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DYNAMICS OF HOMEOMORPHISMS ON MINIMAL SETS GENERATED BY TRIANGULAR MAPPINGS

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The main goal of the paper is the construction of a triangular mapping F of the square with zero topological entropy, possessing a minimal set M such that $F|_M$ is a strongly chaotic homeomorphism, as well as other properties that are impossible for continuous maps on an interval.

To do this we define a parametric class of triangular maps on $Q \times I$, where Q is an infinite minimal set on the interval, which are extendable to continuous triangular maps $F : I^2 \to I^2$. This class can be used to create other examples.

1. INTRODUCTION

Let $I = [0, 1]$ be the closed unit interval. Let $\mathcal{C}$ denote the class of continuous maps $f : I \to I$, and Δ the class of *triangular maps* $F : I^2 \to I^2$, that is, the continuous functions defined by

$$F(x,y) = \bigl(f(x),\ g(x,y)\bigr) = \bigl(f(x),\ g_x(y)\bigr).$$

The map $f \in \mathcal{C}$ is the *base* for F, and $g_x : I \to I$ is a family of continuous maps depending continuously on x. Note that F transforms the *layer* $I_x := \{x\} \times I$ into the layer $I_{f(x)}$.

Triangular maps have much simpler dynamics than continuous maps of the square in general [7]. This is because the projection $\pi_1 : (x,y) \mapsto x$ semiconjugates any $F \in \Delta$ to its base f via $f \circ \pi_1 = \pi_1 \circ F$. This implies, for example, that Sharkovsky's theorem on the coexistence of periodic orbits remains valid in Δ [6]. Moreover, the projection π_1 maps the class $\mathrm{Per}\,(F)$ of periodic points of F onto $\mathrm{Per}\,(f)$, or the class $UR(F)$ of uniformly recurrent points of F onto $UR(f)$. However there are exceptions: homoclinic orbits [7] or isochronically recurrent points [4] of F are not mapped by π_1 *onto* the corresponding classes of f.

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A big difference between the dynamics of maps in $\mathcal{C}$ and in Δ already appears in the simplest cases in which every periodic point of F is a fixed point and the base is linear, see [**5**, **8**] (see also [**7**, Theorem 3]).

However, the class of maps in Δ of type 2^∞ (with respect to the Sharkovsky's ordering) is more interesting. There are, for example, maps in Δ of type 2^∞ with positive topological entropy [**7**] but with recurrent points which are not uniformly recurrent [**5**]. Such maps are impossible in $\mathcal{C}$. In both of the preceeding examples, the map F has a base f of type 2^∞ with an infinite minimal set Q such that F has "bad" behaviour on the set $\pi_1^{-1}(Q) = Q \times I$. (Recall that a set M is a *minimal set* for a map if it is non-empty, closed and invariant and if no proper subset of M has the same properties.)

In the present paper we show that maps of type 2^∞ in Δ, even homeomorphisms on minimal sets, may have very complicated dynamics. Note that if M is a minimal set for F in Δ, then $\pi_1(M)$ is a minimal set for f (this is true for any general semi-conjugacy, see [**11**]), hence $\pi_1(M)$ is either a periodic orbit or a solenoid, that is, a Cantor-type set [**1**]. The first case, however, implies that M is essentially one-dimensional, so non trivial behaviour is possible only of $\pi_1(M)$ is infinite. We shall consider only this case.

In Section 2, starting from a Cantor-type set Q and a map $f : Q \to Q$ of type 2^∞, we define a family $\mathcal{T}$ of functions F of type 2^∞, non-decreasing on any layer and such that $F(Q \times I) \subset Q \times I$. It is always possible to extend each $F \in \mathcal{T}$ to a function $\widetilde{F} \in \Delta$ preserving its type 2^∞ and the monotonicity on each layer. All these functions have zero topological entropy. Then we define a parametric family $\mathcal{T}_0 \subset \mathcal{T}$. This construction is based on an idea from [**5**] and can be further modified to get more general maps.

In Section 3, we construct a subclass $\mathcal{T}_{01}$ of $\mathcal{T}_0$ and prove that the maps in this class have a minimal set containing an interval. (The existence of such maps was already proved in [**5**].)

In Section 4 we show that there are maps in $\mathcal{T}_{01}$ which are distributionally chaotic, and hence, chaotic in the sense of Li and Yorke on a minimal set. Recall that no map in $\mathcal{C}$ having zero topological entropy can be chaotic on a minimal set [**3**].

In Section 5 we prove some results concerning functions in $\mathcal{T}_{01}$ and in other classes $\mathcal{T}_{02} \subset \mathcal{T}_0$ and $\mathcal{T}_1 \subset \mathcal{T}$. These results show properties which are impossible in $\mathcal{C}$.

2. A parametric class of triangular maps

Let $\{0,1\}^{\mathbb{N}}$ be the space of all sequences of two symbols equipped with the following metric ρ: $\rho(\underline{\alpha},\underline{\beta}) := \max\{1/i : \alpha(i) \neq \beta(i)\}$ for any distinct $\underline{\alpha} = \{\alpha(i)\}_{i \geqslant 1}$ and $\underline{\beta} = \{\beta(i)\}_{i \geqslant 1}$ in $\{0,1\}^{\mathbb{N}}$. Since, as is well known, any Cantor-type set Q is homeomorphic to $\{0,1\}^{\mathbb{N}}$, we may identify an element $x \in Q$ with the corresponding sequence $\underline{x} = x(1)x(2)\cdots$.

Consider now the function $f : Q \to Q$ acting on Q as an *adding machine*, that is, for $\underline{\alpha} \in \{0, 1\}^{\mathbf{N}}$, $f(\underline{\alpha}) = \underline{\alpha} + 1000\cdots$ where the adding is in base 2 from the left to right; for example, $f(101100\cdots) = 011100\cdots$, $f(11100\cdots) = 00010\cdots$, and so on. Given a point $\underline{x} \in Q$, the point $f^s(\underline{x}) \in Q$ is represented by the sequence $\underline{x}_s$ obtained by adding (in base 2) the sequence $\underline{x}$ and the eventually zero sequence representing the number s written in base 2 from left to right. It is easy to see that $\omega_f(\underline{x}) = Q$ for any $\underline{x} \in Q$.

Denote by $\mathcal{T}$ the class of maps $F : Q \times I \to Q \times I$, where Q is a Cantor-type set and $F(\underline{x}, y) = \big(f(\underline{x}), g(\underline{x}, y)\big)$ where $f : Q \to Q$ is the adding machine, and $g(\underline{x}, \cdot) : I \to I$ is continuous and non-decreasing for any $\underline{x} \in Q$, and the family $g(\underline{x}, \cdot)$ depends continuously on $\underline{x}$ with respect to the uniform metric. Thus F is continuous on $Q \times I$.

Note that each map $F \in \mathcal{T}$ (and obviously also its monotonic extension $\widetilde{F} \in \Delta$) has topologically entropy $h(F) = 0$. Indeed, we have (see [7]).

$$\sup\big\{h\big(F, I_{\underline{x}}\big);\ x \in Q\big\} + h(f) \geqslant h(F),$$

where $h\big(F, I_{\underline{x}}\big)$ denotes the topological entropy of the map $F : Q \times I \to Q \times I$ with respect to the compact subset $I_{\underline{x}}$, that is, the entropy $h\big(F, I_{\underline{x}}\big)$ is computed only for trajectories starting from $I_{\underline{x}}$. But since F^i is monotonic on $I_{\underline{x}}$ for any i, we have clearly $h\big(F, I_{\underline{x}}\big) = 0$, and of course $h(f) = 0$ since f is of type 2^∞. Thus, $h(F) = 0$.

Now we describe the construction of the mappings of a special subclass $\mathcal{T}_0$ of $\mathcal{T}$.

First we take an increasing sequence of non-negative integers $\{k_i\}_{i=0}^{\infty}$ with $k_0 = 0$ and such that, for all $i \geqslant 1$, $k_i - k_{i-1} - 1 =: m_i \geqslant 1$. Thus $k_n = k_{n-1} + m_n + 1 = m_1 + \cdots + m_n + n$. For any $\underline{x} \in Q$, the digits $x(k_1)$, $x(k_2)$, $\ldots$ are called *control digits* of $\underline{x}$. If

$$\underline{x} = \underbrace{x(1)\cdots x(k_1 - 1)}_{m_1} x(k_1)\cdots x(k_{n-1}) \underbrace{x(k_{n-1} + 1)\cdots x(k_n - 1)}_{m_n} x(k_n)\cdots,$$

we define, for every $n \geqslant 1$,

$$\chi_n(\underline{x}) := \big(x(k_{n-1} + 1), \ldots, x(k_n - 1)\big) \in \{0, 1\}^{m_n} \quad \text{and} \quad \big|\chi_n(\underline{x})\big| := \sum_{i=1}^{m_n} x(k_{n-1} + i)2^{i-1}.$$

Then we consider a family $\Gamma_n := \big\{\varphi(n, j),\ j = 0, \ldots, 2^{m_n} - 1\big\}$ of functions from I into I satisfying the following properties:

(1) each function $\varphi(n, j) \in \Gamma_n$ is continuous and non-decreasing :

(2) $\varphi^r(n, 2^{m_n} - 1) \circ \cdots \circ \varphi^r(n, 0) = Id$ for all $r \geqslant 1$

where Id denotes the identity map. We call any map of Γ_n a map of *rank* n. Moreover we assume that

$$\lim_{n\to+\infty} \max_j \Big\{ \big\| \varphi(n,j) - Id \big\| \Big\} = 0 \tag{3}$$

where $\|\cdot\|$ is the uniform norm.

Finally we define a function $F : Q \times I \to Q \times I$ as follows. Take an arbitrary point $\underline{x} \in Q$. If the first zero control digit of $\underline{x}$ is $x(k_n)$, then we define

$$F(\underline{x}, y) = \big(f(\underline{x}),\ \varphi(n, |\chi_n(\underline{x})|)(y)\big).$$

Otherwise, if $\underline{x}$ has no zero control digits, we set

$$F(\underline{x}, y) = \big(f(\underline{x}),\ y\big).$$

Note that (1) and (3) guarantee the continuity of F in $Q \times I$.

The class $\mathcal{T}_0$ consists of the functions constructed in this way for any possible choice of the parameters k_n and of the families Γ_n.

Let $\pi_2 : (x, y) \mapsto y$ be the projection on the second variable and put $t_n := 2^{k_n - 1}$ for all $n \geqslant 0$. (Note that $t_0 = 1/2$.) Given $F \in \mathcal{T}_0$, for any $i \geqslant 0$ and any $y_0 \in I$, define $y_i := \pi_2\big[F^i(\underline{0}, y_0)\big]$. Then, for any integer $i \geqslant 0$ we have

$$F^i(\underline{0}, y_0) = \big(f^i(\underline{0}), y_i\big) = \big(f^i(\underline{0}),\ \psi(i)(y_0)\big),$$

where $\psi(0) = Id$ and, if $1 \leqslant i < t_n$, $\psi(i)$ is a composition of maps φ of rank not greater than n.

For all $0 \leqslant j \leqslant 2^{m_{n+1}} - 1$ and $0 \leqslant r < t_n$, we have the following relations

$$\psi(2jt_n + r) = \psi(r) \circ \psi(2jt_n), \tag{4}$$

$$\psi\big((2j+1)t_n + r\big) = \psi_j^*(r) \circ \psi(2jt_n), \tag{5}$$

where $\psi_j^*(r)$ is the function obtained from $\psi(r)$ by replacing all maps φ of rank n with $\varphi(n+1, j)$.

Indeed,

$$f^{2jt_n}(\underline{0}) = \underbrace{0\cdots0}_{k_n}\xi(1)\cdots\xi(m_{n+1})0\cdots, \quad f^{(2j+1)t_n}(\underline{0}) = \underbrace{0\cdots01}_{k_n}\xi(1)\cdots\xi(m_{n+1})0\cdots,$$

with $\big|\chi_{n+1}\big(f^{2jt_n}(\underline{0})\big)\big| = \big|\chi_{n+1}\big(f^{(2j+1)t_n}(\underline{0})\big)\big| = \big|\big(\xi(1), \dots, \xi(m_{n+1})\big)\big| = j$. This means that after $2jt_n$ iterations, all the first n control digits are zero and so, for the

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next r iterations, we apply the same functions φ as when starting from $\underline{0}$. Conversely, after $(2j+1)t_n$ iterations the n-th control digit is equal to one and so, during the next r iterations we proceed as in the previous case, but instead of using the functions φ of rank n, we apply the function $\varphi(n+1,j)$. This is exactly what is written in formulas (4) and (5). Obviously, if $r < t_{n-1}$, the function $\psi(r)$ does not contain any map of rank n and so $\psi_j^*(r) = \psi(r)$.

Note that from (5) with $r = 0$ we obtain

$$\psi(t_n) = Id, \quad \psi\big((2j+1)t_n\big) = \psi(2jt_n), \tag{6}$$

Now we prove some identities concerning the functions in $\mathcal{T}_0$.

Lemma 1. *Let $F \in \mathcal{T}_0$. For every $i \geqslant 1$ take $n \geqslant 1$ such that $t_{n-1} \leqslant i < t_n$ and consider the representation of i in the form*

$$i = (2\alpha_{n-1} + \beta_{n-1})t_{n-1} + \cdots + (2\alpha_1 + \beta_1)t_1 + 2\alpha_0 t_0$$

with $0 \leqslant \alpha_s \leqslant 2^{m_{s+1}} - 1$, $\beta_s \in \{0,1\}$ for $0 \leqslant s \leqslant n-1$ and $\beta_0 = 0$. If

$$\nu(i) = \max\{s \leqslant n-1 : \beta_s = 0\}$$

we write

$$i = (2\alpha_{n-1} + 1)t_{n-1} + \cdots + \big(2\alpha_{\nu(i)+1} + 1\big)t_{\nu(i)+1} + 2\alpha_{\nu(i)}t_{\nu(i)} + \theta(i).$$

Then we have

(7)
$$\psi(i) = \psi\big(\theta(i)\big) \circ \varphi^{\gamma(i)}(n,\alpha_{n-1}) \circ \varphi^{2^{k_{n-1}-(n-1)}}(n,\alpha_{n-1}-1) \circ \cdots \circ \varphi^{2^{k_{n-1}-(n-1)}}(n,0)$$

$$\psi_u^*(i) = \psi\big(\theta(i)\big) \circ \varphi^{\overline{\gamma}(i)}(n+1,u) \tag{8}$$

where $\gamma(i) = \sum_{j=\nu(i)}^{n-2} \alpha_j 2^{k_j - j}$ $\big(< 2^{k_{n-1}-(n-1)}\big)$ and $\overline{\gamma}(i) = \gamma(i) + \alpha_{n-1}2^{k_{n-1}-(n-1)}$.

In particular, for all $n \geqslant 2$ and all j with $0 \leqslant j \leqslant 2^{m_n} - 1$,

$$\psi(2jt_{n-1}) = \varphi^{2^{k_{n-1}-(n-1)}}(n,j-1) \circ \cdots \circ \varphi^{2^{k_{n-1}-(n-1)}}(n,0). \tag{9}$$

Proof: First we prove (7) by induction on n. Let $n = 1$, that is, $1 \leqslant i < t_1$; we have

$$\psi(i) = \varphi(1,i-1) \circ \cdots \circ \varphi(1,0).$$

In this case $i = 2\alpha_0 t_0 = \alpha_0$, $\nu(i) = 0$ and $\theta(i) = \gamma(i) = 0$. So (7) is satisfied.

Assume (7) true for n and consider $n+1$. We have to find the representation of $\psi(i)$ for all i with $t_n \leqslant i < t_{n+1}$. Let

$$i = (2\alpha_n + \beta_n)t_n + \cdots + (2\alpha_1 + \beta_1)t_1 + 2\alpha_0 t_0$$

and assume first $\beta_n = 0$. Then, $i = 2\alpha_n t_n + \theta(i)$ and, by (4),

$$\psi(i) = \psi\big(\theta(i)\big) \circ \psi(2\alpha_n t_n).$$

Since in this case $\gamma(i) = 0$, (7) is proved if we show that

$$\psi(2\alpha_n t_n) = \varphi^{2^{k_n - n}}(n+1, \alpha_n - 1) \circ \cdots \circ \varphi^{2^{k_n - n}}(n+1, 0). \tag{10}$$

We prove (10) by induction on α_n. By the induction hypothesis and the representation

$$t_n - 1 = \big(2(2^{m_n} - 1) + 1\big)t_{n-1} + \cdots + 2(2^{m_i} - 1)t_0,$$

we have

(11)

$$\psi(t_n - 1) = \varphi^{\gamma(t_n - 1)}(n, 2^{m_n} - 1) \circ \varphi^{2^{k_{n-1} - (n-1)}}(n, 2^{m_n} - 2) \circ \cdots \circ \varphi^{2^{k_{n-1} - (n-1)}}(n, 0)$$

where $\gamma(t_n - 1) = \sum_{j=0}^{n-2} (2^{m_{j+1}} - 1)2^{k_j - j} = 2^{k_{n-1} - (n-1)} - 1$.

Now, by (5) and (11)

$$\psi\big(t_n + (t_n - 1)\big) = \psi_0^*(t_n - 1) \circ \psi(0) = \psi_0^*(t_n - 1) = \varphi^{2^{k_n - n} - 1}(n+1, 0).$$

Since

$$f^{2t_n - 1}(\underline{0}) = \underbrace{1 \cdots 1}_{k_n} 0 \cdots$$

at the next iteration we apply the map $\varphi(n+1, 0)$, thus

$$\varphi(2t_n) = \varphi^{2^{k_n - n}}(n+1, 0),$$

hence (10) is proved for $\alpha_n = 1$. Assume it is true for $\alpha_n = j$. By (6)

$$\psi\big((2j+1)t_n\big) = \psi(2jt_n)$$

and by (5) and the induction hypothesis we have

$$\begin{aligned}\psi\big((2j+1)t_n + t_n - 1\big) &= \psi_j^*(t_n - 1) \circ \psi(2jt_n) \\ &= \varphi^{2^{k_n - n} - 1}(n+1, j) \circ \varphi^{2^{k_n - n}}(n+1, j-1) \circ \cdots \circ \varphi^{2^{k_n - n}}(n+1, 0).\end{aligned}$$

Since

$$f^{2jt_n+2t_n-1}(\underline{0}) = \underbrace{1\cdots 1}_{k_n}\xi(1)\cdots\xi(m_{n+1})0\cdots$$

with $\left|\chi_{n+1}\left(f^{2jt_n+2t_n-1}(\underline{0})\right)\right| := \left|\left(\xi(1),\ldots,\xi(m_{n+1})\right)\right| = j$, at the next iteration we apply the map $\varphi(n+1,j)$, thus obtaining (10) for $\alpha_n = j+1$. Hence (10) is completely proved.

Assume now $\beta_n = 1$, that is,

$$i = (2\alpha_n+1)t_n + \cdots + \theta(i) = (2\alpha_n+1)t_n + r$$

and observe that $\theta(i) = \theta(r)$.

By (5) and (10) we obtain

$$\psi(i) = \psi^*_{\alpha_n}(r)\circ\psi(2\alpha_n t_n) = \psi^*_{\alpha_n}(r)\circ\varphi^{2^{k_n-n}}(n+1,\,\alpha_n-1)\circ\cdots\circ\varphi^{2^{k_n-n}}(n+1,0).$$

If $t_{n-1}\leqslant r < t_n$, then $\nu(i) = \nu(r)$ and

$$\psi^*_{\alpha_n}(r) = \psi\big(\theta(r)\big)\circ\varphi^{\overline{\gamma}(r)}(n+1,\alpha_n) = \psi\big(\theta(i)\big)\circ\varphi^{\gamma(i)}(n+1,\alpha_n)$$

since $\overline{\gamma}(r) = \sum_{j=\nu(i)}^{n-1}\alpha_j 2^{k_j-j} = \gamma(i)$.

If $r < t_{n-1}$, then $\nu(i) = n-1$, $\alpha_{n-1} = 0$ and so $\overline{\gamma}(r) = \gamma(i) = 0$; in this case

$$\psi^*_{\alpha_n}(r) = \psi\big(\theta(r)\big) = \psi\big(\theta(i)\big).$$

Thus (7) is proved for $n+1$. ∎

3. Properties of minimal sets for maps in $\mathcal{T}$ and $\mathcal{T}_0$

Theorem 1. *No $F\in\mathcal{T}$ can have a minimal set with non-empty interior in $Q\times I$.*

Proof: Assume there is a function $F\in\mathcal{T}$ with a minimal set M containing a non-empty open set G of $Q\times I$. We may assume, without loss of generality, $G\subset Q\times(0,1)$.

Since $\pi_1(M)$ is minimal for the base map, $\pi_1(M) = Q$ and so, for any $\underline{x}\in Q$ the set $M\cap I_{\underline{x}}$ is non-empty. Let $\underline{x}_0\in Q$ and $M_0 := M\cap I_{\underline{x}_0}$; define $y_0 = \max\{y : (\underline{x}_0, y)\in M\}$. By the minimality of M we have $\omega_F(\underline{x}_0,y_0) = M$, hence there is an integer n such that $(\underline{x}_n, y_n) := F^n(\underline{x}_0,y_0)\in G$. Since $\underline{x}_0$ is the unique preimage of $\underline{x}_n$ with respect to f^n and $F(M) = M$ (see [1]), we have $F^n(M_0) = M\cap I_{\underline{x}_n}$. But $F^n|_{M_0}$ is non-decreasing and so $y_n = \max\{y : (\underline{x}_n,y)\in M\}$, contrary to the fact that $\sup\{y : (\underline{x}_n,y)\in M\} > y_n$. ∎

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THEOREM 2. *Suppose that $F \in \mathcal{T}_0$ has a minimal set M containing the layer $I_{\underline{0}}$. Then $F|_M$ is a homeomorphism.*

PROOF: Since M is a compact set and F is continuous, $F|_M$ is a homeomorphism if and only if it is one-to-one on any set $M_{\underline{x}} = M \cap I_{\underline{x}}$, $\underline{x} \in Q$. Consider first the case $\underline{x} \in \operatorname{Orb}(\underline{0})$, that is, $\underline{x} = f^s(\underline{0})$ for some $s \geqslant 0$ and let $t_n > s$. By (6), $\psi(t_n) = Id$ and this implies that at any step $j < t_n$ the function φ to be applied to is injective on $\pi_2\big[F^j(I_{\underline{0}})\big]$. Thus F is injective on $F^s(I_{\underline{0}})$, which, by the minimality of M, equals $M_{\underline{x}}$. Take now an aribtrary point $\underline{x} \in Q \setminus \operatorname{Orb}(\underline{0})$. If all control digits of $\underline{x}$ are equal to one, then the function to be applied to is the identity. Assume now that the first zero control digit of $\underline{x}$ is $x(k_n)$ and take the neighbourhood U of $\underline{x}$ in Q given by all $\underline{t} \in Q$ with the first k_n digits of their representations equal to those of $\underline{x}$, that is, $\underline{t}(i) = \underline{x}(i)$, $1 \leqslant i \leqslant k_n$. Thus, for every $\underline{t} \in U$,

$$F(\underline{t}, y) = \big(f(\underline{t}), \varphi(n, |\chi_n(\underline{t})|)(y)\big) = \big(f(\underline{t}), \varphi(n, |\chi_n(\underline{x})|)(y)\big),$$

that is, the function φ to be applied to is the same for all $\underline{t} \in U$.

Let $\underline{t}_0$ be the first point in $\operatorname{Orb}(\underline{0})$ belonging to U, hence

$$\underline{t}_0 = f^{r_0}(\underline{0}) = x(1)\cdots x(k_n - 1)0\cdots \in U \cap \operatorname{Orb}(\underline{0})$$

with $\underline{t}_0(i) = 0$ for $i \geqslant k_n$. Every $\underline{t} \in U \cap \operatorname{Orb}(\underline{0})$ is of the form $\underline{t} = f^r(\underline{0})$ with $r \geqslant r_0$ and so,

$$f^{r-r_0}(\underline{0}) = \underbrace{0\cdots0}_{k_n}\cdots\,.$$

Hence, for the first r_0 iterations we apply the same maps either starting from $\underline{0}$ or from $f^{r-r_0}(\underline{0})$. This implies that, for every $y \in I$,

$$\pi_2\big[F^{r_0}(\underline{0}, y)\big] = \pi_2\big[F^{r_0}\big(f^{r-r_0}(\underline{0}), y\big)\big]. \tag{12}$$

Define $J := \pi_2[M_{\underline{t}_0}] = \pi_2\big[F^{r_0}(I_{\underline{0}})\big]$; it follows that

$$\begin{aligned}\pi_2[M_{\underline{t}}] = \pi_2\Big[F^{r_0}\big(F^{r-r_0}(I_{\underline{0}})\big)\Big] &= \pi_2\Big[F^{r_0}\big(f^{r-r_0}(\underline{0}),\ \psi(r-r_0)(I)\big)\Big]\\ &\subset \pi_2\Big[F^{r_0}\big(f^{r-r_0}(\underline{0}), I\big)\Big] = \pi_2\big[F^{r_0}(I_{\underline{0}})\big] = J.\end{aligned}$$

By the previous argument concernig the points of the orbit of $\underline{0}$, the map $\varphi\big(n, |\chi_n(\underline{x})|\big)$ is injective on J. So it is sufficient to show that $\pi_2[M_{\underline{x}}] \subset J$. Since, by the hypothesis, $I_{\underline{0}} \subset M$, the minimality of M implies

$$M = \overline{\bigcup_{i=0}^{\infty} F^i(I_{\underline{0}})}$$

whence

$$M_{\underline{x}} \subset \overline{\bigcup\{M_{\underline{t}} : \underline{t} \in \operatorname{Orb}(\underline{0}) \cap U\}}. \tag{13}$$

Since $\pi_2[M_{\underline{t}}] \subset J$ for every $\underline{t} \in \operatorname{Orb}(\underline{0}) \cap U$, by (13) the same holds for the set $\pi_2[M_{\underline{x}}]$. ∎

REMARK. We conjecture that Theorem 2 is still valid for functions $F \in \mathcal{T}$.

Let us denote by σ_δ and τ_δ the following functions depending on the parameter $\delta \in (0,1)$:

$$\sigma_\delta(t) = (1-\delta)t, \quad \tau_\delta(t) = (1-\delta)t + \delta. \tag{14}$$

Now we introduce the subclass $\mathcal{T}_{01}$ of $\mathcal{T}_0$ consisting of those functions $F \in \mathcal{T}_0$ that satisfy the following additional conditions:

$\forall n \geqslant 1 \quad \exists j_n,\ 0 \leqslant j_n \leqslant 2^{m_{n+1}} - 2$, such that:

$$\varphi(2n-1, j_{2n-1})(t) = \sigma_{\delta_{2n-1}}, \quad \varphi(2n, j_{2n})(t) = \tau_{\delta_{2n}}; \tag{15}$$

$$\text{if } j_n > 0 \text{ then, for all } r \geqslant 1, \quad \varphi^r(n, j_n - 1) \circ \cdots \circ \varphi^r(n, 0) = Id. \tag{16}$$

(17)

$$\left\{(1-\delta_{2n+1})^{2^{k_{2n}-(2n)}}\right\}, \quad \left\{(1-\delta_{2n})^{2^{k_{2n-1}-(2n-1)}}\right\} \text{ are sequences dense in } [0,1].$$

(Of course, by (3), we must have also $\lim_{n\to\infty} \delta_n = 0$).

REMARK. Note that given the sequence $\{k_n\}$, it is always possible to construct a (decreasing) sequence $\{\delta_n\}$ converging to 0 and satisfying (17).

Now we prove the following

THEOREM 3. *Every $F \in \mathcal{T}_{01}$ has a minimal set $M \supset I_{\underline{0}}$.*

PROOF: Take a point $(\underline{0}, y_0) \in I_{\underline{0}}$. By (9) and (14)–(17) we have

$$\begin{aligned} y_{2(j_{2n+1}+1)t_{2n}} &= \varphi^{2^{k_{2n}-2n}}(2n+1, j_{2n+1}) \circ \cdots \circ \varphi^{2^{k_{2n}-2n}}(2n+1, 0)(y_0) \\ &= \varphi^{2^{k_{2n}-2n}}(2n+1, j_{2n+1})(y_0) = y_0(1-\delta_{2n+1})^{2^{k_{2n}-2n}} \end{aligned} \tag{18}$$

and similarly

$$y_{2(j_{2n}+1)t_{2n-1}} = \varphi^{2^{k_{2n-1}-(2n-1)}}(2n, j_{2n})(y_0) = 1 + (y_0 - 1)(1-\delta_{2n})^{2^{k_{2n-1}-(2n-1)}}. \tag{19}$$

By the hypotheses on the sequence $\{\delta_n\}$, we have

$$\omega_F(\underline{0}, y_0) \supset I_{\underline{0}}. \tag{20}$$

Set $M = \omega_F(\underline{0}, 0)$ and let $w = (\underline{u}, v) \in M$. Since $F^i(w)$ visits any neighbourhood of $I_{\underline{0}}$, $\omega_F(w)$ contains a point from $I_{\underline{0}}$ and consequently, by (20), $(\underline{0}, 0) \in \omega_F(w)$. This implies $\omega_F(w) \supset M$, that is, M is a minimal set for F, containing $I_{\underline{0}}$. ∎

4. Distributional Chaos

We start this section by defining the notion of distributional chaos.

Let g be a map from a metric space (S, d) into itself. For any pair (x, y) of points of S and any positive integer n, we define a distribution function $\Phi^n_{xy} : \mathbb{R} \to [0,1]$ by

$$\Phi^n_{xy}(t) = \frac{1}{n}\#\{i : 0 \leqslant i < n \text{ and } d(g^i(x), g^i(y)) < t\}.$$

Obviously Φ^n_{xy} is a left-continuous non-decreasing function, $\Phi^n_{xy}(0) = 0$ and $\Phi^n_{xy}(t) = 1$ for all t greater than the maximum of the numbers $d(g^i(x), g^i(y))$, $0 \leqslant i \leqslant n-1$. Note that for the definition of each Φ^n_{xy} we need only to know the first n iterates of g.

Having the whole sequence $\{\Phi^n_{xy}(t)\}_{n \geqslant 1}$ we set

$$\Phi_{xy}(t) = \liminf_{n\to\infty} \Phi^n_{xy}(t), \quad \Phi^*_{xy}(t) = \limsup_{n\to\infty} \Phi^n_{xy}(t).$$

We shall refer to Φ_{xy} as the *lower* and Φ^*_{xy} as the *upper functions of* x and y.

If there is a pair (x, y) of points of S such that $\Phi_{uv}(t) < \Phi^*_{uv}(t)$ for all t in some non degenerate interval, then we say that g is distributionally chaotic (see [**9, 10**]).

The main result of this section is the following.

Theorem 4. *For every ε, $0 < \varepsilon < 1$, there exists a function $F_\varepsilon \in \mathcal{T}_{01}$ such that for $u = (\underline{0}, 0)$ and $v = (\underline{0}, 1)$,*

$$\Phi^*_{uv}(t) = 1,\ 0 < t < 1 \quad \textit{and} \quad \Phi_{uv}(t) \leqslant \varepsilon,\ 0 < t \leqslant 1 - \varepsilon.$$

Proof: Fix $\varepsilon \in (0,1)$. We construct the function F_ε by choosing $j_n = 0$ for all n and the functions $\varphi(n, j) \in \Gamma_n$, depending on integer parameters a_n, b_n and m_n, as follows:

$$\varphi(2n-1, j) = \begin{cases} \sigma_{\delta_{2n-1}}, & 0 \leqslant j < a_{2n-1} \\ Id, & a_{2n-1} \leqslant j < b_{2n-1} \\ \sigma^*_{\delta_{2n-1}}, & b_{2n-1} \leqslant j < a_{2n-1} + b_{2n-1} \\ Id, & a_{2n-1} + b_{2n-1} \leqslant j < 2^{m_{2n-1}} \end{cases}$$

$$\varphi(2n, j) = \begin{cases} \tau_{\delta_{2n}}, & 0 \leqslant j < a_{2n} \\ Id, & a_{2n} \leqslant j < b_{2n} \\ \tau^*_{\delta_{2n}}, & b_{2n} \leqslant j < a_{2n} + b_{2n} \\ Id, & a_{2n} + b_{2n} \leqslant j < 2^{m_{2n}} \end{cases}$$

where $\sigma_{\delta_{2n-1}}$ and $\tau_{\delta_{2n}}$ are the functions defined in (14), $\sigma^*_{\delta_{2n-1}}$ and $\tau^*_{\delta_{2n}}$ are their left-inverses given by

$$\sigma^*_{\delta_{2n-1}}(t) = \min\{1, t/(1-\delta_{2n-1})\}, \quad \tau^*_{\delta_{2n}}(t) = \max\{0, (t-\delta_{2n})/(1-\delta_{2n})\} \quad t \in [0,1],$$

and $\{\delta_n\}$ is a sequence satifying (17) with $\lim_{n\to\infty} \delta_n = 0$.

In this way we get a function $F_\varepsilon \in \mathcal{T}_{01}$. Thus what remains to be chosen are the parameters a_n, b_n and m_n.

Before starting with the choice of these parameters we need some properties of F_ε. Let $J_i := \pi_2\big[F_\varepsilon^i(I_{\underline{0}})\big]$ and $\lambda_i := |J_i| = \big|\psi(i)(I)\big|$.

By applying the functions σ_δ or τ_δ to an interval $J \subset I$, we get

$$\big|\sigma_\delta(J)\big| < |J|, \quad \big|\tau_\delta(J)\big| < |J|.$$

Moreover, for every $j \leqslant 2^{m_n} - 1$ and $s \leqslant r$,

$$\big|\varphi^s(n,j) \circ \varphi^r(n,j-1) \circ \cdots \circ \varphi^r(n,0)(J)\big| \leqslant |J|$$

since first we apply the functions σ_δ or τ_δ and then their left-inverses for a smaller number of times. Thus, by (7) it is easy to prove by induction that for any interval $J \subset I$

$$\big|\psi(i)(J)\big| \leqslant |J|. \tag{21}$$

For any j with $a_{n+1} \leqslant j \leqslant b_{n+1} - 1$ we have $\varphi(n+1, j) = Id$; thus, by (9) we obtain $\psi(2jt_n) = \psi(2a_{n+1}t_n)$. Then, if we set

$$\overline{J} := J_{2a_{n+1}t_n} = \varphi^{2^{k_n - n}}(n+1, a_{n+1} - 1) \circ \cdots \circ \varphi^{2^{k_n - n}}(n+1, 0)(I),$$

by the structure of the family Γ_{n+1} we have

$$\overline{J} = \sigma_{\delta_{n+1}}^{a_{n+1}2^{k_n - n}}(I) \quad \text{or} \quad \overline{J} = \tau_{\delta_{n+1}}^{a_{n+1}2^{k_n - n}}(I) \tag{22}$$

according to whether n is even or odd. By (4), (5), (8) and (21) we have

$$\begin{aligned}
\lambda_i = \big|\psi(i)(I)\big| &= \big|\psi(r) \circ \psi(2jt_n)(I)\big| = \big|\psi(r) \circ \psi(2a_{n+1}t_n)(I)\big| \\
&= \big|\psi(r)\big(\overline{J}\big)\big| \leqslant \big|\overline{J}\big|, \quad i = 2jt_n + r,\ 0 \leqslant r < t_n \\
\lambda_i = \big|\psi(i)(I)\big| &= \big|\psi_j^*(r) \circ \psi(2jt_n)(I)\big| = \big|\psi_j^*(r) \circ \psi(2a_{n+1}t_n)(I)\big| \\
&= \big|\psi_j^*(r)\big(\overline{J}\big)\big| = \big|\psi\big(\theta(r)\big)\big(\overline{J}\big)\big| \leqslant \big|\overline{J}\big|, \quad i = (2j+1)t_n + r,\ 0 \leqslant r < t_n.
\end{aligned}$$

This implies that, if $\lambda_{2a_{n+1}t_n} = \big|\overline{J}\big| < 1/(n+1)$, then

$$\lambda_i \leqslant \lambda_{2a_{n+1}t_n} < \frac{1}{n+1}, \quad 2a_{n+1}t_n \leqslant i < 2b_{n+1}t_n. \tag{23}$$

Now we have to choose the parameters a_n, b_n and m_n. The choice will be made iteratively in order to assure that, for $n \geqslant 1$,

$$\Phi_{uv}^{2b_n t_{n-1}}(1/n) \geqslant 1 - 1/n \quad \text{and} \quad \Phi_{uv}^{t_n}(1-\varepsilon) \leqslant \varepsilon. \tag{24}$$

The relation $\Phi_{uv}^{2b_n t_{n-1}}(1/n) \geqslant 1 - 1/n$ means that the number of i's less than $2b_n t_{n-1}$ for which $\lambda_i < 1/n$ is "almost the same" as $2b_n t_{n-1}$ while $\Phi_{uv}^{t_n}(1-\varepsilon) \leqslant \varepsilon$ means that the number of i's less than t_n for which $\lambda_i < 1-\varepsilon$ is "small" in respect to t_n.

Let $n = 1$. Take $a_1 = b_1 = 1 < 2^{m_1}$. Then we have $\lambda_0 = 1$, $\lambda_1 = 1 - \delta_1$ and $\lambda_i = 1$ for all $2 \leqslant i \leqslant 2^{m_1} = t_1$. So, the first inequality of (24) is trivially satisfied and

$$\Phi_{uv}^{t_1}(1-\varepsilon) = \frac{1}{2^{m_1}} \#\{i : 0 \leqslant i < 2^{m_1} \text{ and } \lambda_i < 1 - \varepsilon\} \leqslant \frac{1}{2^{m_1}}.$$

If we choose m_1 such that $1/2^{m_1} \leqslant \varepsilon$, then the second inequality of (24) is satisfied.

Assuming we have determined a_r, b_r and m_r for all $r \leqslant n$, now we choose the parameters a_{n+1}, b_{n+1} and m_{n+1}. By (22),

$$\lambda_{2a_{n+1}t_n} = (1 - \delta_{n+1})^{a_{n+1}2^{k_n - n}},$$

so we take a_{n+1} so that $\lambda_{2a_{n+1}t_n} < 1/(n+1)$. Now, by (23),

$$\#\{i : 0 \leqslant i < 2b_{n+1}t_n \text{ and } \lambda_i < 1/(n+1)\} \geqslant 2(b_{n+1} - a_{n+1})t_n$$

and so we can take b_{n+1} so that

$$\Phi_{uv}^{2b_{n+1}t_n}\left(\frac{1}{n+1}\right) \geqslant \frac{2(b_{n+1} - a_{n+1})t_n}{2b_{n+1}t_n} \geqslant 1 - \frac{1}{n+1},$$

that is, the first inequality of (24) is satisfied for $n+1$. Assume m_{n+1} has been chosen with $a_{n+1} + b_{n+1} < 2^{m_{n+1}}$ and take $a_{n+1} + b_{n+1} \leqslant j < 2^{m_{n+1}}$. Then $\varphi(n+1, j) = Id$. If $i = 2jt_n + r$ with $0 \leqslant r < t_n$ then, by (4), (9) and the structure of Γ_{n+1} we have $\psi(i) = \psi(r) \circ \psi(2jt_n) = \psi(r)$ and so $\lambda_i = \lambda_r$. The second inequality in (24) implies

$$\#\{i : i = 2jt_n + r \text{ and } 0 \leqslant r < t_n : \lambda_i < 1 - \varepsilon\} = \#\{r : 0 \leqslant r < t_n \text{ and } \lambda_r < 1 - \varepsilon\} \leqslant \varepsilon t_n. \tag{25}$$

Let now $i = (2j+1)t_n + r$ with $0 \leqslant r < t_n$. Again, by (5), (8) (9) and the structure of Γ_{n+1} we obtain

$$\psi(i) = \psi_j^*(r) \circ \psi(2jt_n) = \psi_j^*(r) = \psi(\theta(r))$$

and so $\lambda_i = \lambda_{\theta(r)}$. Note that $\theta(r)$ may assume all values from 0 to $t_{\nu(r)} - 1$ where $0 \leqslant \nu(r) \leqslant n - 1$. (See the notation of Lemma 1.)

Now we intend counting the number of indices r with $0 \leqslant r < t_n$ for which $\nu(r) = l$. This means counting the indices having in their binary representation 0 at the place k_l, 1 in the places k_p, $l < p < n$ and 0 in all places greater or equal to k_n. Thus the required number is $t_n/2^{n-1}$. These indices can be collected in $t_n/t_l 2^{n-l}$ blocks of type l containing the numbers having in their binary representation the same digits in the places greater or equal to k_l, that is, in blocks of indices of the form

$$r = (2\alpha_{n-1} + 1)t_{n-1} + \cdots + (2\alpha_{l+1} + 1)t_{l+1} + 2\alpha_l t_l + s \quad \text{with} \quad 0 \leqslant s < t_l$$

with the same a_q, $l \leqslant q \leqslant n-1$.

For each block B of type l and any index $i = (2j+1)t_n + r$, $r \in B$, by (7) and the structure of Γ_{n+1}, we have $\lambda_i = |\psi(i)(I)| = |\psi(s)(I)| = \lambda_s$ and, by the second inequality in (24), we get

$$\#\{i : i \in B \text{ and } \lambda_i < 1 - \varepsilon\} = \#\{s : 0 \leqslant s < t_l \text{ and } \lambda_s < 1 - \varepsilon\} \leqslant \varepsilon t_l.$$

Since the number of blocks of type l is $t_n/t_l 2^{n-l}$ and $0 \leqslant l \leqslant n-1$ we obtain

$$\#\{i : i = (2j+1)t_n + r,\ 0 \leqslant r < t_n \text{ and } \lambda_i < 1 - \varepsilon\} \leqslant \sum_{l=0}^{n-1} \frac{t_n}{t_l 2^{n-l}} \varepsilon t_l = \varepsilon t_n \sum_{l=0}^{n-1} \frac{1}{2^{n-l}}.$$

From (25) and this inequality we get

$$\#\{i : i = 2jt_n + r,\ 0 \leqslant r < 2t_n \text{ and } \lambda_i < 1 - \varepsilon\} \leqslant \varepsilon t_n \sum_{p=0}^{n} \frac{1}{2^p} = 2t_n\left(1 - 2^{-(n+1)}\right)\varepsilon.$$

Since $t_{n+1} = 2^{m_{n+1}}(2t_n)$, we conclude that

$$\begin{aligned}
\Phi_{uv}^{t_{n+1}}(1-\varepsilon) &= \frac{1}{2^{m_{n+1}}(2t_n)} \#\{i : 0 \leqslant i < 2^{m_{n+1}}(2t_n) \text{ and } \lambda_i < 1 - \varepsilon\} \\
&\leqslant \frac{2(a_{n+1} + b_{n+1})t_n + 2\varepsilon t_n\left(1 - 2^{-(n+1)}\right)\left(2^{m_{n+1}} - (a_{n+1} + b_{n+1})\right)}{2^{m_{n+1}}(2t_n)} \\
&= \frac{(a_{n+1} + b_{n+1})\left(1 - \varepsilon\left(1 - 2^{-(n+1)}\right)\right)}{2^{m_{n+1}}} + \varepsilon\left(1 - 2^{-(n+1)}\right)
\end{aligned}$$

and we choose m_{n+1} so that $\Phi_{uv}^{t_{n+1}}(1-\varepsilon) \leqslant \varepsilon$. ∎

Summarising, by Theorems 2, 3 and 4 we have the following

COROLLARY 5. *For every ε, $0 < \varepsilon < 1$, there exists a function $F_\varepsilon \in \mathcal{T}_{01}$ satisfying the following properties:*

(i) *F_ε has a minimal set $M \subset Q \times I$ such that $F_\varepsilon|_M$ is a homeomorphism;*
(ii) *M contains points u and v such that*

$$\Phi_{uv}^*(t) = 1,\ 0 < t < 1 \quad \text{and} \quad \Phi_{uv}(t) \leqslant \varepsilon,\ 0 < t \leqslant 1 - \varepsilon.$$

Note that the behaviour described in Corollary 5 is impossible in dimension one. Indeed any $f \in \mathcal{C}$ with $h(f) = 0$ is not chaotic (in the sense of Li and Yorke) on any minimal set [**3**].

5. Other results

In this section of the paper we present some other results about the functions in the class $\mathcal{T}_{01}$. Moreover we define other subclasses of $\mathcal{T}$ and we prove some properties of them.

Theorem 6. *For every function F in $\mathcal{T}_{01}$ no point of the layer $I_{\underline{0}}$ is isochronically recurrent.*

Proof: We have to prove that for every $y_0 \in I$ there exists a neighbourhood $U = U_1 \times U_2$ of $(\underline{0}, y_0)$ such that for every positive integer ν, there is an integer r for which $F^{r\nu}(\underline{0}, y_0) \notin U$. The proof is analogous to that in [**4**].

We distinguish two cases: $y_0 < 1$ and $y_0 = 1$.

Let $y_0 < 1$. We choose $U = I \times U_2$ with $\sup U_2 < 1$, and take an integer p such that

$$1 - \frac{1}{p} > \sup U_2.$$

Consider the set

$$A := \Big\{ n : n \geqslant 1 \text{ and } \pi_2\big[F^{2(j_{n+1}+1)t_n}(\underline{0}, 0)\big] > 1 - \frac{1}{p} \Big\}.$$

By (19), the set A contains infinitely many elements and by the monotonicity of the functions $\varphi(n, .)$, for every $t \in I$ we have

$$\pi_2\Big[F^{2(j_{n+1}+1)t_n}(\underline{0}, t)\Big] \geqslant \pi_2\Big[F^{2(j_{n+1}+1)t_n}(\underline{0}, 0)\Big] > 1 - \frac{1}{p}$$

for any $n \in A$.

Take an integer ν and let $2^q + \sum_{i \geqslant q+1} c_i 2^i$, $c_i \in \{0, 1\}$, $q \geqslant 0$, be its binary representation. Fix any $n \in A$; since

$$2^{k_n - 2^{m_{n+1}} q} \nu 2^{m_{n+1}} = \overbrace{\underbrace{0 \cdots 0}_{k_n} 1 \underbrace{0 \cdots 0}_{m_{n+1}}}^{k_{n+1}} \cdots,$$

for $r = 2(j_{n+1} + 1) t_n 2^{k_n - 2^{m_{n+1}} q} \nu 2^{m_{n+1}} - 1$ we have

$$r\nu = \underbrace{0 \cdots 0}_{k_n} \xi(1) \cdots \xi(m_{n+1}) 0 \cdots,$$

with $j_{n+1} + 1 = \Big|\big(\xi(1), \ldots, \xi(m_{n+1})\big)\Big|$. Since $\mu = r\nu - 2(j_{n+1} + 1)t_n = \underbrace{0 \cdots 0}_{k_{n+1}} \cdots$, we have $f^{\mu}(\underline{0}) = \underbrace{0 \cdots 0}_{k_{n+1}} \cdots$ and $F^{r\nu}(\underline{0}, y_0) = F^{2(j_{n+1}+1)t_n}(f^{\mu}(\underline{0}), y_{\mu})$.

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Since $\pi_2\left[F^{2(j_{n+1}+1)t_n}(f^\mu(\underline{0}),t)\right] = \pi_2\left[F^{2(j_{n+1}+1)t_n}(\underline{0},t)\right]$ for every $t \in I$, by the definition of A we get

$$\pi_2\left[F^{r\nu}(\underline{0},y_0)\right] = \pi_2\left[F^{2(j_{n+1}+1)t_n}(f^\mu(\underline{0}),y_\mu)\right] = \pi_2\left[F^{2(j_{n+1}+1)t_n}(\underline{0},y_\mu)\right]$$
$$\geqslant \pi_2\left[F^{2(j_{n+1}+1)t_n}(\underline{0},0)\right] > 1 - \frac{1}{p}.$$

In the case $y_0 = 1$ we proceed in a similar way starting from a neighbourhood $U = I \times U_2$ with $\inf(U_2) > 0$ and using formula (18). ∎

REMARK. The previous result shows that for each function in $\mathcal{T}_{01}$ the point $\underline{0}$ is isochronically recurrent for the base map while it is not the projection of any isochronically recurrent point of the triangular map F.

Hence, from Theorems 2, 4 and 6 we get the following

COROLLARY 7. *For every ε, $0 < \varepsilon < 1$, there exists a function $F_\varepsilon \in \mathcal{T}_{01}$ such that:*

(i) *F_ε has a minimal set $M \supset I_{\underline{0}}$;*
(ii) *$F_\varepsilon|_M$ is a homeomorphism;*
(iii) *no point of $I_{\underline{0}}$ is isochronically recurrent;*
(iv) *$\Phi^*_{uv}(t) = 1$ for $0 < t < 1$ and $\Phi_{uv}(t) \leqslant \varepsilon$ for $0 < t \leqslant 1 - \varepsilon$, where $u = (\underline{0}, 0)$ and $v = (\underline{0}, 1)$.*

Now we define another subclass $\mathcal{T}_{02}$ of $\mathcal{T}_0$ as follows: for every $n \geqslant 1$

$$\varphi(n,0) = \varphi(n,1) = Id,$$
$$\varphi^r(n, 2^p - 1) \circ \cdots \circ \varphi^r(n,0) = Id, \ 1 \leqslant p \leqslant m_n, \ \text{for all } r \geqslant 1.$$

THEOREM 8. *For every $F \in \mathcal{T}_{02}$ we have*

$$\lim_{s \to \infty} F^{2^s}(\underline{0}, y_0) = (\underline{0}, y_0)$$

for every $y_0 \in I$.

PROOF: Formulas (6) and (9) and the definition of $\mathcal{T}_{02}$ imply $y_{2^s} = y_0$ for every integer s. ∎

COROLLARY 9. *There exists $F \in \mathcal{T}_{02}$ such that:*

(i) *for every $y_0 \in I$, $\lim_{s \to \infty} F^{2^s}(\underline{0}, y_0) = (\underline{0}, y_0)$;*
(ii) *F has a minimal set $M \supset I_{\underline{0}}$;*
(iii) *$F|_M$ is a homeomorphism;*
(iv) *no point of $I_{\underline{0}}$ is isochronically recurrent.*

PROOF: By Theorems 2 and 6, the only thing to be proved is that $\mathcal{T}_{01} \cap \mathcal{T}_{02} \neq \emptyset$. It is enough to take in the definition of $\mathcal{T}_{01}$, $j_n = 2$ for all n, $\varphi(n,0) = \varphi(n,1) = Id$ and choose $\varphi(n, j_n + 1) = \varphi(n,3)$ as the left inverse of $\varphi(n,2)$. All other functions can be chosen equal to the identity. ∎

REMARK. A one-dimensional map f has zero topological entropy if and only if the set $\{x \in I : \lim_{s\to\infty} f^{2^s}(x) = x\}$ coincides with the set of the isochronically recurrent points [**7**, Table 1]. We recall (see Section 2) that our maps have zero topological entropy and so properties (i) and (iv) of Corollary 9 show a completely different behaviour with respect to the one-dimensional case.

To present the last results of the paper we introduce another subclass $\mathcal{T}_1$ of $\mathcal{T}$. Let $\{k_i\}_{i=1}^{\infty}$ be an increasing sequence if positive integers with $k_i - i \to +\infty$, and $\{\varphi_i\}_{i=1}^{\infty}$ a sequence of mappings from I into I of the form

$$\varphi_i(t) = t^{s_i}, \quad \text{with } s_i > 0,\ \lim_{i\to\infty} s_i = 1.$$

As in the definition of the class $\mathcal{T}_0$, the digits $x(k_1), x(k_2), \ldots$ are called control digits of $\underline{x} \in Q$. We define a function $f : Q \times I \to Q \times I$ as follows:

If the first zero control digit of $\underline{x}$ is $x(k_n)$,

$$F(\underline{x}, y) = \big(f(\underline{x}), \varphi_n(y)\big);$$

otherwise $F(\underline{x}, y) = \big(f(\underline{x}), y\big)$. The condition $\lim_{i\to\infty} s_i = 1$ assures the continuity of F. Moreover, it is easy to recognise that F is a homeomorphism of $Q \times I$ onto itself.

THEOREM 10. *There exists a function $F \in \mathcal{T}_1$ with the following properties:*

(i) *for any $w \in \{\underline{0}\} \times (0,1)$ we have $\omega_F(w) = Q \times I$;*
(ii) *F has two minimal sets, namely $Q \times \{0\}$ and $Q \times \{1\}$;*
(iii) $\{\underline{0}\} \times (0,1) \subset \mathrm{Rec}\,(F) \backslash UR(F)$;
(iv) *for any $u \in \{\underline{0}\} \times (0,1)$ and $v = (\underline{0}, 0)$ or $v = (\underline{0}, 1)$,*

$$\Phi^*_{uv}(t) = 1, \quad \Phi_{uv}(t) = 0, \quad t \in (0,1); \tag{26}$$

hence F is distributionally chaotic.

PROOF: (i) Since the functions φ_i commute, the value $F^m(\underline{0}, y_0) = \big(f^m(\underline{0}), y_m\big)$ depends only on the number of times any function φ_i is applied.

Given a positive number r, take n so that $k_n \leqslant r < k_{n+1}$. Then the points $f^i(\underline{0})$, $0 \leqslant i < 2^r$ are represented by all the 2^r sequences

$$a_1, \cdots a_r 0 \cdots, \quad a_i \in \{0,1\},\ 1 \leqslant i \leqslant r$$

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which have the $(n+1)$-th control digit equal zero and so the only functions that may enter in the expression of y_i, $1 \leqslant i \leqslant 2^r$, are $\varphi_1, \dots, \varphi_{n+1}$. The number of times the function φ_i, $1 \leqslant i \leqslant n+1$, enters the expression of y_{2^r} equals the number of sequences $a_1 \cdots a_r 0 \cdots$ having $a_{k_i} = 0$ and $a_{k_s} = 1$ for all $1 \leqslant s < i$. This number is 2^{r-i} for $1 \leqslant i \leqslant n$, and 2^{r-n} for $i = n+1$. So, we have

$$y_{2^{k_n}} = \varphi_1^{2^{k_n-1}} \circ \varphi_2^{2^{k_n-2}} \circ \cdots \circ \varphi_{n-1}^{2^{k_n-(n-1)}} \circ \varphi_n^{2^{k_n-n}} \circ \varphi_{n+1}^{k_n-n}(y_0).$$

Since

$$f^{2^{k_n}}(\underline{0}) = \underbrace{0 \cdots 0}_{k_n} 10 \cdots ,$$

for the next 2^{k_n} iterations we use exactly the same functions as starting from $\underline{0}$. We may proceed in this way until the k_{n+1} digit is zero. Thus, for all m with $2 \leqslant m \leqslant 2^{k_{n+1}-k_n-1}$ we get

$$y_{m2^{k_n}} = \varphi_1^{m2^{k_n-1}} \circ \varphi_2^{m2^{k_n-2}} \circ \cdots \circ \varphi_{n-1}^{m2^{k_n-(n-1)}} \circ \varphi_n^{m2^{k_n-n}} \circ \varphi_{n+1}^{m2^{k_n-n}}(y_0). \tag{27}$$

In order to construct the function F we start by imposing on the sequence $\{s_i\}$ the additional condition

$$s_{2i-1}^2 s_{2i} = 1, \quad i \geqslant 1. \tag{28}$$

This implies

$$\varphi_{2i-1}^2 \circ \varphi_{2i} = Id, \quad i \geqslant 1.$$

Hence, by (27) and (28) for all $n \geqslant 1$ we obtain

$$\begin{cases} y_{m2^{k_{2n}}} = \varphi_{2n+1}^{m2^{k_{2n}-2n}}(y_0) = y_0^{s_{2n+1}^{\left(m2^{k_{2n}-2n}\right)}}, & 1 \leqslant m \leqslant 2^{k_{2n+1}-k_{2n}-1} \\ y_{m2^{k_{2n-1}}} = \varphi_{2n}^{m2^{k_{2n-1}-2n}}(y_0) = y_0^{s_{2n}^{\left(m2^{k_{2n-1}-2n}\right)}}, & 1 \leqslant m \leqslant 2^{k_{2n}-k_{2n-1}-1}. \end{cases} \tag{29}$$

We want to show that it is possible to choose the sequence of parameters $\{s_n)$ in order to assure that

$$\forall y_0 \in (0,1), \quad \omega_F\big((\underline{0}, y_0)\big) \supset I_{\underline{0}}.$$

Since the ω-limit sets are strongly F-invariant, this implies $\omega_F\big((\underline{0}, y_0)\big) = Q \times I$.

To this aim it is enough to assure that the values given by (29) with $m = 1$ are dense in I and this is equivalent to requiring that

$$2^{k_{2n}-2n} \log(s_{2n+1}) \quad \text{is dense in } (-\infty, +\infty). \tag{30}$$

If $\{s_{2n+1}\}$ is a sequence satisfying (30) and

$$\lim_{n\to+\infty} s_{2n+1} = 1, \tag{31}$$

then the whole sequence $\{s_n\}$ constructed by using (28) satisfies the required property $\lim\limits_n s_n = 1$. To satisfy (30) and (31) we define the sequence $\{s_{2n+1}\}_{n=1}^{\infty}$ by

$$\log(s_{2n+1}) = \frac{\sigma_n}{2^{k_{2n}-2n}} \tag{32}$$

where $\{\sigma_n\}_{n=1}^{\infty}$ is a sequence dense in $(-\infty, +\infty)$ satisfying

$$\left|\frac{\sigma_n}{2^{k_{2n}-2n}}\right| < \frac{1}{n}. \tag{33}$$

So (i) is proved. In the following, given the sequence $\{\sigma_n\}$, we show how to construct the sequence $\{k_n\}$ in order to satisfy (33).

Property (ii) is obvious. By (i), every point $w \in \{\underline{0}\} \times (0,1)$ is recurrent and, by (ii), $\omega_F(w)$ is not a minimal set. So w is not uniformly recurrent (see [**1**]). Now we prove (iv). In order to assure (26) we take sequence $\{\sigma_n\}$, dense in $(-\infty, +\infty)$ and such that $\sigma_{2n-1} < 0$ and $\sigma_{2n} > 0$.

Now we recursively define the sequences $\{k_n\}$ and $\{s_n\}$.

We start the recursive process by taking k_1 arbitrarily, $s_1 = s_2 = 1$ and $k_2 > k_1$ satisfying (33). Assume now we have constructed k_i, s_i for $i \leqslant 2n$ so that (33) is satisfied. By (28) and (32) we immediately get s_{2n+1} and s_{2n+2}.

Suppose now n even [n odd]. We take $0 < \rho_n < 1/2n$ such that, for $y_0 \leqslant \rho_n$ [$y_0 \geqslant 1 - \rho_n$] and for all j with $0 \leqslant j < 2^{k_{2n}}$, we have

$$y_j < \frac{1}{2n} \quad \left[y_j > 1 - \frac{1}{2n}\right]. \tag{34}$$

This is possible since only a finite number of continuous functions enter in the expression of y_j and for them both points 0 and 1 are fixed.

Then we find an integer p so that, for $y_0 = 1 - (1/2n)$ $[y_0 = (1/2n)]$,

$$y_{p2^{k_{2n}}} = \varphi_{2n+1}^{p2^{(k_{2n}-2n)}}(y_0) < \rho_n \quad \left[y_{p2^{k_{2n}}} = \varphi_{2n+1}^{p2^{(k_{2n}-2n)}}(y_0) > 1 - \rho_n\right]. \tag{35}$$

Now, we choose k_{2n+1} so that

$$\frac{p2^{k_{2n}}}{2^{k_{2n+1}-1}} < \frac{1}{2n} \tag{36}$$

and $k_{2n+2} > k_{2n+1}$ satisfying (33).

Now we show that the function F constructed in this way satisfies (26). Take $y_0 \in (0,1)$ and n_0 such that $y_0 \in \left[(1/2n_0), 1-(1/2n_0)\right]$. Choose n even and greater than n_0. For every r, $p2^{k_{2n}} \leqslant r \leqslant 2^{k_{2n+1}-1}$ we can write $r = m2^{k_{2n}} + j$ with $p \leqslant m \leqslant 2^{k_{2n+1}-k_{2n}-1}$ and $0 \leqslant j < 2^{k_{2n}}$. So, by (29), (34) and (35)

$$y_{m2^{k_{2n}}} \leqslant y_{p2^{k_{2n}}} < \rho_n \quad \text{and} \quad y_r < \frac{1}{2n}.$$

Thus

$$\#\left\{i : 0 \leqslant i < 2^{k_{2n+1}-1} \text{ and } y_i < \frac{1}{2n}\right\} \geqslant 2^{k_{2n+1}-1} - p2^{k_{2n}},$$

and so, by (36)

$$\Phi_{uv}^{2^{k_{2n+1}-1}}\left(\frac{1}{2n}\right) \geqslant 1 - \frac{p2^{k_{2n}}}{2^{k_{2n+1}-1}} > 1 - \frac{1}{2n}.$$

Hence we conclude that $\Phi^*_{uv}(t) = 1$ for $t \in (0,1)$.

Similarly, if we take n odd, we get

$$\Phi_{uv}^{2^{k_{2n+1}-1}}\left(1 - \frac{1}{2n}\right) < \frac{1}{2n},$$

and so $\Phi_{uv}(t) = 0$ for $t \in (0,1)$. ∎

REMARK. Again the properties proved in Theorem 10 are impossible in $\mathcal{C}$. Indeed for a one-dimensional map f with $h(f) = 0$ we have $\mathrm{Rec}\,(f) = UR(f)$ and each ω-limit set contains only one minimal set. The next theorem shows something more: the existence of a triangular map F with $h(F) = 0$ having an ω-limit set containing infinitely many minimal sets.

THEOREM 11. *There exists a triangular map F of type 2^∞, strictly increasing on any layer I_x, having an ω-limit set containing uncountably many minimal sets.*

PROOF: By [**2**, Theorems 6.2, 6.5] there exists a function $f \in \mathcal{C}$ of type 2^∞ having an infinite ω-limit set $\widetilde{Q} \supset Q$ containing isolated points and such that $\widetilde{Q}\backslash Q$ is a single orbit disjoint from Q. Moreover, this function acts as the adding machine on Q and for every $x \in \widetilde{Q}\backslash Q$ we have $\omega_f(x) = Q$. We take such a function as base of the triangular map we are constructing. We choose $p_0 \in \widetilde{Q}\backslash Q$ with $\mathrm{Orb}\,(p_0) = \widetilde{Q}\backslash Q$ and associate to it the zero sequence. Then we code $\mathrm{Orb}\,(p_0)$ by associating to each point $p_n = f^n(p_0)$ the corresponding sequence $f^n(\underline{0})$. Now we define $\widetilde{F}(x,y) = \big(f(x), g_x(y)\big)$ on $\widetilde{Q} \times I$ as follows: for $x \in Q$, $g_x = Id$ and for $x \in \widetilde{Q}\backslash Q$, g_x as in the construction of the class $\mathcal{T}_1$ on the corresponding points of $\mathrm{Orb}\,(\underline{0})$. Arguing as in the proof of Theorem 9 we get $\omega_{\widetilde{F}}(z) = Q \times I$ for any $z \in \left(\widetilde{Q}\backslash Q\right) \times (0.1)$. Clearly, any set $Q \times \{a\}$ is a minimal set for $\widetilde{F}$ contained in $\omega_{\widetilde{F}}(z)$. It is easy to see that it is possible to extend $\widetilde{F}$ continuously to a triangular map $F : I^2 \to I^2$ increasing on any layer. ∎

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