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On sequences of lattice packings

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In this note we establish theorems on compactness of lattice packings.

I. Introduction

1. Let $a^1, \ldots, a^d$ be a linearly independent set of vectors in d-dimensional euclidean space E^d. A set Λ is called a d-dimensional lattice with basis $a^1, \ldots, a^d$ if its elements are all vectors of the form $\lambda_1 a^1 + \ldots + \lambda_d a^d$ where the λ_i are integers. The determinant of Λ, $d(\Lambda)$, is defined by

$$d(\Lambda) = \left|\det\left(a^1, \ldots, a^d\right)\right| .$$

This definition is, in fact, independent of the basis taken for Λ.

2. An infinite sequence of lattices $\{\lambda_n\}$ is said to converge to a lattice Λ if each Λ_n has a basis $a_n^1, \ldots, a_n^d$ and Λ has a basis $a^1, \ldots, a^d$ such that $\lim_{n\to\infty} a_n^j = a^j$ $(1 \le j \le d)$. With this definition of convergence, $d(\Lambda)$ becomes a continuous function, for $\Lambda_n \to \Lambda$ implies $d(\Lambda_n) \to d(\Lambda)$. Mahler's selection (compactness) theorem for lattices states (see Mahler [4]):

Let $\{\lambda_n\}$ be an infinite sequence of lattices satisfying the following two conditions: there are constants K_1 and K_2 such that

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(i) $d(\Lambda_n) \le K_1$ for all lattices Λ_n ,

(ii) $\|a\| \ge K_2 > 0$ for every $a \ne 0 \in \Lambda_n$.

Then $\{\Lambda_n\}$ contains a convergent subsequence.

In other words it is possible to select from a *bounded* sequence of lattices a subsequence which tends to a lattice Λ . (A sequence of lattices is said to be bounded if it satisfies (i) and (ii).)

3. Let S be a point-set in E^d . A lattice Λ is called *S-admissible if no point of* Λ *except possibly* 0 *is an inner point of* S . The *critical* determinant $\Delta(S)$ of S is defined as the infimum of $d(\Lambda)$ taken over all S-admissible lattices Λ . A *critical lattice* for S is one which is S-admissible and for which $d(\Lambda) = \Delta(S)$.

For the existence of a critical lattice of a set Mahler [5] has the following theorem.

Let S *be a point set in* E^d *and let* $0 < \Delta(S) < \infty$. *Then* S *possesses a critical lattice if and only if there exists a bounded infinite sequence of S-admissible lattices* $\{\Lambda_n\}$ *such that*

$$\lim_{n\to\infty} d(\Lambda_n) = \Delta(S) .$$

4. Let S be a set and $P = \{P_n\}$ a sequence of points in E^d . If the sets $\{S+P_n\}$ do not overlap (that is, interiors do not meet) then P_n is said to provide a *packing* for S . When $\{P_n\}$ is a lattice Λ , then we have an (S, Λ) *lattice packing.* For a central symmetric convex set S the following relationship holds: Λ is a lattice packing for S if and only if Λ is admissible for $2S$.

We say that $\{S+P_n\}$ *cover* the whole space if each point of space lies in at least one of the sets of $\{S+P_n\}$.

5. A distance function on bounded subsets of E^d is defined by $D(S_1, S_2) = \inf\{\varepsilon \ge 0 : S_1 \subset N(S_2, \varepsilon), S_2 \subset N(S_1, \varepsilon)\}$, where $N(S, \varepsilon)$ is the ε-neighbourhood of S . This distance function defines a metric on the

closed bounded subsets of E^d . For this metric, Blaschke has the following convergence theorem (Blaschke [1]).

Let $\{S_n\}$ *be an infinite sequence of compact convex subsets of* E^d *which are bounded; that is, contained in some solid sphere. Then* $\{S_n\}$ *contains a subsequence which converges to a compact convex set.*

II. Sequence of packings

Let

$L = \{(S, \Lambda_S) : S$ is central symmetric convex set and Λ_S is a packing lattice of $S\}$.

Now L can be viewed as a subset of the cartesian product $A \times B$ where $A = \{S\}$ and $B = \{\Lambda_S\}$, and so to have the subspace topology derived from the product (metric) topology on $A \times B$. Thus (S_n, Λ_{S_n}) tends to, say, (S_0, Λ_{S_0}) if and only if S_n tends to S_0 and Λ_{S_n} to Λ_{S_0} in the sense defined in 2 and 5 of the introduction. We ask *when is* L *compact*? If $L \equiv A \times B$, that is, every element of $A \times B$ is a lattice packing, then by Tychonoff's product theorem L is compact if and only if A and B are compact. But this is a very special case. It occurs if every element of $\{\Lambda_S\}$ provides a packing for every element of $\{S\}$. In the theorem below the above question is answered for the case that L is a map; that is, there is a map $\phi : A \to B$ such that $(S, \Lambda_S) \in L$ if and only if $\Lambda_S = \phi(S)$.

THEOREM 1. *Let* A *and* B *be compact and* $\phi : A \to B$ *as described above. Then* L *is compact if and only if* ϕ *is continuous.*

Proof. Let ϕ be continuous and $\{(S_n, \Lambda_{S_n})\}$ be a sequence in L . Because A is compact $\{S_n\}$ has a convergent subsequence $\{S_{n_k}\}$ which converges to S , say. Since ϕ is continuous $\Lambda_{S_{n_k}} = \phi(S_{n_k})$ converges to $\Lambda_S = \phi(S)$.

Thus $\left\{\left(S_{n_k}, \Lambda_{S_{n_k}}\right)\right\}$ is a convergent subsequence of the original sequence. Therefore L is compact.

On the other hand let L be compact and $\{S_n\}$ be a sequence in A converging to S. We have to show that $\Lambda_{S_n} = \phi(S_n)$ tends to $\Lambda_S = \phi(S)$. Suppose, by way of contradiction, that there is a subsequence $\{S_{n_k}\}$ such that no $\Lambda_{S_{n_k}}$ is within a certain distance of Λ_S. Since L is compact, there is a subsequence $\left\{\left(S_{n_{k_l}}, \Lambda_{S_{n_{k_l}}}\right)\right\}$ of $\left\{\left(S_{n_k}, \Lambda_{S_{n_k}}\right)\right\}$ which converges. Now $S_{n_{k_l}}$ tends to S and since each $\Lambda_{S_{n_{k_l}}}$ does not lie within a certain distance of Λ_S, $\left\{\Lambda'_{S_{n_{k_l}}}\right\}$ tends to $\Lambda'_S \neq \Lambda_S$. However, this sequence tends to (S, Λ'_S) and $(S, \Lambda'_S) \in L$ which contradicts L being a map. Therefore ϕ is continuous.

An interesting illustration of the theorem occurs in discussion of Voronoi domains. In Groemer [2] it is shown that corresponding to a lattice Λ there is a unique Voronoi domain $V(\Lambda)$;

$$V(\Lambda) = \{x \in E^d : \|x\| \leq \|x-g\| \text{ for all } g \in \Lambda\} .$$

Let A be a compact set of Voronoi domains and B the corresponding compact set of lattices. Then $L = \{(V(\Lambda), \Lambda) : V(\Lambda) \in A\}$ is compact for Groemer has essentially shown $V(\Lambda) \to \Lambda$ is continuous.

REMARK 1. Theorem 1 remains true if we replace the sequence of lattice *packings* by sequence of lattice *coverings* or by the same token the theorem is true for a sequence of any lattice *distribution* of sets S in E^d.

REMARK 2. In Theorem 1 we showed that the compactness of $\{S\}$ and $\{\Lambda_S\}$ and the continuity of $\phi : \{S\} \to \{\Lambda_S\}$ implies that $\{(S, \Lambda_S)\}$ is compact. Similarly we can show that the compactness of $\{S\}$ and

$\{(S, \Lambda_S)\}$ implies the compactness of $\{\Lambda_S\}$, or the compactness of $\{\Lambda_S\}$ and $\{S, \Lambda_S\}$ implies the compactness of $\{S\}$.

REMARK 3. By definition, the density function for a packing (S, Λ) is

$$\lim_{\rho(B)\to\infty} \frac{\tau(B,S,\Lambda,z)}{V(B)} = \frac{V(S)}{d(\Lambda)} ,$$

where $V(S)$ is the volume of S , B is an arbitrary bounded convex body, $\rho(B)$ is the radius of largest sphere contained in B , z is an arbitrary point, and $\tau(B, S, \Lambda, z)$ is the total volume of the bodies $S + x$ with $x \in \Lambda + z$ and $S + x \subset B$. (For particulars see Lekkerkerker [3, page 165].)

It can be seen that this density function is continuous in (S, Λ) ; so in a compact lattice packing there is a packing which has maximum density. In particular let Λ_0 be a packing lattice for a given convex set S . Put $\lambda = \{\Lambda : (S, \Lambda)$ is a packing and such that $d(\Lambda) \leq d(\Lambda_0)\}$. Then $L = \{(S, \Lambda) : \Lambda \in \lambda\}$ is a compact set of packings, which assumes a densest packing for S .

III. Sequences of admissible lattices

Let Λ be S-admissible. Then (S, Λ) will be called a *pair*. If Λ is also critical for S then (S, Λ) is called a *critical pair*. A sequence (S_n, Λ_n) is called *bounded* if $\{S_n\}$ and $\{\Lambda_n\}$ are bounded.

THEOREM 2. *Let* $\{(S_n, \Lambda_n)\}$ *be a bounded sequence of pairs where the* S_n *are closed convex. Then* $\{(S_n, \Lambda_n)\}$ *has a convergent subsequence such that its limit* $\{S, \Lambda\}$ *is a pair. If the* (S_n, Λ_n) *are also critical pairs, then the limit of this subsequence is a critical pair.*

Proof. (i) Since $\{S_n\}$ and $\{\Lambda_n\}$ are bounded we may then select a subsequence of pairs $\{(S_{n_k}, \Lambda_{n_k})\}$, a convex set S and a lattice Λ such that $S_{n_k} \to S$, $\Lambda_{n_k} \to \Lambda$. We show Λ is S-admissible. Suppose not. Then there exists $x \in \text{int}\, S$, $x \neq 0$, such that $x \in \Lambda$. Since the S_{n_k} tend to S and since the S_{n_k} and S are convex, there exist a number N

and a neighbourhood U of x such that

(1) $$n_k > N \text{ implies } x \in U \subset \operatorname{int} S_{n_k} .$$

Also we have

(2) $$\lim \Lambda_{n_k} = \Lambda .$$

From (1) and (2) we obtain a contradiction to Λ_{n_k} being S_{n_k}-admissible. Therefore Λ is S-admissible; that is, (S, Λ) is a pair.

(ii) Suppose further that the (S_n, Λ_n) are critical. Then $\Delta(S_n) = d(\Lambda_n)$. Thus

$$d(\Lambda) = \lim_{n_k \to \infty} d(\Lambda_{n_k}) = \lim_{n_k \to \infty} \Delta(S_{n_k})$$
$$= \Delta(S)$$

by Mahler's continuity result on critical lattices.

Therefore Λ is critical for S .

COROLLARY. *Let* $\{S_n\}$ *be a bounded sequence of central symmetric convex sets and* $\{\Lambda_n\}$ *a bounded sequence of lattices. If* $\{(S_n, \Lambda_n)\}$ *is a sequence of pairs then the sequence of lattice packings* $\{(\frac{1}{2}S_n, \Lambda_n)\}$ *is compact.*

REMARK. In Mahler [5] it has essentially been shown that the sequence of pairs $\{S, \Lambda_n\}$ is compact where S is a point set such that $0 < \Delta(S) < \infty$ and $\{\Lambda_n\}$ is bounded.

This is a special case of the sequence of pairs $\{S_n, \Lambda_n\}$ for we can simply put $S_n = S$ for all n . But in Theorem 2, S_n cannot be extended to non-convex sets, for example to star shaped sets. In the theorem we have used Blaschke's compactness theorem for convex sets and the continuity of the Δ function. To extend the theorem to star-shaped bodies we would want a compactness theorem corresponding to Blaschke's while retaining the continuity of the Δ function. Now Mahler [4] has shown that for Δ to

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be continuous for star bodies, a topology κ must be used. This topology κ is *strictly* finer than the usual Hausdorff-Blaschke topology for closed sets, say τ. It is known that τ is a compact topology. The following result from topology shows that κ cannot be compact.

If τ is a coarser topology than κ on a set X, and if τ is Hausdorff and κ compact, then $\tau = \kappa$ (see, for instance, Rudin [7, p. 61]).

On the other hand, Theorem 1 can be extended to non-convex sets S_n provided $\{S_n\}$ is compact. For instance, Melzak [6] has obtained a compactness theorem for a certain class of star-shaped bodies in E^3. For this family Theorem 1 is valid.

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