x_0,r)}|dy$$

$$\le\Big(\int_{\Delta_i}|\Omega(x-y)|^s dy\Big)^{\frac{1}{s}}\Big(\int_{\Delta_i}\frac{|D^\gamma A(y)-(D^\gamma A)_{B(x_0,r)}f(y)|^{s'}}{|x-y|^{(n-\alpha)s'}}dy\Big)^{\frac{1}{s'}}$$

$$\le C\sum_{j=1}^{\infty}(2^{j+1}r)^{\alpha-\frac{n}{s'}}\Big(\int_{B(x_0,2^{j+1}r)}|D^\gamma A(y)-(D^\gamma A)_{B(x_0,r)}|^{s'}|f(y)|^{s'}dy\Big)^{\frac{1}{s'}}.$$

Applying Hölder's inequality again, we get

$$\Big(\int_{B(x_0,2^{j+1}r)}|D^\gamma A(y)-(D^\gamma A)_{B(x_0,r)}|^{s'}|f(y)|^{s'}dy\Big)^{\frac{1}{s'}}\le$$
$$C\|f\|_{L_p(w^p,B(x_0,2^{j+1}r))}\|(D^\gamma A(y)-(D^\gamma A)_{B(x_0,r)})w(\cdot)^{-1}\|_{L_{\frac{s'p}{(p-s')}}(B(x_0,2^{j+1}r))}.$$

Consequently,

$$I_2\le C\sum_{|\gamma|=m-1}\|D^\gamma A\|_*\sum_{j=1}^{\infty}\int_{2^{j+1}r}^{2^{j+2}r}(1+\ln\frac{2^{j+1}r}{r})(2^{j+1}r)^{\alpha-\frac{n}{s'}}\|f\|_{L_p(w^p,B(x_0,t))}$$
$$\times\|(D^\gamma A(y)-(D^\gamma A)_{B(x_0,r)})w(\cdot)^{-1}\|_{L_{\frac{s'p}{(p-s')}}(B(x_0,t))}dt$$
$$\le C\sum_{|\gamma|=m-1}\|D^\gamma A\|_*\int_{2r}^{\infty}\|f\|_{L_p(w^p,B(x_0,t))}$$
$$\times\|(D^\gamma A(y)-(D^\gamma A)_{B(x_0,r)})w(\cdot)^{-1}\|_{L_{\frac{s'p}{(p-s')}}(B(x_0,t))}\frac{dt}{t^{\frac{n}{s'}-\alpha+1}}.$$

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By $w(x)^{s'} \in A(\frac{p}{s'}, \frac{q}{s'})$ and Lemma 2.1(ii), we know that $w(x)^{-\frac{s'p}{(p-s')}} \in A_{1+\frac{s'p}{(p-s')q}}$. Then it follows from (2.6) and (3.4) that

$$\begin{aligned}
&\Big\| \big(D^\gamma A(y) - (D^\gamma A)_{B(x_0,r)} \big) w(\cdot)^{-1} \Big\|_{L_{\frac{s'p}{(p-s')}}(B(x_0,t))} \frac{dt}{t^{\frac{n}{s'}-\alpha+1}} \\
&\leq \Big(\int_{B(x_0,r)} \big| (D^\gamma A(y) - (D^\gamma A)_{B(x_0,r)}) \big|^{\frac{s'p}{(p-s')}} w^{-\frac{s'p}{(p-s')}}(y) dy \Big)^{\frac{(p-s')}{s'p}} \\
&\leq C \| D^\gamma A \|_* \Big(1 + \ln \frac{t}{r} \Big) \Big(w^{-\frac{s'p}{(p-s')}} \big(B(x_0,t) \big) \Big)^{\frac{(p-s')}{s'p}} \\
&= C \| D^\gamma A \|_* \Big(1 + \ln \frac{t}{r} \Big) \| w^{-1} \|_{L_{\frac{s'p}{(p-s')}}(B(x_0,t))} \\
&\leq C \| D^\gamma A \|_* \Big(1 + \ln \frac{t}{r} \Big) t^{\frac{n}{s'}-\alpha} \big(w^q (B(x_0,t)) \big)^{-\frac{1}{q}}.
\end{aligned}$$

Thus,

$$I_2 \leq C \sum_{|\gamma|=m-1} \| D^\gamma A \|_* \int_{2r}^{\infty} \Big(1 + \ln \frac{t}{r} \Big) \| f \|_{L_p(w^p, B(x_0,t))} \big(w^q (B(x_0,t)) \big)^{-\frac{1}{q}} \frac{1}{t} dt.$$

Combining the estimates of I_1 and I_2, we get

$$\begin{aligned}
&|T^A_{\Omega,\alpha} f_2(x)| \leq \\
&\qquad C \sum_{|\gamma|=m-1} \| D^\gamma A \|_* \int_{2r}^{\infty} \Big(1 + \ln \frac{t}{r} \Big) \| f \|_{L_p(w^p, B(x_0,t))} \big(w^q (B(x_0,t)) \big)^{-\frac{1}{q}} \frac{1}{t} dt.
\end{aligned}$$

Then we get

$$\begin{aligned}
\| T^A_{\Omega,\alpha} f_2 \|_{L_q(w^q, B(x_0,r))} &\leq C \sum_{|\gamma|=m-1} \| D^\gamma A \|_* \big(w^q (B(x_0,r)) \big)^{\frac{1}{q}} \\
&\quad \times \int_{2r}^{\infty} \Big(1 + \ln \frac{t}{r} \Big) \| f \|_{L_p(w^p, B(x_0,t))} \big(w^q (B(x_0,t)) \big)^{-\frac{1}{q}} \frac{1}{t} dt.
\end{aligned}$$

This completes the proof of Theorem 1.8. ∎

Proof of Theorem 1.9 We consider (1.3) first. Since $f \in M_{p,\varphi_1}(w^p, \mathbb{R}^n)$, by (2.3) and the fact that $\| f \|_{L_p(w^p, B(x_0,t))}$ is a non-decreasing function of t, we get

$$\begin{aligned}
\frac{\| f \|_{L_p(w^p, B(x_0,t))}}{\operatorname{essinf}_{0<t<\tau<\infty} \varphi_1(x_0,\tau) (w^p(B(x_0,\tau)))^{\frac{1}{p}}} &\leq \operatorname*{esssup}_{0<t<\tau<\infty} \frac{\| f \|_{L_p(w^p, B(x_0,t))}}{\varphi_1(x_0,\tau) (w^p(B(x_0,\tau)))^{\frac{1}{p}}} \\
&\leq \operatorname*{esssup}_{0<\tau<\infty} \frac{\| f \|_{L_p(w^p, B(x_0,\tau))}}{\varphi_1(x_0,\tau) (w^p(B(x_0,\tau)))^{\frac{1}{p}}} \\
&\leq \| f \|_{M_{p,\varphi_1}(w^p, \mathbb{R}^n)}.
\end{aligned}$$

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For $s' < p < \infty$, since (φ_1, φ_2) satisfies (1.2), we have

$$(3.5)\int_r^\infty (1+\ln\frac{t}{r})\|f\|_{L_p(w^p,B(x_0,t))}(w^q(B(x_0,t)))^{-\frac{1}{q}}\frac{dt}{t}$$
$$\le \int_r^\infty (1+\ln\frac{t}{r})\frac{\|f\|_{L_p(w^p,B(x_0,t))}}{\operatorname{essinf}_{t<\tau<\infty}\varphi_1(x_0,\tau)(w^p(B(x_0,\tau)))^{\frac{1}{p}}}$$
$$\times \frac{\operatorname{essinf}_{t<\tau<\infty}\varphi_1(x_0,\tau)(w^p(B(x_0,\tau)))^{\frac{1}{p}}}{(w^q(B(x_0,t)))^{\frac{1}{q}}}\frac{dt}{t}$$
$$\le C\|f\|_{M_{p,\varphi_1}(w^p,\mathbb{R}^n)}\int_r^\infty (1+\ln\frac{t}{r})\frac{\operatorname{essinf}_{t<\tau<\infty}\varphi_1(x_0,\tau)(w^p(B(x_0,\tau)))^{\frac{1}{p}}}{(w^q(B(x_0,t)))^{\frac{1}{q}}}\frac{dt}{t}$$
$$\le C\|f\|_{M_{p,\varphi_1}(w^p,\mathbb{R}^n)}\varphi_2(x_0,r).$$

Then by (1.1) and (3.5), we get

$$\|T^A_{\Omega,\alpha}f\|_{M_{q,\varphi_2}(w^q,\mathbb{R}^n)} = \sup_{x_0\in\mathbb{R}^n,r>0}\varphi_2(x_0,r)^{-1}(w^q(B(x_0,r)))^{-\frac{1}{q}}\|T^A_{\Omega,\alpha}f\|_{L_q(w^q,B(x_0,r))}$$
$$\le C\sum_{|\gamma|=m-1}\|D^\gamma A\|_*\sup_{x_0\in\mathbb{R}^n,r>0}\varphi_2(x_0,r)^{-1}$$
$$\times\int_r^\infty (1+\ln\frac{t}{r})\|f\|_{L_p(w^p,B(x_0,t))}\big(w^q(B(x_0,t))\big)^{-\frac{1}{q}}\frac{1}{t}dt$$
$$\le C\sum_{|\gamma|=m-1}\|D^\gamma A\|_*\|f\|_{M_{p,\varphi_1}(w^p,\mathbb{R}^n)}.$$

Hence, we have completed the proof of (1.3).

We are now in a place of proving (1.4) in Theorem 1.9.

Remark 3.1 The conclusion of (1.4) is a direct consequence of Lemma 3.2 and (1.3). In order to do this, we need to define an operator by

$$\widetilde{T}^A_{|\Omega|,\alpha}(|f|)(x) = \int_{\mathbb{R}^n}\frac{|\Omega(x-y)|}{|x-y|^{n-\alpha+m-1}}|R_m(A;x,y)||f(y)|dy \qquad 0<\alpha<n,$$

where $\Omega \in L_s(S^{n-1})(s>1)$ is homogeneous of degree zero in $\mathbb{R}^n$.

Using the idea of proving [4, Lemma 2], we can obtain the following pointwise relation.

Lemma 3.2 *Let $0<\alpha<n$ and $\Omega\in L_s(S^{n-1})(s>1)$. Then we have*

$$M^A_{\Omega,\alpha}f(x) \le \widetilde{T}^A_{|\Omega|,\alpha}(|f|)(x) \qquad \text{for } x\in\mathbb{R}^n.$$

In fact, for any $r>0$, we have

$$\widetilde{T}^A_{|\Omega|,\alpha}(|f|)(x) \ge \int_{|x-y|<r}\frac{|\Omega(x-y)|}{|x-y|^{n-\alpha+m-1}}|R_m(A;x,y)||f(y)|dy$$
$$\ge \frac{1}{r^{n-\alpha+m-1}}\int_{|x-y|<r}|\Omega(x-y)||R_m(A;x,y)||f(y)|dy.$$

Taking the supremum for $r > 0$ on the inequality above, we get

$$\widetilde{T}^{A}_{|\Omega|,\alpha}(|f|)(x) \geq M^{A}_{\Omega,\alpha} f(x) \qquad \text{for } x \in \mathbb{R}^n.$$

From the process proving (1.3), it is easy to see that the conclusions of (1.3) also hold for $\widetilde{T}^{A}_{|\Omega|,\alpha}$. Combining this with Lemma 3.2, we can immediately obtain (1.4), which completes the proof. ∎

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