Canad. Math. Bull. Vol. 16 (2), 1973

ON THE IRREGULAR SETS OF A TRANSFORMATION GROUP

BY
S. K. KAUL

We assume throughout that (X, T, π) is a transformation group [2], where X is a topological space which is always assumed to be regular and Hausdorff. We call a point $x \in X$ regular under T if for any open set U in X and any subset G of T such that $\overline{xG} \subseteq U$, there exists an open set V containing x, such that $VG \subseteq U$ [7]. Let $R(X)$ denote the interior of the set of all the regular points of X under T, and $I(X)$ the set of *irregular* points of X under T, that is the set of points which are not regular under T. Let C denote the set of all the components of $\overline{I(X)}$. In this paper we wish to study C. We shall say that C has property Q if for any $A \in C$ and any open set U containing A there exists an open set V, such that $A \subseteq V \subseteq U$ and $\overline{I(X)} \cap \text{bdy } V = \phi$, where bdy is the boundary. In case each component of $\overline{I(X)}$ is a singleton property Q is the same as zero dimensionality. In view of Theorem A below the results here extend the results of [3] to general topological spaces.

We say that $A \in C$ is *irregular* if there exists a net $t = \{t_\alpha, \alpha \in D, \geq\}$ in T such that the net $xt = \{xt_\alpha, \alpha \in D, \geq\}$ converges to a point of A for some $x \in R(X)$. Although theorems (4.1) and (4.4) are of interest in themselves the main results in this study are:

THEOREM A. *Let (X, T, π) be a transformation group. Let X be a locally compact and locally connected space, $\overline{xT}$ be compact for each $x \in X$, $\overline{I(X)}$ be compact, and $R(X)$ be connected and dense in X. If C has property Q and there is an $A \in C$ which is irregular, then,* (a) *each member of C which is invariant under T is irregular, and* (b) *C has at most two invariant sets.*

T is said to be discrete on a subset E of X if there exists a net t of distinct elements in T such that, $xt \to x$ (xt converges to x) for each $x \in E$. T is called *strongly discrete* if it is discrete on $R(X)$, and simply *discrete* if it is discrete on X. By $C(x, t)$ we shall denote the set of all the accumulation points of the net xt for a net t in T and $x \in X$. T is said to be discontinuous if for any given net t in T, $C(x, t)$ lies entirely in $\overline{I(X)}$ for each $x \in R(X)$.

THEOREM B. *Let (X, T, π) be a transformation group. Let X be a locally compact, locally connected, separable and first countable space, $\overline{xT}$ be compact for each $x \in X$,*

Received by the editors November 26, 1970 and, in revised form, December 3, 1971.

https://doi.org/10.4153/CMB-1973-039-2 Published online by Cambridge University Press

$\overline{I(X)}$ be compact, and $R(X)$ be connected and dense in X. Let T be strongly discrete. If C is finite then C has at most two elements.

REMARK. If C is finite then it has property Q, since X is regular and Hausdorff.

1. LEMMA (1.1). *Let $t=\{t_\alpha, \alpha \in D, \geq\}$ and $s=\{s_\beta, \beta \in E, \geq\}$ be nets in T and let $at \to b$ and $bs \to c$, where a, b, c are in X. If $ts=\{t_\alpha s_\beta, (\alpha, \beta) \in D \times E\}$ is the product directed net, then*

(i) *$c \in C(a, ts)$, and*

(ii) *if b is regular under T, then $ats \to c$.*

Proof. Proof of (i) is straightforward. We prove (ii). Let U be an open set containing c. Let W be an open set such that $c \in W \subseteq \overline{W} \subseteq U$. Let $\delta \in E$ be such that for any $\beta \in E$ and β beyond δ (that is $\beta \geq \delta$), $bs_\beta \in W$. Let $G=\{s_\beta : \beta \in E$ and β beyond $\delta\}$. Then, $\overline{bG} \subseteq U$. From regularity of T at b there exists an open set V containing b, such that, $VG \subseteq U$. Let $\gamma \in D$ be such that $at_\alpha \in V$ for α beyond γ $(\alpha \geq \gamma)$. Then clearly for any (α, β) in the product directed set beyond (γ, δ), $at_\alpha s_\beta \in U$. This proves (ii), and the proof of the lemma is complete.

LEMMA (1.2). *Let $\{x_\alpha, \alpha \in D, \geq\}$ and $\{y_\alpha, \alpha \in D, \geq\}$ be nets in X converging respectively to x and y. If T is regular at x and for each $\alpha \in D$ there exists a $g_\alpha \in T$, such that, $x_\alpha g_\alpha = y_\alpha$, then the net $\{xg_\alpha, \alpha \in D, \geq\}$ converges to y.*

Proof. Suppose $\{xg_\alpha\}$ does not converge to y. Then there exists an open set U containing y such that the net is frequently in its complement. Let V, W be open sets, such that, $y \in W \in \overline{W} \subseteq V \subseteq \bar{V} \subseteq U$, and $G=\{g_\alpha : \alpha \in D$ and $xg_\alpha \in X - \bar{V}\}$. Then $\overline{xG} \subseteq X - \overline{W}$, and from the regularity of T at x there exists an open set 0 containing x, such that, $0G \subseteq X - \overline{W}$. Let $\alpha \in D$ be such that for $\alpha \geq \gamma$, $\alpha \in D$, $x_\alpha \in 0$ and $y_\alpha \in W$. But there exists a $\delta \geq \gamma$, $\delta \in D$, such that $g_\delta \in G$, and hence $y_\delta = x_\delta g_\delta \notin W$. This is a contradiction. Hence $\{xg_\alpha\}$ converges to y and the proof is complete.

Lemma (1.2) gives immediately:

COROLLARY (1.2). *If $t=\{t_\alpha\}$ is a net in T such that, for some $x \in X$, xt converges to y and T is regular at y, then $yt^{-1} \to x$, where t^{-1} is the net $\{t_\alpha^{-1}\}$.*

LEMMA (1.3). *Let $\overline{xT}$ be compact for each $x \in X$. If $t=\{t_\alpha, \alpha \in D, \geq\}$ is a net in T, then the set $E(t)=\{x \in R(X) : C(x, t) \subseteq R(X)\}$ is both open and closed in $R(X)$.*

Proof. Let $x \in E(t)$. Since $C(x, t)$ is compact and $R(X)$ is open, there exists open sets V and W, such that, $C(x, t) \subseteq W \subseteq \overline{W} \subseteq V \subseteq \bar{V} \subseteq R(X)$. Let $G=\{t_\alpha : \alpha \in D$

https://doi.org/10.4153/CMB-1973-039-2 Published online by Cambridge University Press

and $xt_\alpha \in W\}$. Then $\overline{xG} \subseteq V$, and from the regularity of T at x there exists an open set U containing x such that $UG \subseteq V$. Hence for any $y \in U$, $\overline{yG} \subseteq R(X)$. Since G contains all t_α for $\alpha \geq \alpha_0$, for some $\alpha_0 \in D$, it follows that $C(y, t) \subseteq R(X)$ for each $y \in U$. This proves that $E(T)$ is open.

Suppose a is an accumulation point of $E(t)$ in $R(X)$ and that $C(a, t) \cap \overline{I(X)} \neq \varnothing$. Then there exists a subset $s=\{s_\alpha, \alpha \in E, \geq\}$ of t, such that, $as \to b \in \overline{I(X)}$. Since $\overline{bT}$ is compact, we may assume, without loss of generality, that bs^{-1} converges, say to c. It is easy to see that $I(X)$ is invariant under T, and, hence, so is $\overline{I(X)}$. Thus $c \in \overline{I(X)}$. Since $a \neq c$ there exist open sets V and W, such that, $c \in W \subseteq \overline{W} \subseteq V$ and $a \notin \bar{V}$. Let $G=\{g \in T : ag \in W\}$. Since T is regular at a, we may choose an open set U containing a such that $UG \subseteq V$. Let also $U \cap V = \varnothing$. Let $y \in E(t) \cap U$, and $r=\{r_\alpha, \alpha \in F\}$ be a subnet of s, such that, yr converges, say to z. Since $y \in E(t)$, $z \in R(X)$. Hence $y\, rr^{-1} \to y$ [Lemma (1.1), (ii)]. But since r is a subset of s, $ar \to b$ and $br^{-1} \to c$. Hence a rr^{-1} is frequently in W [Lemma (1.1), (i)], and therefore G contains a subset which is cofinal in rr^{-1}. This implies, since $y \in U$, that $y\, rr^{-1}$ is frequently in V, which is a contradiction. Hence $C(a, t) \cap I(X) = \varnothing$ and $a \in E(t)$. This proves that $E(t)$ is closed in $R(X)$, and completes the proof of the lemma.

Lemma (1.4). *Suppose $\overline{xT}$ is compact for each $x \in X$ and X is metrizable. Then $x \in X$ is regular under T if and only if it is equicontinuous with respect to any metric on X compatible with the topology of X.*

Proof. Suppose d is a metric on X compatible with its topology. For any set $A \subseteq X$ we shall denote by $U(A, \varepsilon)$ the ε-neighbourhood of A for any $\varepsilon > 0$. We also denote $U(\{x\}, \varepsilon)$ by $U(x, \varepsilon)$.

Suppose T is regular at $x \in X$. Let $\varepsilon > 0$ be given. Let $\{U_i : 1 \leq i \leq n\}$ be a finite open covering of $\overline{xT}$, where $U_i = U(x_i, \varepsilon/4)$ for some $x_i \in \overline{xT}$, $1 \leq i \leq n$. Let $G_i = \{t \in T : xt \in U_i\}$. Then $\overline{xG_i} \subseteq W_i = U(x_i, \varepsilon/2)$, and $T = \bigcup_{i=1}^n G_i$. Since n is finite, by the regularity of T at x, there exists a $\delta > 0$, such that, $U(x, \delta)G_i \subseteq W_i$. Hence if $d(x, y) < \delta$ then $d(xt, yt) < \varepsilon$ for each $t \in T$, and T is equicontinuous at x.

Suppose that T is equicontinuous at x. Let U be open and $G \subseteq T$ be such that $\overline{xG} \subseteq U$. Since $\overline{xG}$ is compact there exists an $\varepsilon > 0$, such that, $U(\overline{xG}, \varepsilon) \subseteq U$. Let $\delta > 0$ be such that, for each $t \in T$, $d(x, y) < \delta$ implies that $d(xt, yt) < \varepsilon$. Then clearly $U(x, \delta)G \subseteq U$, and T is regular at x. This completes the proof.

2. Lemma (2.1). *Suppose X is a locally connected and locally compact space, and $\overline{I(X)}$ is zero dimensional. If $y \in I(X)$ and $\overline{yT}$ is compact then there exists a net t in T such that for some $z \in R(X)$, $zt \to y$.*

Proof. Since $y \in I(X)$, there exists an open set V such that for some subset G of T, $\overline{yG} \subseteq V$ but there does not exist an open set U containing y such that $UG \subseteq V$.

https://doi.org/10.4153/CMB-1973-039-2 Published online by Cambridge University Press

Since $I(X)$ is invariant under T, $yG \subseteq I(X)$. Since $\overline{yG}$ is compact, $I(X)$ is zero dimensional and X is locally compact, there exists an open set $W \subseteq V$, such that, $\overline{W}$ is compact, $\overline{yG} \subseteq W$ and $I(X) \cap \text{bdry } W = \varnothing$, where bdry is the boundary. Thus if U is any connected open set containing y, $Ug \cap \text{bdry } W \neq \varnothing$ for some $g \in G$. Let $\{V_\alpha : \alpha \in D\}$ be a neighbourhood system of connected open sets at y, and for any $\alpha, \beta \in D$ let $\alpha \geq \beta$ if and only if $V_\alpha \subseteq V_\beta$. Let $x_\alpha \in V_\alpha$ and $g_\alpha \in G$ be such that, $x_\alpha g_\alpha = y_\alpha \in \text{bdry } W$. Then the net $\{y_\alpha, \alpha \in D\}$ has a convergent subnet, $\{y_\alpha, \alpha \in E\}$ converging, say to $z \in \text{bdry } W \subseteq R(X)$. Since $\{y_\alpha g_\alpha^{-1}, \alpha \in E\}$ converges to y, it follows from Corollary (1.2) that $\{zg_\alpha^{-1}, \alpha \in E\} \to y$. This proves the lemma.

Lemma (2.2) *Let $\overline{xT}$ be compact for each $x \in X$, and $A \subseteq X$ be closed and invariant under T. If there exists a net t in T such that for some $x \in X - A$, $xt \to y \in A$, then $y \in I(X)$.*

Proof. t contains a subnet $s = \{s_\alpha, \alpha \in E\}$ such that, ys^{-1} converges. Since A is invariant under T and closed if $ys^{-1} \to z$, then $z \in A$. Let U, V, W be open sets such that, $z \in V$ and $\bar{V} \subseteq U$, W contains x and is disjoint with U. If $G = \{s_\alpha^{-1} \in s : ys_\alpha^{-1} \in V\}$, then $\overline{yG} \subseteq U$, but for no open set 0 containing y is $0G \subseteq U$, since the net xt and therefore xs is eventually in 0. Hence $y \in I(X)$. This proves the lemma.

Theorem (2.3). *Suppose X is a locally compact and locally connected space, $\overline{I(X)}$ is zero dimensional, $\overline{xT}$ is compact for each $x \in X$ and $R(X)$ is connected. Then given a $y \in I(X)$ there exists a net t in T, such that, for each $x \in R(X)$, $xt \to y$.*

Proof. From Lemma (2.1) there exists a net t in T such that $x_0 t \to y$ for some $x_0 \in R(X)$. Since $R(X)$ is connected and $x_0 \in E(t)$, it follows from Lemma (1.3) that $C(x, t) \cap R(X) = \varnothing$ for each $x \in R(X)$. Now let U be an open set containing y such that $\overline{I(X)} \cap \text{bdry } U = \varnothing$. Then it is easy to see that the set $E(t, U) = \{x \in R(X) : C(x, t) \subseteq U\}$ is open. $E(t, U)$ is also closed in $R(X)$: For let a be an accumulation point of $E(t, U)$ in $R(X)$ and suppose that $C(a, t) \nsubseteq U$. Then there exists a subnet s of t, such that, $as \to b \notin U$. Since at has no accumulation point in $R(X)$, $b \in \overline{I(X)}$ and hence $b \notin \bar{U}$. Let V be an open set containing b and disjoint with U. From the regularity of T at a there exists an open set 0 containing a, such that, for any $z \in 0$, zs is eventually in V. But 0 contains a point $z \in E(U, t)$ contradicting that $C(z, t) \subseteq U$. Hence $E(t, U)$ is also closed.

Thus, from above, $R(X) = E(t, U)$. Since for any given $z \neq y$ there exists an open set U not containing z, it is clear that $xt \to y$ for each $x \in R(X)$. This completes the proof of the theorem.

Theorem (2.4). *Suppose X is a locally connected and locally compact space, $\overline{I(X)}$ is zero dimensional, $\overline{xT}$ is compact for each $x \in X$ and $R(X)$ is connected. Then $I(X)$ is closed.*

https://doi.org/10.4153/CMB-1973-039-2 Published online by Cambridge University Press

Proof. Let y be an accumulation point of $I(X)$, and $\{y_\alpha, \alpha \in D\}$ be a net in $I(X)$ converging to y. For each $\alpha \in D$, let $t_\alpha=\{t_{\alpha\beta}, \beta \in E_\alpha\}$ be a net in T such that, $xt_\alpha \to y_\alpha$ for each $x \in R(X)$ [Theorem (2.3)]. Fix an $x \in R(X)$, then $\lim_\alpha \lim_\beta xt_{\alpha\beta}=y$. Hence from [8, p. 69, Theorem 4], there exists a net t in T such that $xt \to y$. Then, since $\overline{I(X)}$ is invariant under T and $x \notin I(X)$, from Lemma (2.2), $y \in I(X)$. Hence $I(X)$ is closed, and the proof is complete.

THEOREM (2.5). *Under the hypotheses of Theorem* (2.3) $I(X)$ *has at most two minimal sets.*

Proof. From Theorem (2.4), $I(X)$ is closed. The proof can be completed using Theorem (2.3) and following the proof of the theorem in [3, p. 62].

3. Let C denote the set of all the components of $\overline{I(X)}$ of a transformation group (X, T, π). Let $Y=C \cup \{\{x\}:x \in R(X)\}$ have the quotient topology, and $p:X \to Y$ be the natural projection.

LEMMA (3.1) *If C has property Q, then p is a closed map and $\overline{I(X)}p$ is zero dimensional.*

Proof. It is easy to see that if C has property Q then Y is an upper semi-continuous decomposition of X, and therefore p is a closed map. To show that $\overline{I(X)}p$ is zero dimensional notice that if V is an open set containing an $A \in C$, such that, $\overline{I(X)} \cap \text{bdry } V=\varnothing$, then Vp is open and $\overline{I(X)}p \cap \text{bdry } Vp=\varnothing$. This implies, using property Q, that $\overline{I(X)}p$ is zero dimensional. This proves the lemma.

Since $\overline{I(X)}$ is invariant under T, for any component A of $\overline{I(X)}$, At is again a component of $\overline{I(X)}$ for each $t \in T$. Consequently if each $A \in C$ is compact then (Y, T, ρ), where $\rho: Y\times T \to Y$ is defined by $(Y, T)\rho=xt\rho$, for any $(y, t) \in Y \times T$ where $x \in y\rho^{-1}$, is a transformation group [2, p. 7, Definition (1.39)] provided C has property Q. Furthermore, the following diagram

(*)
$$\begin{array}{ccc} X\times T & \xrightarrow{\pi} & X \\ \Big\downarrow{\scriptstyle p\times 1} & & \Big\downarrow{\scriptstyle p} \\ Y\times T & \xrightarrow{\rho} & Y \end{array}$$

where 1 is the identity on T, is commutative.

Henceforth in this section we assume that X is locally connected and locally compact, C has property Q and each $A \in C$ is compact, so that, Y is a locally compact and locally connected space, and furthermore that $\overline{xT}$ is compact for each $x \in X$, so that $\overline{yT}$ is compact for each $y \in Y$.

Let $I(Y)$ denote the set of all the irregular points of (Y, T, ρ) and $R(Y)= Y-\overline{I(Y)}$.

https://doi.org/10.4153/CMB-1973-039-2 Published online by Cambridge University Press

LEMMA (3.2). *$\overline{I(Y)}$ is zero dimensional and is contained in $\overline{I(X)}p$.*

Proof. Since $R(X)$ is open in X, and p is $1-1$ on $R(X)$, the restriction of p to $R(X)$, $p \mid R(X)$, is a homeomorphism. Hence from (*), $R(X)p \subseteq R(Y)$. This implies that $I(Y) \subseteq \overline{I(X)}p$. Hence $\overline{I(Y)}$ is zero dimensional, since $\overline{I(X)}p$ is zero dimensional [Lemma (3.1)].

THEOREM (3.3). *$A \in C$ is irregular if and only if $Ap \in I(Y)$.*

Proof. If $A \in C$ is irregular then by definition there exists a net t in T such that for some $x \in R(X)$, xt converges to a point of A. Hence $xpt \to Ap$. Since, $y=xp \in R(Y)$ [Lemma (3.1)], and $Ap \in \overline{I(X)}p$ which is closed and invariant under T, from (*), from Lemma (2.2), $Ap \in I(Y)$.

Conversely, suppose $z=Ap \in I(Y)$. Since $\overline{I(X)}p$ is zero dimensional [Lemma (3.1)], and $\overline{zT}$ is compact, from Lemma (2.1) there exists a net t in T such that for some $y \in R(X)p$, $yt \to z$. Hence since $\overline{xt}$ is compact, if $x=yp^{-1}$, then $x \in R(X)$, and for a subnet s of t, xs converges to a point of A. This completes the proof.

Proof of Theorem A. Let $A \in C$ be an irregular set, and $Ap=a$. Then $a \in I(Y)$ [Theorem (3.3)]. Since $R(X)$ is connected and dense in X, $R(Y)$ is connected. Hence from Theorem (2.3), there exists a net t in T, such that, for each $y \in R(Y)$, $yt \to a$. If $B \in C$ is not irregular, then $Bp \notin I(Y)$ [Theorem (3.4)]. From Theorem (2.4), $I(Y)$ is closed, and hence $Bp \in R(Y)$. Therefore $Bpt \to a$ and Bp is not invariant under T, and from (*) B is not invariant under T. This proves (a).

Since from (a) each invariant element $A \in C$ is irregular, it follows that $\{Ap\}$ is a minimal set in (Y, T, ρ). But as there can be at most two such minimal sets [Theorem (2.5)], C can have at most two invariant elements. This proves (b), and completes the proof of the theorem.

4. THEOREM (4.1). *Suppose (X, T, π) is a transformation group, where X is a separable and first countable space, $\overline{xT}$ is compact for each $x \in X$, and $I(X)=\varnothing$. If T is discrete then it is finite.*

Proof. Let $\{a_n : n=1, 2, \ldots\}$ be a dense set in X and let T be not finite. Since $\overline{a_1T}$ is compact and X is first countable there exists a sequence $s_1=\{s_{1n} : n=1, 2, \ldots\}$ of distinct elements in T, such that, the sequence a_1s_1 converges. Inductively we get a subsequence s_m of s_{m-1}, $m \geq 2$, such that, a_ms_m converges. Hence by the diagonal process there exists a sequence s in T, such that, a_ns converges for each a_n, $n= 1, 2, \ldots$. Since T is regular at each a_n, for the product directed net ss^{-1}, $a_nss^{-1} \to a_n$ for each a_n [Lemma (1.1)]. Again, from the first countability of X, we get a sequence t in T, such that, $a_nt \to a_n$ for each a_n: For consider a neighbourhood

https://doi.org/10.4153/CMB-1973-039-2 Published online by Cambridge University Press

system $U_n=\{U_{nm}:m=1,2,\ldots\}$ of a_n, and let $t_n=s_{i(n)}s_{j(n)}^{-1}\in s\times s^{-1}$ such that, $t_n\neq t_k$ and $a_k s_{i(n)}s_{j(n)}^{-1}\in U_{kn}$ for all $k\leq n$, $n=1,2,\ldots$. Then $t=\{t_n:n=1,2,\ldots\}$ is the required sequence. Hence xt converges for each $x\in X$ [7, p. 465, Lemma (3.2)], and from the proof of Theorem (3.2) [7, p. 465] it follows that $xt\to x$ for each $x\in X$. This proves the lemma.

THEOREM (4.2). *Suppose X is a separable and first countable space, $\overline{xT}$ is compact for each $x\in X$, $I(X)\neq\varnothing$ and T is strongly discrete. Then C has at least one irregular set.*

Proof. Since $I(X)\neq\varnothing$, T is not finite. There must exist a point $x\in R(X)$ such that $\overline{xT}\cap\overline{I(X)}\neq\varnothing$, for otherwise, $R(X)$ being invariant under T, the transformation group $(R(X), T, \pi)$ satisfies all conditions of Theorem (4.1) and hence is not discrete, that is (X, T, π) is not strongly discrete. This is a contradiction. Hence there exists a sequence t in T, such that, $xt\to y\in\overline{I(X)}$. If A is the component of $\overline{I(X)}$ containing y, then A is irregular. This completes the proof.

COROLLARY (4.2). *Assume the hypotheses of Theorem A, and, furthermore, that X is separable and first countable and T is strongly discrete. Then*

(a) each $A\in C$ which is invariant under T is irregular, and

(b) C has at most two invariant sets.

Proof. From Theorem (4.2) C has at least one irregular set. The results then follow from Theorem A. This completes the proof.

The following lemma is essentially a result of Roberson [9].

LEMMA (4.3) *Suppose X is a locally compact, locally connected and first countable space, $\overline{I(X)}$ is zero dimensional, y and z are points of $\overline{I(X)}$, such that, $z\notin\overline{yT}$, $R(X)$ is connected and $\overline{xT}$ is compact for each $x\in X$. If t is a sequence in T, such that, $xt\to z$ for some $x\in R(X)$, then it has a subsequence s, such that, $xs^{-1}\to y$ for each $x\in X$. Consequently $\overline{yT}=\{y\}$.*

Proof. Consider a sequence of connected open sets $\{V_n:n=1,2,\ldots\}$ which is a neighbourhood system at y. Let U be an open set containing z, such that, $\bar{U}$ is compact and disjoint with $\overline{yT}$, and $\overline{I(X)}\cap\text{bdry }U=\varnothing$. Since, from Lemma (2.1), $R(X)$ is dense in X, there exists a $y_n\in V_n\cap R(X)$ for each n. Let $m(n)\geq n$ be an integer, such that, $y_n t_{m(n)}\in U$ (this is possible from Theorem (2.3)), so that, since $yt_{m(n)}\in\overline{yT}$, and $V_n t_{m(n)}$ is connected, $V_n t_{m(n)}\cap\text{bdry }U\neq\varnothing$. Let x_n be a point in this intersection. Assuming without loss of generality that $x_n\to x$, since $x\in R(X)$, we get from Lemma (1.2) that $\{xt_{m(n)}^{-1}:n=1,2,\ldots\}\to y$. Thus $s^{-1}=\{t_{m(n)}^{-1}:n=1,2,\ldots\}$ is the required sequence and $xs^{-1}\to y$ for each $x\in R(X)$ [Theorem (2.3)].

https://doi.org/10.4153/CMB-1973-039-2 Published online by Cambridge University Press

If $y' \in \overline{yT}$ and $y' \neq y$, then since $xs \to z$ for each $x \in R(X)$, we can get, as above, a subsequence s' of s, such that, $xs'^{-1} \to y'$ for each $x \in X$. But this is impossible. This completes the proof.

Proof of Theorem B. Suppose $I(X) \neq \varnothing$. Then from Theorem (4.2), C has at least one irregular element. Hence $I(Y) \neq \varnothing$ (Theorem (3.3)). Since $I(Y)$ and $\overline{I(X)}p$ are both invariant under T, the set $E = \overline{I(X)}p - I(Y)$ is also invariant under T. If $A \in C$ is not irregular, then $a = Ap \in E$. Let $b \in I(Y)$, and t be a net in T such that for each $y \in R(Y)$, $yt \to b$ [Theorem (2.3)]. Since $E \subseteq R(Y)$, $at \to b$. But since $\overline{I(X)}p$ is finite at eventually takes the value b. But, since $b \notin E$, $a \in E$, and E is invariant under T, this is impossible. Hence $E = \varnothing$, and each $A \in C$ is irregular [Theorem (3.3)]. Since C is finite, so is $I(Y)$. Then as in Lemma 2 [8], using Lemma (4.3), we can show that $I(Y)$ has at most two points. Hence C has at most two elements. This completes the proof.

THEOREM (4.4). *Suppose X is a separable and first countable space, $\overline{xT}$ is compact for each $x \in X$, and $R(X)$ is connected. Then T is discontinuous if and only if it is strongly discrete.*

Proof. If T is discontinuous then it is obviously strongly discrete. To prove the converse, suppose that T is strongly discrete but not discontinuous. Then, since X is first countable, there exists a sequence t in T, such that, for some $x \in R(X)$, $C(x, t) \subseteq R(X)$. Consequently, from Lemma (1.3), $C(x, t) \subseteq R(X)$ for each $x \in R(X)$. Hence, just as in the proof of Theorem (4.1), using the separability of X, we can get a sequence s, such that, $xs \to x$ for each $x \in R(X)$, contradicting the fact that T is strongly discrete. Therefore, T is discontinuous and the proof is complete.

REFERENCES

1. J. Dugundji, *Topology*, Allyn and Bacon, Boston, Mass., 1968.
2. W. H. Gottschalk and G. A. Hedlund, *Topological dynamics*, Colloq. Publ., American Math. Soc., Providence, R.I., 1955.
3. W. J. Gray and F. A. Roberson, *On the near equicontinuity of transformation groups*, Proc. Amer. Math. Soc. **23** (1969), 59–63.
4. T. Homma and S. Kinoshita, *On homeomorphisms which are regular except for a finite number of points*, Osaka J. Math. **7** (1955), 29–38.
5. S. K. Kaul, *On almost regular homeomorphisms*, Canad. J. Math. **20** (1968), 1–6.
6. ——, *On a transformation group*, Canad. J. Math. **21** (1969), 935–941.
7. ——, *Compact subsets in function spaces*, Canad. Math. Bull. **12** (1969), 461–466.
8. J. L. Kelley, *General topology*, Van Nostrand, Princeton, N.J., 1955.
9. F. A. Roberson, *Some theorems on the structure of near equicontinuous transformation groups*, Canad. J. Math. **23** (1971), 421–425.

UNIVERSITY OF SASKATCHEWAN,
REGINA, SASKATCHEWAN