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ELEMENTS OF ORDER COXETER NUMBER +1 IN CHEVALLEY GROUPS

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1. Introduction. Following the notation and the definitions in [**1**], let $L(K)$ be the Chevalley group of type L over a field K, W the Weyl group of L and h the Coxeter number, i.e., the order of Coxeter elements of W. In a letter to the author, John McKay asked the followıng question: If $h + 1$ is a prime, is there an element of order $h + 1$ in $L(C)$? In this note we give an affirmative answer to this question by constructing an element of order $h + 1$ (prime or otherwise) in the subgroup $L_{\mathbf{Z}} = \langle x_r(1) | r \in \Phi \rangle$ of $L(K)$, for any K.

Our problem has an immediate solution when $L = A_n$. In this case $h = n + 1$ and the $(n + 1) \times (n + 1)$ matrix

$$M = \begin{bmatrix} 1 & 1 & 1 & . & . & . & 1 & 1 \\ -1 & 0 & 0 & . & . & . & 0 & 0 \\ 0 & -1 & 0 & . & . & . & 0 & 0 \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & -1 & 0 \end{bmatrix}$$

has order $2(h + 1)$ in $SL_{n+1}(K)$. This seemingly trivial solution turns out to be a prototype of general solutions in the following sense. Using the usual identification (see, for example, [**1**], p. 185), one may write

$$(1) \quad M = \phi_{\alpha_1}\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \phi_{\alpha_2}\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \cdots \phi_{\alpha_n}\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the fundamental roots of A_n (in the usual order). We shall see that if the α_i's in (1) are replaced by fundamental roots of any L (of rank n) then we again have $M^{2(h+1)} = I$ in $L(K)$. A rather amazing fact is that our proof is valid for all types but A_n (n even).

Let us state our theorem.

THEOREM. *Let* $\Pi = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ *be a fundamental system of roots of* L *and let*

$$(2) \quad M = \phi_{\alpha_1}\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \phi_{\alpha_2}\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \cdots \phi_{\alpha_n}\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then $M^{2(h+1)} = 1$, *where* h *is the Coxeter member of* L.

The proof will be given in Section 4.

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2. Orderings of the α_i's. We first note that the order of the α_i's appearing in (1) is inessential because of

LEMMA 2.1. *If M' is obtained from M by permuting $\alpha_1, \alpha_2, \ldots, \alpha_n$ in* (2), *then M and M' are conjugate in $L_{\mathbf{Z}}$.*

Proof. We recall the well known argument used in the proof (for example, [**1**], p. 157) of the conjugacy of the Coxeter elements. A virtually identical argument will yield the lemma.

We find it convenient (and essential in the proof when $L = A_n$, D_n, n odd or E_6) to choose a particular order of the α_i's in (2).

Let $\Pi = A \cup B$ be the partition of Π into two subsets each of which contains mutually orthogonal roots. Then, for any S, T and $r, s \in A$ or B, $r \neq s$, we have

(4) $\phi_r(S)\phi_s(T) = \phi_s(T)\phi_r(S)$.

The right element to deal with will be

(5) $$M = \prod_{a \in A} \phi_a\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \prod_{b \in B} \phi_b\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}.$$

3. Coxeter elements and the involution w_0. Let Φ^+ (or Φ^-) be the set of positive (or negative) roots of L. Let w_0 be the element of W such that $w_0(\Phi^+) = \Phi^-$, or equivalently, $l(w_0) = |\Phi^+|$, where l is the minimal length function. We recall that

3.1. (1) If $L = B_n$, C_n, D_n (n even), F_4, E_7, E_8 or G_2, then $w_0 = -I$, i.e.,

$$w_0(\alpha_i) = -\alpha_i$$

for all $\alpha_i \in \Pi$.

(2) If $L = A_n$, D_n (n odd) or E_6, then

$$w_0(\alpha_i) = -\rho(\alpha_i)$$

where ρ is the symmetry of the Dynkin diagram ([**1**], p. 200).

Let w be a Coxeter element of W. The order h of w is even except for the case when $L = A_n$, n even. We put $h = 2k$. Then simple computations show that

3.2 (1) If $w_0 = -I$ then $w^k = w_0$ for any Coxeter element w.

(2) If $L = A_n$ (n odd), D_n (n odd) or E_6 then ($w^k = w_0$ is no longer true for an arbitrary Coxeter element w and)

$$(w_A w_B)^k = w_0$$

where

$$w_A w_B = \prod_{a \in A} w_a \prod_{b \in B} w_b$$

with $\Pi = A \cup B$ as in Section 2.

We need the following lemma on the orbits of Coxeter elements w.

LEMMA 3.3. *Suppose that h is even (i.e., $L \neq A_n$, n even) and*

$$w = w_{\alpha_1} w_{\alpha_2} \ldots w_{\alpha_n}$$

is a Coxeter element enjoying the property $w^k = w_0$. Let

$$\beta_1 = \alpha_1,$$

$$\beta_j = w_{\alpha_1} w_{\alpha_2} \ldots w_{\alpha_{j-1}}(\alpha_j),\ 2 \leqq j \leqq n.$$

Then

$$w^i(\beta_j) > 0$$

for all $0 \leqq i \leqq k - 1$ and $1 \leqq j \leqq n$.

Proof. Since $w^k = w_0$ and $l(w_0) = |\Phi^+| = kl(w)$ ([**1**], p. 165), $l(w^i) = il(w)$ for all $0 \leqq i \leqq k$. Suppose that i is the smallest nonnegative integer such that

$$w^i(\beta_j) < 0$$

for some β_j. Then

$$w^i w_{\alpha_1} w_{\alpha_2} \ldots w_{\alpha_{j-1}}(\alpha_j) < 0$$

and for the smallest j satisfying this relation, we have (cf [**1**], p. 18),

$$l(w^i w_{\alpha_1} \ldots w_{\alpha_{j-1}} w_{\alpha_j}) = l(w^i w_{\alpha_1} \ldots w_{\alpha_{j-1}}) - 1.$$

Then $l(w^{i+1}) < (i + 1)l(w)$, and hence $i > k - 1$.

COROLLARY 3.4. *If $w = w_A w_B$ (and h is even) then*

$$w^i(a) > 0$$

$$w^i w_A(b) > 0$$

for all $a \in A$, $b \in B$ and $0 \leqq i \leqq k - 1$.

4. Proof of theorem. In this section, we assume that $L \neq A_n$, $n = 1$ or n even, and by M and w we shall mean

$$M = \prod_{a \in A} \phi_a \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \prod_{b \in B} \phi_b \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix},$$

$$w = w_A w_B.$$

For any $T \in SL_2(K)$, let

$$\phi_A(T) = \prod_{a\in A} \phi_a(T), \quad \phi_B(T) = \prod_{b\in B} \phi_b(T).$$

Then by virtue of (4), we have

$$\phi_R(S)\phi_R(T) = \phi_R(ST)$$

for any S, $T \in SL_2(K)$ and $R = A$ or B. Let

$$x_R = \phi_R\begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix} = \prod_{r\in R} x_r(-1),$$

$$x_{-R} = \phi_R\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix} = \prod_{r\in R} x_{-r}(1),$$

$$\omega_R = \phi_R\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, \qquad R = A, B.$$

Then, from

$$\begin{pmatrix}1 & 1\\ -1 & 0\end{pmatrix} = \begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} = \begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$$

we obtain

$$(6) \qquad M = x_A\omega_A x_B\omega_B$$

and

$$M = x_{-A}^{-1}x_A^{-1}x_{-B}^{-1}x_B^{-1}.$$

Since $x_{\alpha_i}(s)$ and $x_{-\alpha_j}(t)$ commute, for any fundamental roots α_i, α_j, $\alpha_i \neq \alpha_j$, we have

$$(7) \qquad \begin{aligned} &x_A^{-1}x_{-B}^{-1} = x_{-B}^{-1}x_A^{-1} \quad \text{and}\\ &M = x_{-A}^{-1}x_{-B}^{-1}x_A^{-1}x_B^{-1}.\end{aligned}$$

Note also that, for $R = A$ or B and $r \in \Phi$,

$$\omega_R x_R \omega_R^{-1} = x_{-R},$$

$$\omega_R x_r(t)\omega_R^{-1} = x_{w_R(r)}(\pm t).$$

Thus if we let $\omega = \omega_A\omega_B$ and $\omega_0 = \omega^k$, then

$$\omega x_r(t)\omega^{-1} = x_{w(r)}(\pm t),$$

$$\omega_0 x_r(t)\omega_0^{-1} = x_{w_0(r)}(\pm t)$$

by virtue of 3.2.

Now take the kth power of M written as (6). If we put $x_A(\omega_A x_B\omega_A^{-1}) = y$ then $M = y\omega$ and

$$M^k = y(\omega y\omega^{-1}) \ldots (\omega^{k-1}y\omega^{-k+1})\omega_0.$$

Since

$$\omega^i y \omega^{-i} = \prod_{a \in A} x_{w^i(a)}(\pm 1) \prod_{b \in B} x_{w^i w_A(b)}(\pm 1),$$

Corollary 3.4 shows that

$$\omega^i y \omega^{-i} \in U$$

for all $0 \leqq i \leqq k - 1$, where U is the unipotent subgroup $\langle x_r(t) | r \in \Phi^+ \rangle$, as usual. Hence

$$M^k = \tilde{y}\omega^k$$

with $\tilde{y} \in U$. Then

(8) $$M^h = M^{2k} = \tilde{y}(\omega_0 \tilde{y} \omega_0^{-1})\omega_0^2$$

with $\tilde{y} \in U$, $\omega_0 \tilde{y} \omega_0^{-1} \in U^-$, $\omega_0^2 \in H$ where

$$U^- = \langle x_r(t) | r \in \Phi^- \rangle$$

and H the Cartan subgroup, again, as usual. On the other hand the second expression (7) of M gives us

(9) $$M^{-1} = (x_B x_A)(x_{-B} x_{-A})$$

with $x_B x_A \in U$ and $x_{-B} x_{-A} \in U^-$. Since such a decomposition of an element of $L(K)$ is unique, (8) and (9) give us a clear indication of the next steps.

Let

$$y = x_A \omega_A x_B \omega_A^{-1},\ y' = x_{-A} \omega_A x_{-B} \omega_A^{-1},\ x = x_B x_A,\ x' = x_{-B} x_{-A},$$

so that $M^{-1} = xx'$. First we verify

(10) $$x = y(\omega x \omega^{-1})y'^{-1}$$

(11) $$x' = y'(\omega x' \omega^{-1})y^{-1}.$$

Proof. Simple substitutions give us

$$y(\omega x \omega^{-1})y'^{-1} = x_A \omega_A x_B (\omega_B x_B x_A x_B^{-1} \omega_B^{-1}) x_A^{-1} \omega_A^{-1}.$$

Then from a simple calculation of 2×2 matrices we obtain

$$\omega_B x_B = x_B^{-1} x_{-B}^{-1}.$$

Consequently,

$$\omega_B x_B x_A x_B^{-1} \omega_B^{-1} = x_B^{-1} x_{-B}^{-1} x_A x_{-B} x_B = x_B^{-1} x_A x_B.$$

Hence

$$\begin{aligned} y(\omega x \omega^{-1})y'^{-1} &= x_A \omega_A x_B (x_B^{-1} x_A x_B) x_A^{-1} \omega_A^{-1} \\ &= x_A(\omega_A x_A x_B x_A^{-1} \omega_A^{-1}) \\ &= x_A(x_A^{-1} x_B x_A) = x_B x_A. \end{aligned}$$

Similarly we can verify (11).

By substituting the right hand side of (10) into x, we obtain

$$x = y(\omega y\omega^{-1})(\omega^2 x\omega^{-2})(\omega y'\omega^{-1})^{-1}y'^{-1}$$

and repeating this substitution further we obtain

$$x = \tilde{y}(\omega_0 x\omega_0^{-1})\tilde{y}'^{-1}$$

where

$$\tilde{y} = y(\omega y\omega^{-1}) \dots (\omega^{k-1}y\omega^{-k+1}), \quad \text{(as above)}$$
$$\tilde{y}' = y'(\omega y'\omega^{-1}) \dots (\omega^{k-1}y'\omega^{-k+1}).$$

Since x, $\tilde{y} \in U$ and $\omega_0 x\omega_0^{-1}$, $\tilde{y}' \in U^-$ we have

(12) $x = \tilde{y}$,

(13) $\omega_0 x\omega_0^{-1} = \tilde{y}'$.

Similarly, we can obtain, from (11)

(14) $x' = \tilde{y}'$,

(15) $\omega_0 x'\omega_0^{-1} = \tilde{y}$.

Hence

(16) $x' = \omega_0 x\omega_0^{-1} = \omega_0\tilde{y}\omega_0^{-1}$.

Then (8), (9), (12) and (16) give us

$$M^h\omega_0^2 = M^{-1}.$$

Since $\omega_0^2 x_r(t)\omega_0^{-2} = x_r(\pm t)$ for all $r \in \Phi$, ω_0^2 is (loosely speaking) a diagonal element with entries 1 or -1, and hence $\omega_0^4 = 1$. However, in our case we can say a little more about ω_0^2. From (12), (13), (14), and (15), we obtain that

$$\omega_0^2 x\omega_0^{-2} = x.$$

Hence we have

$$\omega_0^2 x_\alpha(-1)\omega_0^{-2} = x_\alpha(-1)$$

for all $\alpha \in \Pi$, which in turn, implies that

$$\omega_0^2 x_r(t)\omega_0^{-2} = x_r(t)$$

for all $r \in \Phi$. Hence ω_0^2 is an element in the center of $L(K)$.

This completes the proof of our theorem for $L \neq A_n$, $n = 1$ or n even. For the cases when $L = A_1$, or A_n, n even, we have to be content with the matrix M in Section 1.

5. Some remarks. (i) The identity (12) may be regarded as an identity for the commutator $x_A^{-1}x_Bx_Ax_B^{-1}$ written as the product of $x_r(t)$ with $t \neq 0$ for all r with $h(r) \geqq 2$, because (16) shows that

$$\omega_0 x_\alpha(-1)\omega_0^{-1} = \omega_\alpha x_\alpha(-1)\omega_\alpha^{-1} \quad (\alpha \in \Pi)$$

and hence the last factor $\omega^{k-1}y\omega^{-k+1}$ of $\tilde{y}$ is equal to $(\omega_B^{-1}x_A\omega_B)x_B$.

(ii) If $L = G_2$ then $G_2(K)$ may be regarded as the automorphism group of the octanion algebra over K. Then the cyclic permutation of the seven basis elements ($\neq 1$) is an element of order $h + 1 = 7$.

(iii) For the case when $L = E_7$ or E_8, all the prime torsions of $L(\mathbf{Z})$ are given in [**2**], hence the existence of elements of order $h + 1$. I wish to thank Professor Eckmann for bringing this paper to my attention.

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