

EMBEDDING SEMIRINGS IN SEMIRINGS WITH MULTIPLICATIVE UNIT

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(Received 28 November 1966)

A *topological semiring* is a system $(S, +, \cdot)$ where $(S, +)$ and (S, \cdot) are topological semigroups and \cdot distributes across $+$ as in a ring; that is, for all x, y, z in S ,

$$\begin{aligned}x \cdot (y+z) &= (x \cdot y) + (x \cdot z), \\(x+y) \cdot z &= (x \cdot z) + (y \cdot z).\end{aligned}$$

The operations $+$ and \cdot are called *addition* and *multiplication* respectively.

If S is any topological semiring and we adjoin to S an element 0 as an isolated point and let 0 be a multiplicative zero and either an additive unit or an additive zero for $S' = S \cup \{0\}$, then it can be easily seen that S' is also a topological semiring. Thus it is always possible to embed S in a semiring S' with multiplicative zero; also S' is compact when S is.

Selden has shown in Theorem 7 of [6] (see also [7]) that each additive group in a compact semiring with multiplicative unit must be totally disconnected. This means that the semiring $(C, +, \cdot)$, where $(C, +)$ is the circle group and $x \cdot y = 0$ for all $x, y \in C$, cannot be embedded in a compact semiring with multiplicative unit. We investigate here conditions under which it is possible to embed a semiring in a semiring with multiplicative unit. In particular, we derive in Theorem 3 a necessary and sufficient condition for the embedding of a compact additively commutative semiring in a compact semiring with multiplicative unit to be possible. The special case of embedding a compact ring in a compact ring with unit is also dealt with.

It is first necessary to establish some points of notation. If x is a member of an additive semigroup and n is a positive integer, we shall use nx to mean the semigroup sum of n elements each equal to x . Note that if 1 is a multiplicative left unit of a semiring then n denotes the semiring sum of n elements each equal to 1 . Hence nx also equals the product of n and x . For any x in $(S, +, \cdot)$, let

¹ This paper is part of the author's Ph.D. thesis, written under the supervision of Dr. J. H. Michael.

$$\begin{aligned} 0[+](x) &= \{nx \mid n \text{ a positive integer}\}, \\ \Gamma[+](x) &= 0[+](x)^-, \\ K[+](x) &= \bigcap_{n=1}^{\infty} \{mx \mid m \geq n\}^-, \end{aligned}$$

where $-$ denotes topological closure. Then $\Gamma[+](x)$ is a commutative (additive) semigroup. If it is compact, then $K[+](x)$ is the minimal ideal of $\Gamma[+](x)$ and is also the maximal additive group in $\Gamma[+](x)$ (see [5], Theorem 3.1.1 or [2], Theorems 3.3 and 3.4). A semiring is said to be *additively Γ -compact* if $\Gamma[+](x)$ is compact for each x in S .

S is said to be a *subsemiring* of a topological semiring T if and only if for each x, y in S , both $x+y$ and $x \cdot y$ are in S ; S , given the relative topology, is a topological semiring. We shall say that a topological semiring S can be *embedded* in a topological semiring T if there is a subsemiring S_1 of T which is topologically isomorphic with S ; note that S_1 need not be topologically closed in T .

If S is an additively Γ -compact subsemiring of a topological semiring T then, for each x in S , the closure of $0[+](x)$ in the relative topology of S is compact so that it is also compact in the topology of T . Because T is Hausdorff, it follows that the closure of $0[+](x)$ in the relative topology of S is the same as its closure in the topology of T . Thus we can use the symbol $\Gamma[+](x)$ without fear of confusion.

The first result is a topological extension of a familiar construction (see, for example, [4], page 49).

THEOREM 1. *Any additively commutative topological semiring S can be embedded in some additively commutative topological semiring T with multiplicative unit. Further, T can be made locally compact when S is locally compact.*

PROOF. We first adjoin to S an element α as an isolated point and let α be an additive unit and a multiplicative zero for $S' = S \cup \{\alpha\}$. Let N denote the (locally compact) semiring of non-negative integers with ordinary addition and multiplication. Then we put $T = N \times S'$ and define addition $*$ and multiplication \circ on T by

$$\begin{aligned} (n_1, x_1) * (n_2, x_2) &= (n_1 + n_2, x_1 + x_2), \\ (n_1, x_1) \circ (n_2, x_2) &= (n_1 n_2, x_1 x_2 + n_1 x_2 + n_2 x_1), \end{aligned}$$

where $0x = \alpha$ for all x in S' . It is a simple matter to check that $(T, *, \circ)$ is a topological semiring which is locally compact when S is locally compact. Also $(1, \alpha)$ is a multiplicative unit for T and $\{0\} \times S$ is a subsemiring topologically isomorphic with S .

In what follows we are concerned with additively Γ -compact semirings.

For each x , $\Gamma[+](x)$ is then a compact monothetic semigroup and the structure of all such semigroups, first found by Hewitt in [2], is given in Theorems 3.1.6 and 3.1.7 of [5]. In broad outline, our procedure is to construct what is roughly speaking the largest relevant compact monothetic semigroup and use it in a cartesian product with S' (as in Theorem 1) to be the space in which S is embedded when S is additively commutative.

LEMMA 1. *If S is a topological semiring then $\Gamma[+](x)$ is a subsemiring if and only if $x^2 \in \Gamma[+](x)$.*

PROOF. Suppose $x^2 \in \Gamma[+](x)$. Then if p, q are positive integers, $(pq)x^2 \in \Gamma[+](x)$; but $(pq)x^2 = (px)(qx)$ and so $(px)(qx) \in \Gamma[+](x)$. Hence $\{0[+](x)\}^2 \subset 0[+](x)$ and it follows from the continuity of multiplication that $\{\Gamma[+](x)\}^2 \subset \Gamma[+](x)$.

LEMMA 2. *If S is a topological semiring and x is a multiplicative idempotent, then $\Gamma[+](x)$ is a subsemiring for which x is a multiplicative unit.*

PROOF. It follows from Lemma 1 that $\Gamma[+](x)$ is a subsemiring. Also, for any positive integer m , $(mx) \cdot x = mx^2 = mx = x \cdot (mx)$. Thus x is a multiplicative unit for $0[+](x)$ and hence for $\Gamma[+](x)$.

LEMMA 3. *Let S be a topological semiring containing an element x for which $\Gamma[+](x)$ is compact. Then for each y in S ,*

- (i) $y \cdot \Gamma[+](x) = \Gamma[+](yx)$ and $\Gamma[+](x) \cdot y = \Gamma[+](xy)$;
- (ii) $y \cdot K[+](x) = K[+](yx)$ and $K[+](x) \cdot y = K[+](xy)$.

PROOF. The mapping $\phi : S \rightarrow S$ given by $\phi(z) = yz$ is continuous and $\phi(0[+](x)) = 0[+](yx)$. It follows from the compactness of $0[+](x)^-$ that

$$\phi(0[+](x)^-) = 0[+](yx)^- = \Gamma[+](yx)$$

(see Corollary 2, page 101 and Prop. 9, page 61 of [1]). Hence $\Gamma[+](yx)$ is compact and so $K[+](yx)$ exists. Because $K[+](x)$ is an additive group and ϕ is an additive homomorphism onto $y \cdot K[+](x)$, we see that $y \cdot K[+](x)$ is an additive group, so that $y \cdot K[+](x) \subset K[+](yx)$, the maximal additive group in $\Gamma[+](yx)$. On the other hand, if $z \in \Gamma[+](yx)$ and $w \in y \cdot K[+](x)$, then there are z_1, w_1 in $\Gamma[+](x), K[+](x)$ respectively with $z = yz_1, w = yw_1$, and therefore $z+w = y(z_1+w_1) \in y \cdot K[+](x)$ since $K[+](x)$ is an additive ideal of $\Gamma[+](x)$. Thus $y \cdot K[+](x)$, being an additive ideal of $\Gamma[+](yx)$, contains the minimal such ideal $K[+](yx)$, and the result follows.

We now make the construction previously referred to.

EXAMPLE. Let P be the set of prime integers and put

$$N_1 = \prod_{\rho \in P} \Delta_\rho$$

where, for each $\rho \in P$, Δ_ρ is the (compact) ring of ρ -adic integers (see, for example, [3], § 10) and N_1 is given the product topology. If we give N_1 coordinate-wise addition and multiplication, N_1 is a compact ring. For any $\rho \in P$ we let 0_ρ be the additive unit and 1_ρ be the multiplicative unit of Δ_ρ . Then let $u \in N_1$ be such that u_ρ , the coordinate of u in Δ_ρ , is equal to 1_ρ for all $\rho \in P$. Clearly u is a multiplicative unit for N_1 . It also follows that $\{nu | n \text{ a positive integer}\}$ is dense in N_1 ; we prove this in Lemma 4 below. Thus the additive group of N_1 is monothetic with generator u .

Let N_2 be the set of positive integers and put $N_3 = N_1 \cup N_2$. We can make N_3 a compact monothetic additive semigroup by proceeding as in Theorem 3.1.7 of [5] (see also Theorem 5.3 of [2]). We take addition in N_2 to be ordinary addition, put

$$x + m = x + mu = m + x \text{ if } x \in N_1 \text{ and } m \in N_2,$$

and retain the same addition in N_1 . We define a topology on N_3 by letting each point in N_2 be isolated and, for each x in N_1 , we take

$$\left\{ V_n^*(x) \mid \begin{array}{l} V_n^*(x) = V(x) \cup \{m | m \geq n \text{ and } mu \in V(x)\}, \text{ where} \\ V(x) \text{ is any neighbourhood of } x \text{ in } N_1 \text{ and } n \geq 1 \end{array} \right\}$$

as the set of all neighbourhoods of x . It is shown in [5], Theorem 3.1.7 (see also [2], Theorem 5.3) that N_3 is a compact additive semigroup with $N_2^- = \Gamma[+](1) = N_3$ and $K[+](1) = N_1$.

Finally we make N_3 a semiring by taking ordinary multiplication on N_2 , putting

$$m \cdot x = x \cdot m = mx \text{ if } x \in N_1, m \in N_2,$$

and retaining the multiplication on N_1 . It is not difficult to check that N_3 becomes a compact semiring with multiplicative unit 1.

LEMMA 4. *Let u and N_1 be as in the example above. Then $\Gamma[+](u) = N_1$.*

PROOF. It follows from Theorem 25.16 of [3] that there is an element v of N_1 such that

$$\{nv | n \text{ any integer}\}$$

is dense in N_1 and so, as shown on page 109 of [5] (see also § 2 of [2]), $\Gamma[+](v) = N_1$. For any $\rho \in P$, let v_ρ be the coordinate of v which is in Δ_ρ . Because $\Gamma[+](v_\rho)$ must be Δ_ρ , it follows from Lemma 3 that

$$\Delta_\rho = \Gamma[+](v_\rho) = \Gamma[+](1_\rho \cdot v_\rho) = \Gamma[+](1_\rho) \cdot v_\rho = \Delta_\rho \cdot v_\rho$$

since $\Gamma[+](1_\rho) = \Delta_\rho$ (see § 10.6 of [3]). Thus there exists an element w_ρ of Δ_ρ with $1_\rho = w_\rho v_\rho$. Let w be the element of N_1 which has its coordinate in Δ_ρ equal to w_ρ for all $\rho \in P$; then $wv = u$. Hence, since u is a multiplicative unit for N_1 ,

$$N_1 = uN_1 = (wv)N_1 = w(vN_1) \subset wN_1 \subset N_1$$

and we see that $wN_1 = N_1$. Thus, by Lemma 3,

$$\Gamma[+](u) = \Gamma[+](wv) = w\Gamma[+](v) = wN_1 = N_1.$$

Our main effort is devoted to proving the following theorem.

THEOREM 2. *Let S be an additively commutative and additively Γ -compact semiring. The following are equivalent.*

- (i) *S can be embedded in an additively commutative and additively Γ -compact semiring with multiplicative unit.*
- (ii) *S can be embedded in an additively Γ -compact semiring with multiplicative left unit.*
- (iii) *There is a continuous extension $\Psi : N_3 \times S \rightarrow S$ of the mapping $\psi : N_2 \times S \rightarrow S$ defined by $\psi(n, x) = nx$ for $n \in N_2, x \in S$.*

Clearly (i) implies (ii). We shall prove the theorem by showing that (ii) \Rightarrow (iii) and (iii) \Rightarrow (i). That (ii) \Rightarrow (iii) follows from the following more general result.

LEMMA 5. *Let S be any additively Γ -compact subsemiring of a topological semiring T with multiplicative left unit α such that $\Gamma[+](\alpha)$ is compact. Then there is a continuous additive extension $\Psi : N_3 \times S \rightarrow S$ of the mapping $\psi : N_2 \times S \rightarrow S$ defined by $\psi(n, x) = nx$. (Note that Ψ is uniquely determined by ψ because N_2 is dense in N_3 .)*

PROOF. As $\Gamma[+](\alpha)$ is a compact subsemiring in which α is a multiplicative unit (Lemma 2), it is a consequence of [6], Theorem 7 that $K[+](\alpha)$ is totally disconnected, and hence 0-dimensional (Theorem 3.5 of [3]). Because $K[+](\alpha)$ is a monothetic additive group with generator $\alpha' = \alpha + e$, where e is the unit of $K[+](\alpha)$ ([5], Theorem 3.1.2 or [2], Theorems 3.2 and 3.4), it follows from Theorem 25.16 of [3] that $(K[+](\alpha), +)$ is topologically isomorphic with a cartesian product $\times_{\rho \in P} A_\rho$ where for each ρ in P , A_ρ is either the trivial group with one member or the group $Z(\rho^{r_\rho})$ of residues modulo ρ^{r_ρ} for some integer $r_\rho \geq 1$ or the group Δ_ρ . (If $K[+](\alpha)$ is finite then it is cyclic and the result follows from Theorem 17, Chapter III of [9].) In what follows we shall assume that $K[+](\alpha)$ is identical with $\times_{\rho \in P} A_\rho$. For each $\rho \in P$ we introduce a multiplication \circ on A_ρ as the natural ring multiplication and denote by 1_ρ its multiplicative unit. We give $\times_{\rho \in P} A_\rho$ the coordinate-wise multiplication, for which the element β , whose coordinate in A_ρ is 1_ρ for all ρ , is a multiplicative unit. It is clear that for each $\rho \in P$ there is a continuous additive homomorphism ϕ'_ρ from Δ_ρ onto A_ρ with $\phi'_\rho(1_\rho) = 1_\rho$. (If A_ρ is isomorphic with $Z(\rho^{r_\rho})$, put

$$\phi'_\rho(x) = x_0 + x_1\rho + \dots + x_{r_\rho-1}\rho^{r_\rho-1}$$

for each $x = (x_0, x_1, x_2, \dots)$ in Δ_ρ .) If for each x in N_1 we let x_ρ be the coordinate of x in Δ_ρ and we define a function $\phi' : N_1 \rightarrow K[+](\alpha)$ by putting the coordinate of $\phi'(x)$ in A_ρ equal to $\phi'_\rho(x_\rho)$ for all $\rho \in P$ and all x in N_1 , then ϕ' is clearly a continuous additive homomorphism with $\phi'(u) = \beta$ and $\phi'(N_1) = K[+](\alpha)$. The mapping $f : K[+](\alpha) \rightarrow K[+](\alpha)$ given by $f(x) = \alpha' \circ x$ is a continuous additive homomorphism and $f(\beta) = \alpha' \circ \beta = \alpha'$. Also $f(K[+](\alpha))$ is closed and contains $0[+](\alpha')$ and hence its closure $\Gamma[+](\alpha') = K[+](\alpha)$; hence f maps $K[+](\alpha)$ onto $K[+](\alpha)$. We now see that the mapping $\phi : N_1 \rightarrow K[+](\alpha)$ given by $\phi(x) = f(\phi'(x))$ is a continuous additive homomorphism of N_1 onto $K[+](\alpha)$ for which

$$\phi(u) = f(\phi'(u)) = f(\beta) = \alpha'.$$

We extend ϕ to become $\Phi : N_3 \rightarrow \Gamma[+](\alpha)$ by putting $\Phi(n) = n\alpha$ if $n \in N_2$. It is not difficult to show that Φ is a continuous additive homomorphism of N_3 onto $\Gamma[+](\alpha)$. (However the proof must be split into two cases according as $\Gamma[+](\alpha) \setminus K[+](\alpha)$ is infinite or finite, and use must be made of the characterization of $\Gamma[+](\alpha)$ in either [5], Theorems 3.1.7 and 3.1.6 respectively or in [2], Theorem 5.6.)

Finally, we put $\Psi(y, x) = \Phi(y) \cdot x$ (where \cdot is multiplication in T) for each y in N_3 and x in S . For each x in S ,

$$\Psi(N_3 \times \{x\}) \subset \Phi(N_3) \cdot x = \Gamma[+](\alpha) \cdot x = \Gamma[+](\alpha x) = \Gamma[+](x) \subset S,$$

and so Ψ maps $N_3 \times S$ into S . Clearly Ψ is continuous and, for all n in N_2 ,

$$\Psi(n, x) = \Phi(n) \cdot x = n\Phi(1) \cdot x = (n\alpha) \cdot x = n(\alpha \cdot x) = nx = \psi(n, x).$$

We now show that (iii) \Rightarrow (i).

LEMMA 6. *Let S be an additively commutative and additively Γ -compact semiring for which there is a continuous extension $\Psi : N_3 \times S \rightarrow S$ of $\psi : N_2 \times S \rightarrow S$ defined by $\psi(n, x) = nx$. Then S can be embedded in an additively commutative and additively Γ -compact semiring with multiplicative unit.*

PROOF. We first adjoin an element γ as an isolated point to S so that γ is an additive unit and a multiplicative zero for $S' = S \cup \{\gamma\}$; then S' is an additively Γ -compact semiring. We also adjoin an element 0 as an isolated point to N_3 so that 0 is an additive unit and a multiplicative zero for $N_4 = N_3 \cup \{0\}$; then N_4 is a compact semiring. We extend Ψ to $N_4 \times S'$ by putting $\Psi(0, x) = \Psi(y, \gamma) = \gamma$ for all x in S' , y in N_4 ; then Ψ is continuous on $N_4 \times S'$. Let $T = N_4 \times S'$ and define addition $*$ and multiplication \circ on T by

$$\begin{aligned} (y_1, x_1) * (y_2, x_2) &= (y_1 + y_2, x_1 + x_2), \\ (y_1, x_1) \circ (y_2, x_2) &= (y_1 y_2, x_1 x_2 + \Psi(y_1, x_2) + \Psi(y_2, x_1)). \end{aligned}$$

Then clearly $*$, \circ are continuous and $*$ is associative and commutative. To complete the proof that T is a semiring we first note that, for all x_1, x_2 in S' and y_1, y_2 in N_4 ,

- (a) $[\Psi(y_1, x_1)] \cdot x_2 = \Psi(y_1, x_1 x_2) = x_1 \cdot [\Psi(y_1, x_2)];$
- (b) $\Psi[y_1, \Psi(y_2, x_1)] = \Psi(y_1 y_2, x_1);$
- (c) $\Psi(y_1, x_1 + x_2) = \Psi(y_1, x_1) + \Psi(y_1, x_2);$
- (d) $\Psi(y_1 + y_2, x_1) = \Psi(y_1, x_1) + \Psi(y_2, x_1).$

(These properties are clear for all x_1, x_2 in S' and y_1, y_2 in $N_2 \cup \{0\}$ because Ψ is an extension of ψ . The results follow from the continuity of Ψ and the fact that N_2 is dense in N_3 .) It is a simple matter to use (a)–(d) and the commutativity of $+$ to check the distributive laws and the associativity of \circ .

That T is additively Γ -compact follows because

$$\begin{aligned} \Gamma[*](y, x) &= \{n(y, x) | n \geq 1\}^- \\ &= \{(ny, nx) | n \geq 1\}^- \subset \Gamma[+](y) \times \Gamma[+](x). \end{aligned}$$

Also, $(1, \gamma)$ is a multiplicative unit for T since

$$\begin{aligned} (y, x) \circ (1, \gamma) &= (y \cdot 1, x\gamma + \Psi(y, \gamma) + \Psi(1, x)) \\ &= (y, \gamma + \gamma + x) = (y, x) = (1, \gamma) \circ (y, x). \end{aligned}$$

Finally we note that $\{0\} \times S$ is a subsemiring which is topologically isomorphic with S since

$$\begin{aligned} (0, x_1) * (0, x_2) &= (0 + 0, x_1 + x_2) = (0, x_1 + x_2), \\ (0, x_1) \circ (0, x_2) &= (0 \cdot 0, x_1 x_2 + \Psi(0, x_2) + \Psi(0, x_1)) \\ &= (0, x_1 x_2 + \gamma + \gamma) = (0, x_1 x_2). \end{aligned}$$

When S is compact we have the following result.

THEOREM 3. *Let S be a compact additively commutative semiring. Then the following are equivalent.*

- (i) *S can be embedded in a compact additively commutative semiring with multiplicative unit.*
- (ii) *S can be embedded in an additively Γ -compact semiring with multiplicative left unit.*
- (iii) *There is a continuous extension $\Psi : N_3 \times S \rightarrow S$ of the mapping $\psi : N_2 \times S \rightarrow S$ defined by $\psi(n, x) = nx$.*
- (iv) *The mapping $\psi : N_2 \times S \rightarrow S$ is uniformly continuous.*

PROOF. As S is compact, the semiring T constructed in Lemma 6 is compact; hence the equivalence of (i), (ii) and (iii). Because $N_3 \times S$ is a compact Hausdorff space, it can be regarded as a uniform space (see Theorem

1, page 225 of [1]). Then if $N_2 \times S$ is given its relative uniformity as a subset of $N_3 \times S$, the equivalence of (iii) and (iv) follows from Corollary 2 to Theorem 2, page 228 of [1].

Our analysis includes the embedding of rings in rings with unit as a special case.

If S is any topological ring and M is the ring of integers then S can be embedded in the product ring $M \times S$, which is locally compact when S is; the construction is given in [4], page 49, and is similar to that in Theorem 1.

In an additively Γ -compact ring, $\Gamma[+](x)$ is a group and so is identical with $K[+](x)$. (This follows immediately from the Corollary to Theorem 1.1.10 of [5]. Alternatively, $\Gamma[+](x)$ contains an additive idempotent (the additive unit of $K[+](x)$); this idempotent must be the additive unit of the ring and so it follows from [8], Theorem 3.2 that $\Gamma[+](x)$ is a group.) In this case we have the following analogue of Lemma 5.

LEMMA 7. *Let S be an additively Γ -compact subring of a topological semi-ring T with multiplicative left unit α such that $\Gamma[+](\alpha)$ is compact. Then there is a continuous extension $\bar{\chi}: N_1 \times S \rightarrow S$ of the mapping $\chi: \{nu | n \geq 1\} \times S \rightarrow S$ defined by $\chi(nu, x) = nx$.*

PROOF. Let $\bar{\chi}$ be the restriction to $N_1 \times S$ of the function $\Psi: N_3 \times S \rightarrow S$ defined in Lemma 5. Then for each positive integer n and each x in S ,

$$\bar{\chi}(nu, x) = \Phi(nu) \cdot x = n\Phi(u) \cdot x = (n\alpha') \cdot x = n(\alpha'x).$$

But

$$\alpha'x = (\alpha + e)x = \alpha x + ex = x + ex,$$

where e is the additive identity of $K[+](\alpha)$. Now

$$ex + ex = (e + e)x = ex$$

and

$$ex \in K[+](\alpha) \cdot x = K[+](\alpha x) = K[+](x)$$

so that ex is the additive identity of $K[+](x)$. However $K[+](x) = \Gamma[+](x)$ because S is a ring and so $x + ex = x$. Thus $\bar{\chi}(nu, x) = nx$ which means that $\bar{\chi}$ is an extension of χ .

Conversely, if S is an additively Γ -compact ring for which $\bar{\chi}$ of Lemma 7 exists, then we can construct an additively Γ -compact ring on $N_1 \times S$ (using $\bar{\chi}$ in place of the Ψ of Lemma 6), and S is topologically isomorphic with $\{\epsilon\} \times S$ where ϵ is the additive unit of N_1 . Also, if γ is the additive unit of S , (u, γ) is a multiplicative unit for the ring $N_1 \times S$.

Hence we have the following results.

THEOREM 4. *Let S be an additively Γ -compact ring. Then the following are equivalent.*

- (i) S can be embedded in an additively Γ -compact ring with unit.
- (ii) S can be embedded in an additively Γ -compact semiring with left unit.
- (iii) There is a continuous extension $\bar{\chi}: N_1 \times S \rightarrow S$ of the mapping $\chi: \{nu|n \geq 1\} \times S \rightarrow S$ defined by $\chi(nu, x) = nx$.

THEOREM 5. *Let S be a compact ring. Then the following are equivalent.*

- (i) S can be embedded in a compact ring with unit.
- (ii) S can be embedded in an additively Γ -compact semiring with left unit.
- (iii) There is a continuous extension $\bar{\chi}: N_1 \times S \rightarrow S$ of the mapping $\chi: \{nu|n \geq 1\} \times S \rightarrow S$ defined by $\chi(nu, x) = nx$.
- (iv) The mapping $\chi: \{nu|n \geq 1\} \times S \rightarrow S$ is uniformly continuous.

References

- [1] N. Bourbaki, *Topologie générale* (Actualités scientifiques et industrielles no. 1142, 1961).
- [2] E. Hewitt, 'Compact monothetic semigroups', *Duke Math. J.* **23** (1956), 447–457.
- [3] E. Hewitt and K. A. Ross, *Abstract harmonic analysis*, Vol. 1 (Springer-Verlag, Berlin, 1963).
- [4] A. G. Kurosh, *Lectures on general algebra* (Chelsea, New York, 1963).
- [5] A. B. Paalman-de Miranda, *Topological semigroups* (Mathematisch Centrum, Amsterdam, 1964).
- [6] J. Selden, *Theorems on topological semigroups and semirings* (Doctoral Dissertation, University of Georgia, 1963).
- [7] J. Selden, 'A note on compact semirings' *Proc. Amer. Math. Soc.* **17** (1966), 882–886.
- [8] A. D. Wallace, 'The structure of topological semigroups', *Bull. Amer. Math. Soc.* **61** (1955), 95–112.
- [9] H. Zassenhaus, *The theory of groups* (2nd ed., Chelsea, New York, 1958).

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