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PAPER

On notions of compactness, object classifiers, and weak Tarski universes

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Abstract

We prove a correspondence between κ-small fibrations in simplicial presheaf categories equipped with the injective or projective model structure (and left Bousfield localizations thereof) and relatively κ-compact maps in their underlying quasi-categories for suitably large regular cardinals κ. We thus obtain a transition result between weakly universal small fibrations in the (type-theoretic) injective Dugger–Rezk-style standard presentations of model toposes and object classifiers in Grothendieck ∞-toposes in the sense of Lurie.

Keywords: relative compactness; object classifiers; Tarski universes; presentable ∞-categories; combinatorial model categories

1. Introduction

A Grothendieck ∞-topos $\mathcal{M}$ is an accessible left exact localization of the presheaf $(\infty, 1)$-category $\hat{\mathcal{C}}$ over some small $(\infty, 1)$-category $\mathcal{C}$ (Lurie, 2009, Section 6.1). Hence, it is presented by a model topos $\mathbb{M}$ given by a left exact left Bousfield localization of the simplicial category $\mathbf{sPsh}(\mathbf{C})$ of simplicially enriched presheaves on an associated small simplicial category $\mathbf{C}$, equipped with either the projective or the injective model structure (Rezk, 2010, Section 6). In the following, we prove a correspondence between κ-small fibrations in such model toposes $\mathbb{M}$ and relatively κ-compact maps in their associated Grothendieck ∞-toposes $\mathcal{M} = \mathrm{Ho}_\infty(\mathbb{M})$ for regular cardinals κ large enough. This is motivated by the interpretation of univalent Tarski universes defined in Martin-Löf-type theory (The Univalent Foundations Program 2013) as univalent fibrations universal for the class of κ-small fibrations for suitable cardinals κ (Cisinski 2014; Shulman 2019*a*) and their intended interpretation as object classifiers in higher topos theory for relatively κ-compact maps as developed in Lurie (2009, Section 6.1.6). Therefore, even though we prove an analogous (but slightly weaker result) result for the projective model structure and arbitrary localizations, the main result of this paper is the following.

Theorem 3.21. *Let* $\mathbf{C}$ *be a small simplicial category,* T *be a set of arrows in the underlying category of* $\mathbf{sPsh}(\mathbf{C})$*, and* $\mathbb{M}$ *be the left Bousfield localization* $\mathcal{L}_T\mathbf{sPsh}(\mathbf{C})_{\mathrm{inj}}$ *of* $\mathbf{sPsh}(\mathbf{C})$ *equipped with the injective model structure at the set* T *(in the classic sense of Hirschhorn 2003, Section 3.3). Assume that the localization is left exact, and let* κ *be a sufficiently large regular cardinal. Then a morphism* $f \in \mathrm{Ho}_\infty(\mathbb{M})$ *is relatively* κ*-compact if and only if there is a* κ*-small fibration* $g \in \mathrm{sPsh}(\mathbf{C})$ *such that* $g \simeq f$ *in* $\mathrm{Ho}_\infty(\mathbb{M})$.

Corollary 4.1. *Let $\mathbb{M} = \mathcal{L}_T\mathbf{sPsh}(\mathbf{C})_{\mathrm{inj}}$ be a left exact left Bousfield localization as in Theorem 3.21, and let κ be a sufficiently large regular cardinal. Then, a relatively κ-compact map $p \in \mathrm{Ho}_\infty(\mathbb{M})$ is a classifying map for all relatively κ-compact maps in $\mathrm{Ho}_\infty(\mathbb{M})$ if and only if there is a univalent κ-small fibration $\pi \in \mathbb{M}$ which is weakly universal for all κ-small fibrations in $\mathbb{M}$ such that $p \simeq \pi$ in $\mathrm{Ho}_\infty(\mathbb{M})$.*

As a prerequisite for the proof we are giving, in Section 2, we show that up to DK-equivalence every simplicial category can be replaced by the localization of a suitable well-founded poset as already observed by Shulman in (2017). This allows us to replace simplicial presheaf categories over arbitrary small simplicial categories $\mathbf{C}$ by localizations of simplicial presheaf categories over such posets I. In Section 3, we will use that such model categories come equipped with a theory of minimal fibrations that will allow us to present relatively κ-compact maps in their underlying quasi-category by κ-small fibrations. Those can be pushed forward to κ-small projective fibrations in our original presheaf category over $\mathbf{C}$ making use of Dugger's ideas about universal homotopy theories in Dugger (2001*b*). The move to the injective model structure then follows by Shulman's recent observation (Shulman, 2019*a*, Section 8) that the cobar construction on simplicial presheaf categories takes projective fibrations to injective ones. In Section 4, we explain the relevance of this result for the semantics of homotopy type theory in higher topos theory, as Theorem 3.21 is necessary to translate Tarski universes in the syntax to object classifiers in an ∞-topos (when using the common semantics via type-theoretic model categories given in Shulman 2015, Section 4).

2. Direct Poset Presentations of Simplicial Categories

In the following, simplicial categories – that is simplicially enriched categories - will be denoted by bold faced letters $\mathbf{C}$ and ordinary categories will be distinguished by blackboard letters $\mathbb{C}$. $\mathbf{S}$ denotes the (simplicial) category of simplicial sets. By a simplicial presheaf over $\mathbf{C}$, we mean a simplicially enriched presheaf $X\colon \mathbf{C}^{op} \to \mathbf{S}$. Simplicial presheaves and simplicial natural transformations form part of a simplicial category $\mathbf{sPsh}(\mathbf{C})$ (via the usual end-construction, denoted $[\mathbf{C}^{op}, \mathbf{S}]$ in Kelly 2005, Section 2.2) whose underlying ordinary category will be denoted by $\mathrm{sPsh}(\mathbf{C})$.

Mike Shulman noted in Shulman (2017, Lemma 0.2) that every quasi-category can be presented by the localization of a direct – in other words, well-founded – poset of degree at most ω.[1] Since the note is unpublished, in this section, we present a slightly stronger variation of his observation (with an accordingly slightly different proof) and discuss the resulting presentations of associated presheaf $(\infty, 1)$-categories. Although the following sections only will require the fact that every $(\infty, 1)$-category can be presented by the localization of an Eilenberg–Zilber category (Berger and Moerdijk 2011), proving the stronger condition of posetality only requires about as much work as the Eilenberg–Zilber condition itself.

Recall the following constructions and notation from Barwick and Kan (2012*b*). A *relative category* is a pair $(\mathbb{C}, V)$ such that $\mathbb{C}$ is a category and V is a subcategory of $\mathbb{C}$. A *relative functor* $F\colon (\mathbb{C}, V) \to (\mathbb{D}, W)$ is a functor $F\colon \mathbb{C} \to \mathbb{D}$ of categories such that $F[V] \subseteq W$. The relative functor F is a *relative inclusion* if its underlying functor of categories is an inclusion and $V = W \cap \mathbb{C}$. The category of small relative categories and relative functors is denoted by RelCat.

There are two canonical inclusions of the category Cat of small categories into RelCat; for a category $\mathbb{C}$ and its discrete wide subcategory $\mathbb{C}_0$, we obtain the associated minimal relative category $\check{\mathbb{C}} := (\mathbb{C}, \mathbb{C}_0)$ and the associated maximal relative category $\hat{\mathbb{C}} := (\mathbb{C}, \mathbb{C})$.

In Barwick and Kan (2012*b*, Section 4), Barwick and Kan introduce a combinatorial subdivision operation ξ : RelPos $\to$ RelPos on relative posets (considered as posetal relative categories) and an associated bisimplicial nerve construction N_ξ : RelCat $\to s\mathbf{S}$ giving rise to the adjoint pair:

$$s\mathbf{S} \underset{N_\xi}{\overset{K_\xi}{\rightleftarrows}} \mathrm{RelCat}. \tag{1}$$

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The left adjoint K_ξ is given by $K_\xi(\Delta[m, n]) = \xi([\check{m}] \times [\hat{n}])$ on representables and left Kan extension along the Yoneda embedding. The authors of Barwick and Kan (2012*b*) have shown that the category RelCat inherits a transferred model structure from the Reedy model structure $(s\mathbf{S}, R_v)$ which turns the pair (K_ξ, N_ξ) into a Quillen-equivalence. By construction, the set $K_\xi[\mathcal{I}_v]$ forms a set of generating cofibrations for the model structure in question, where

$$\mathcal{I}_v := \{\partial\Delta[n, m] \hookrightarrow \Delta[n, m] \mid n, m \geq 0\}$$

denotes the generating set of monomorphisms in $s\mathbf{S}$. Left Bousfield localization of both sides of (1) induces a model structure (RelCat, BK) such that (K_ξ, N_ξ) is a Quillen-equivalence to Rezk's model structure $(s\mathbf{S}, \mathrm{CS})$ for complete Segal spaces.

It follows that the underlying quasi-category of the model category RelCat is equivalent to the quasi-category of small $(\infty, 1)$-categories (Lurie 2017, Definition 1.3.4.15). We will denote the thus associated small $(\infty, 1)$-category to a relative category $(\mathbb{C}, V)$ by $\mathrm{Ho}_\infty(\mathbb{C}, V)$ whenever it is not necessary to specify a specific model of $(\infty, 1)$-category theory.

A central notion of Barwick and Kan (2012*b*) is that of a "Dwyer map" in RelCat. A relative functor $F\colon (\mathbb{C}, V) \to (\mathbb{D}, W)$ is a *Dwyer inclusion* if F is a relative inclusion such that $\mathbb{C}$ is a sieve in $\mathbb{D}$ and such that the cosieve $Z\mathbb{C}$ generated by $\mathbb{C}$ in $\mathbb{D}$ comes equipped with a strong deformation retraction $Z\mathbb{C} \to \mathbb{C}$. The relative functor F is a *Dwyer map* if it factors as an isomorphism followed by a Dwyer inclusion, see Barwick and Kan (2012*b*, Section 3.5) for more details.

A major insight of the authors was that the generating cofibrations:

$$K_\xi(\partial\Delta[n, m]) \hookrightarrow K_\xi(\Delta[n, m]) \tag{2}$$

of the model category (RelCat, BK) are Dwyer maps of (finite) relative posets (Barwick and Kan 2012*b*, Proposition 9.5). It follows that every cofibration in (RelCat, BK) is a Dwyer map and that every cofibrant object is a relative poset (Barwick and Kan 2012*b*, Theorem 6.1).

Proposition 2.1. *The underlying category of a cofibrant object in* (RelCat, BK) *is a direct (i.e. well-founded) poset of degree at most ω (i.e. it comes equipped with a strictly monotone degree function to ω).*

Proof. Since the empty relative category $\emptyset$ is a relative direct poset, it suffices to show that for every cofibration $(\mathbb{P}, V) \hookrightarrow (\mathbb{Q}, W)$ where $(\mathbb{P}, V)$ is a relative direct poset of degree at most ω also $(\mathbb{Q}, W)$ is a relative direct poset of degree at most ω. We show this by induction along the small object argument as follows.

The generating cofibrations (2) are maps between finite relative posets and such are clearly direct. Both Dwyer maps and relative posets are closed under coproducts and under pushouts between relative posets by Barwick and Kan (2012*b*, Proposition 9.2). Since Dwyer inclusions are inclusions of sieves, it is easy to see that both constructions preserve well-foundedness and the existence of a ω-valued degree function, too. Suppose we are given a transfinite composition of Dwyer maps $A_\alpha \to A_\beta$ for $\alpha < \beta \leq \lambda$ ordinals and A_α relative inverse posets of degree at most ω. Then, as stated in the proof of Barwick and Kan (2012*b*, Proposition 9.6), the colimit A_λ is a relative poset. Suppose $a = (a_i \mid i < \omega)$ is a descending sequence of arrows in A_λ and let $\alpha < \lambda$ such that $a_0 \in A_\alpha$. Then the whole sequence a is contained in A_α, because the inclusion $A_\alpha \hookrightarrow A_\lambda$ is a Dwyer map by Barwick and Kan (2012*b*, Proposition 9.3) and so $A_\alpha \subseteq A_\lambda$ is a sieve. Therefore, the sequence a is finite. For the same reason, the objects in A_α still have finite degree.

In particular, every free cofibration $\emptyset \hookrightarrow (\mathbb{P}, V)$ – that is every transfinite composition of pushouts of coproducts of generating cofibrations with domain $\emptyset$ – yields a relative direct poset $(\mathbb{P}, V)$ of degree at most ω. But every cofibration $\emptyset \hookrightarrow (\mathbb{Q}, W)$ is a retract of such, and hence every cofibrant object in RelCat is a relative direct poset of degree at most ω. □

Remark 2.2. The same proof shows that the cofibrant objects in the Thomason model structure on Cat are direct posets, using Thomason's original observation that the cofibrant objects in the Thomason model structure are posetal in the first place.

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Let $F_\Delta\colon \mathrm{Cat} \to \mathbf{S}\text{-Cat}$ be the Bar construction obtained in the standard way by the monad resolution associated with the free category functor F from the category RGraph of reflexive graphs to Cat (Dwyer and Kan 1980*b*, Section 2.5). Let $U\colon \mathrm{Cat} \to \mathrm{RGraph}$ be the corresponding right adjoint forgetful functor. Recall that F_Δ itself is not the left adjoint to the "underlying category" functor $(\,\cdot\,)_0\colon \mathbf{S}\text{-Cat} \to \mathrm{Cat}$, but instead, as often remarked in the literature, a cofibrant replacement thereof. Furthermore, recall from Dwyer and Kan (1980*b*, Section 4) the (standard) simplicial localization functor:

$$\mathcal{L}_\Delta\colon \mathrm{RelCat} \to \mathbf{S}\text{-Cat}$$

which takes a relative category $(\mathbb{C}, V)$ to the simplicial category given in degree $n \geq 0$ by:

$$\mathcal{L}_\Delta(\mathbb{C}, V)_n = F_\Delta(\mathbb{C})_n[F_\Delta(V)_n^{-1}].$$

The functor $\mathcal{L}_\Delta\colon (\mathrm{RelCat}, \mathrm{BK}) \to \mathbf{S}\text{-Cat}$ is part of an equivalence of associated homotopy theories (Barwick and Kan 2012*a*) so that $\mathrm{Ho}_\infty(C, V) \simeq \mathcal{L}_\Delta(C, V)$. Indeed, it is pointwise equivalent to the *hammock localization* of a relative category (Dwyer and Kan 1980*a*, Section 2) and hence presents the $(\infty, 1)$-categorical localization of $\mathbb{C}$ at V (Mazel-Gee 2019, Remark 3.2, Theorem 3.8). Yet it has the benefit of being a strict enriched construction: it is the localization of the simplicial category $F_\Delta(\mathbb{C})$ at the subcategory $F_\Delta(V)_0 \subseteq F_\Delta(\mathbb{C})_0$ in the following sense.

Lemma 2.3. *For every relative category $(\mathbb{C}, V)$ and every (potentially large) simplicial category $\mathbf{D}$, the canonical simplicial functor $j\colon F_\Delta(\mathbb{C}) \to \mathcal{L}_\Delta(\mathbb{C}, V)$ induces an isomorphism:*

$$j^*\colon \mathbf{S}\text{-Cat}(\mathcal{L}_\Delta(\mathbb{C}, V), \mathbf{D}) \to \mathbf{S}\text{-Cat}_{F_\Delta(V)}(F_\Delta(\mathbb{C}), \mathbf{D}) \tag{3}$$

of hom-categories.

Here, we consider $\mathbf{S}$-Cat as a 2-category given by $\mathbf{S}$-enriched categories, $\mathbf{S}$-enriched functors, and $\mathbf{S}$-enriched natural transformations (Kelly 2005, Section 1.2). The right-hand side of (3) denotes the full subcategory of $\mathbf{S}\text{-Cat}(F_\Delta(\mathbb{C}), \mathbf{D})$ spanned by those functors which take the arrows of $F_\Delta(V)_0$ to isomorphisms in $\mathbf{D}_0$. Thus, Lemma 2.3 states that $F_\Delta(V)_0 \subseteq F_\Delta(\mathbb{C})_0$ is "$\mathbf{S}$-well localizable" in the sense of Wolff (1973).

Lemma 2.3 and Corollary 2.4 are not strictly necessary for the results of this paper, but may be beneficial to motivate the simplicial localization construction.

Proof of Lemma 2.3. First, each $F_\Delta(V)_n$ is generated as a category by the iterated degeneracies of $F_\Delta(V)_0$. Thus, a simplicial functor $F_\Delta(\mathbb{C}) \to \mathbf{D}$ takes all arrows in $F_\Delta(V)_n$ to isomorphisms in $\mathbf{D}_n$ for all $n \geq 0$ if and only if F_0 takes the arrows in $F_\Delta(V)_0$ to isomorphisms in $\mathbf{D}_0$. Using the universal property of the ordinary categorical localization $F_\Delta(\mathbb{C})_n[F_\Delta(V)_n^{-1}]$ at each degree $n \geq 0$, it follows that (3) is bijective on objects.

Second, recall that $\mathbf{S}$-Cat is cartesian closed (Kelly 2005, 2.3). Thus, Cordier's simplicial nerve construction $N_\Delta\colon \mathbf{S}\text{-Cat} \to \mathbf{S}$ with left adjoint $\mathfrak{C}$ (Lurie 2009, Section 1.1.5) yields a simplicial enrichment of $\mathbf{S}$-Cat itself. Thus, the fact that j^* is fully faithful follows from the same objectwise argument but applied to the simplicial category $[\mathfrak{C}(\Delta^1), \mathbf{D}]$. Indeed, enriched natural transformations of simplicial functors $\mathcal{L}_\Delta(\mathbb{C}, V) \to \mathbf{D}$ stand in 1-1 correspondence to 1-simplices in the nerve $N_\Delta([\mathcal{L}_\Delta(\mathbb{C}, V), \mathbf{D}])$. Such correspond bijectively to simplicial functors $\mathfrak{C}(\Delta^1) \to [\mathcal{L}_\Delta(\mathbb{C}, V), \mathbf{D}]$, where $\mathfrak{C}(\Delta^1)$ is the simplicial category with two objects $0, 1$, and $\mathfrak{C}(\Delta^1)(0, 1) \cong \mathfrak{C}(\Delta^1)(0, 0) \cong \mathfrak{C}(\Delta^1)(1, 1) \cong \Delta^0$ and $\mathfrak{C}(\Delta^1)(1, 0) \cong \emptyset$. The latter functors in turn stand in 1-1 correspondence to simplicial functors $\mathcal{L}_\Delta(\mathbb{C}, V) \to [\mathfrak{C}(\Delta^1), \mathbf{D}]$ by Kelly (2005, Section 2.3). Such functors stand in 1-1 correspondence via restriction along j to simplicial functors of type $F_\Delta(\mathbb{C}) \to [\mathfrak{C}(\Delta^1), \mathbf{D}]$ which take arrows in $F_\Delta(V)_0$ to isomorphisms in $[\mathfrak{C}(\Delta^1), \mathbf{D}]_0$ by the first part. These are exactly the enriched natural transformations in $\mathbf{S}\text{-Cat}_{F_\Delta(V)}(F_\Delta(\mathbb{C}), \mathbf{D})$. □

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Corollary 2.4. *For* $(\mathbb{C}, V) \in \mathrm{RelCat}$ *and* $j\colon F_\Delta(\mathbb{C}) \to \mathcal{L}_\Delta(\mathbb{C}, V)$, *the associated localization functor of simplicial categories, the induced restriction:*

$$j^*\colon \mathbf{sPsh}(\mathcal{L}_\Delta(\mathbb{C}, V)) \to \mathbf{sPsh}(F_\Delta(\mathbb{C})) \tag{4}$$

of simplicial presheaf categories is fully faithful.

Proof. Given simplicial presheaves $X, Y \in \mathbf{sPsh}(\mathcal{L}_\Delta(\mathbb{C}, V))$, we want to show that the induced map:

$$j^*(X, Y)\colon \mathbf{sPsh}(\mathcal{L}_\Delta(\mathbb{C}, V))(X, Y) \to \mathbf{sPsh}(F_\Delta(\mathbb{C}))(j^*X, j^*Y)$$

of simplicial sets is a bijection on all degrees $n \geq 0$. But for each such integer $n \geq 0$, the map $j^*(X, Y)_n$ is isomorphic to the restriction:

$$j^*(X, Y)\colon \mathrm{sPsh}(\mathcal{L}_\Delta(\mathbb{C}, V))(X \otimes \Delta^n, Y) \to \mathrm{sPsh}(F_\Delta(\mathbb{C}))(j^*(X \otimes \Delta^n), j^*Y),$$

since simplicial presheaf categories are cotensored over $\mathbf{S}$, and j^* is cocontinuous. Thus, it is bijective by Lemma 2.3 applied to $\mathbf{D} = \mathbf{S}$. □

Hence, the map $j\colon F_\Delta(\mathbb{C}) \to \mathcal{L}_\Delta(\mathbb{C}, V)$ induces both a localization:

$$(j_!, j^*)\colon \mathbf{sPsh}(F_\Delta(\mathbb{C})) \to \mathbf{sPsh}(\mathcal{L}_\Delta(\mathbb{C}, V))$$

and a colocalization:

$$(j^*, j_*)\colon \mathbf{sPsh}(\mathcal{L}_\Delta(\mathbb{C}, V)) \to \mathbf{sPsh}(F_\Delta(\mathbb{C}))$$

between simplicial presheaf categories. Equipping both sides with either the injective or the projective model structure, the restriction j^* becomes a left, respectively, right Quillen functor (Lurie 2009, Proposition A.3.3.7). By Dwyer and Kan (1987, Theorem 2.2) applied to the map

$$j\colon (F_\Delta(\mathbb{C}), F_\Delta(V)) \to (\mathcal{L}_\Delta(C, V), \mathcal{L}_\Delta(C, V)^{\cong})$$

of relative simplicial categories (Barwick and Kan 2012*a*, Section 2.2), the restriction (4) remains fully faithful on associated homotopy theories. More precisely, equipping both sides with the projective model structure, the pair $(j_!, j^*)$ becomes a homotopy localization and induces a Quillen-equivalence:

$$(j_!, j^*)\colon \mathbf{sPsh}(\mathcal{L}_\Delta(\mathbb{C}, V))_{\mathrm{proj}} \to \mathcal{L}_{y[F_\Delta(V)_0]}\mathbf{sPsh}(F_\Delta(\mathbb{C}))_{\mathrm{proj}}, \tag{5}$$

where the right-hand side denotes the according left Bousfield localization. Dually, equipping both sides in (4) with the injective model structure, the pair (j^*, j_*) becomes a homotopy colocalization.

The simplicial localization functor $\mathcal{L}_\Delta\colon \mathrm{RelCat} \to \mathbf{S}\text{-}\mathrm{Cat}$ has a homotopy-inverse, the "delocalization" or "flattening"

$$\flat\colon \mathbf{S}\text{-}\mathrm{Cat} \to \mathrm{RelCat},$$

given by the Grothendieck construction of a given simplicial category $\mathbf{C}$ considered as a simplicial diagram $\mathbf{C}\colon \Delta^{op} \to \mathrm{Cat}$. This functor was introduced in Dwyer and Kan (1987, Theorem 2.5) and is analyzed in detail in Barwick and Kan (2012*a*).

Now, given a simplicial category $\mathbf{C}$, consider its delocalization $\flat(\mathbf{C}) \in \mathrm{RelCat}$. Cofibrantly replacing $\flat(\mathbf{C})$ with some pair $(\mathbb{P}, V)$ in RelCat yields a direct relative poset $(\mathbb{P}, V)$ weakly equivalent, that is, Rezk-equivalent in the language of Barwick and Kan (2012*a*) – to $\flat(\mathbf{C})$. Hence, by Barwick and Kan (2012*a*, Theorem 1.8), the simplicial localization $\mathcal{L}_\Delta(\mathbb{P}, V) \in \mathbf{S}\text{-}\mathrm{Cat}$ is DK-equivalent to the original simplicial category $\mathbf{C}$. That means there is a zig-zag of DK-equivalences of the form:

$$\mathbf{C} \xrightarrow{f_1} \cdots \xleftarrow{f_n} \mathcal{L}_\Delta(\mathbb{P}, V). \tag{$*$}$$

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By Lurie (2009, Proposition A.3.3.8) or Dwyer and Kan (1987, Theorem 2.1) and the sequence of maps in (∗), we obtain a zig-zag of simplicial Quillen-equivalences:

$$\mathbf{sPsh}(\mathcal{L}_\Delta(\mathbb{P}, V))_{\text{proj}} \underset{f_n^*}{\overset{(f_n)_!}{\rightleftarrows}} \ldots \underset{f_1^*}{\overset{(f_1)_!}{\leftrightarrows}} \mathbf{sPsh}(\mathbf{C})_{\text{proj}}.$$

Further recall from Dwyer and Kan (1980*b*, Proposition 2.6) that for every category $\mathbb{C}$, the canonical projection $\varphi\colon F_\Delta\mathbb{C} \to \mathbb{C}$ is a DK-equivalence of simplicial categories. So, to summarize, we have gathered the following chain of Quillen-equivalences.

Proposition 2.5. *Let* $\mathbf{C}$ *be a small simplicial category. Then there is a direct relative poset* $(\mathbb{P}, V)$ *of degree at most* ω *together with a zig-zag of DK-equivalences:*

$$\mathbf{C} \to \cdots \leftarrow \mathcal{L}_\Delta(\mathbb{P}, V)$$

in $\mathbf{S}$*-Cat which induces a zig-zag of simplicial Quillen-equivalences of the form:*

$$\begin{array}{ccccc} \mathcal{L}_{y[V]}\mathbf{sPsh}(\mathbb{P}) & \underset{\varphi^*}{\overset{\varphi_!}{\leftrightarrows}} & \mathcal{L}_{y[F_\Delta(V)_0]}\mathbf{sPsh}(F_\Delta\mathbb{P}) & \underset{j^*}{\overset{j_!}{\rightleftarrows}} & \mathbf{sPsh}(\mathcal{L}_\Delta(\mathbb{P}, V)) \\ & & & & \| \\ \mathbf{sPsh}(\mathbf{C}) & \underset{f_1^*}{\overset{(f_1)_!}{\rightleftarrows}} & \ldots & \underset{f_n^*}{\overset{(f_n)_!}{\leftrightarrows}} & \mathbf{sPsh}(\mathcal{L}_\Delta(\mathbb{P}, V)), \end{array}$$

where all simplicial presheaf categories are equipped with the projective model structure.

Proof. The only part left to show is that $\varphi\colon F_\Delta\mathbb{C} \to \mathbb{C}$ induces a Quillen-equivalence of given left Bousfield localizations, but this follows directly from Dwyer and Kan (1980*b*, Proposition 2.6) together with Dwyer and Kan (1987, Corollary 3.8). □

3. Compactness in Combinatorial Model Categories

We start with some facts about compactness in presheaf categories. Given a small category $\mathbb{C}$, we denote the cardinality of $\mathbb{C}$ by:

$$|\mathbb{C}| := \sum_{C, C' \in \mathbf{C}} |\mathrm{Hom}_{\mathbb{C}}(C, C')|.$$

Given a (set-valued) presheaf $X \in \widehat{\mathbb{C}}$, its cardinality is denoted by:

$$|X| := \sum_{C \in \mathbb{C}} |X(C)|.$$

Given a regular cardinal $\kappa > |\mathbb{C}|$, recall that a presheaf $X \in \widehat{\mathbb{C}}$ is κ-small if $|X| < \kappa$, that is if all its values $X(C)$ have cardinality smaller than κ. A map $f\colon X \to Y$ in $\widehat{\mathbb{C}}$ is κ-small if all its pullbacks along maps $Z \to Y$ with κ-small domain Z are κ-small presheaves. Equivalently, $f\colon X \to Y$ is κ-small if and only if for all objects $C \in \mathbb{C}$, the function $f(C)\colon X(C) \to Y(C)$ of sets has κ-small fibers.

Given a small simplicial category $\mathbf{C}$, we also denote the cardinality of $\mathbf{C}$ by:

$$|\mathbf{C}| := \sum_{C, C'} |\mathrm{Hom}_{\mathbf{C}}(C, C')|$$

where the cardinality of the hom-objects $\mathbf{C}(C, C') \in \mathbf{S}$ is given by the cardinality of presheaves defined above. Accordingly, given a regular cardinal $\kappa > |\mathbf{C}|$, a simplicial presheaf $X \in \mathrm{sPsh}(\mathbf{C})$ is κ-small if all its values $X(C)$ are κ-small. A simplicial natural transformation $f\colon X \to Y$ in $\mathrm{sPsh}(\mathbf{C})$

is κ-small if all its pullbacks along maps $Z \to Y$ with κ-small domain Z are κ-small simplicial presheaves.

Remark 3.1. Again, a map $f\colon X \to Y$ in sPsh($\mathbf{C}$) is κ-small if and only if for all $C \in \mathbf{C}$, the map $f(C)\colon X(C) \to Y(C)$ is a κ-small map of simplicial sets.

The category sPsh($\mathbf{C}$) is locally finitely presentable (Kelly 1982, Examples 3.4, Proposition 7.5), generated by (finite colimits of) the objects $yC \otimes \Delta^n$ for $C \in \mathbf{C}$ and $n \geq 0$ which we refer to in the following as the generators. When an ordinary category $\mathbb{C}$ is considered as a discrete simplicial category, we have an obvious isomorphism between sPsh($\mathbb{C}$) and the set-valued presheaf category $\widehat{\mathbb{C} \times \Delta^{op}}$.

Notation 3.2. To develop a general theory of accessible categories, Adámek and Rosický introduce for pairs of regular cardinals $\mu < \kappa$ the *sharply larger* relation (Adámek and Rosický 1994, Definition 2.12) and a special case "$\ll$" in Adámek and Rosický (1994, Example 2.13.(4)) which is used as well in Lurie (2009, Definition 5.4.2.8) to develop a theory of accessible $(\infty, 1)$-categories. Here, $\mu \ll \kappa$ if for all cardinals $\kappa_0 < \kappa$ and $\mu_0 < \mu$, also $\kappa_0^{\mu_0} < \kappa$.

The order "$\ll$" is chosen in such a way that whenever $\mu \ll \kappa$ holds, then $\mu < \kappa$ and μ-accessibility of a quasi-category $\mathcal{C}$ implies κ-accessibility of $\mathcal{C}$ (Lurie 2009, Proposition 5.4.2.11). As noted in Lurie (2009), the order is unbounded in the class of regular cardinals as for any regular cardinal μ we have $\mu \ll \sup\{\tau^\mu \mid \tau < \mu\}^+$. In particular, we always find a regular cardinal sharply larger than a given regular μ. In fact, if μ is regular, then μ^+ is already sharply larger than μ (Adámek and Rosický 1994, Examples 2.13.(2)). Whenever $\lambda < \mu$ is a regular cardinal and $\mu \ll \kappa$, then also $\lambda \ll \kappa$. Thus, for any set X of cardinals, there is a regular cardinal μ such that $\kappa \ll \mu$ for all $\kappa \in X$.

Recall that an object C in an accessible category $\mathbb{C}$ is *κ-compact* if its associated corepresentable preserves κ-directed colimits. A map f in $\mathbb{C}$ is *relatively κ-compact* if the pullback of f along any map with κ-compact domain in $\mathcal{C}$ is itself again κ-compact.

Lemma 3.3. *Let $\mathbf{C}$ be a small simplicial category and $\kappa \gg |\mathbf{C}|$ an infinite regular cardinal. Then*

1. *An object $X \in$ sPsh($\mathbf{C}$) is κ-compact if and only if it is κ-small.*
2. *A map $f \in$ sPsh($\mathbf{C}$) is relatively κ-compact if and only if it is κ-small.*

Proof. Let $\mathbf{C}$ be a small simplicial category. For Part 1, recall that a presheaf X is κ-compact if and only if it is a retract of a κ-small directed colimit of finite colimits of the generators $yC \otimes \Delta^n$ via Adámek and Rosický (1994, Remark 2.15). But $\kappa \gg |\mathbf{C}|$ being infinite and regular implies that all generators $yC \otimes \Delta^n$ are κ-small, and hence so are all their finite colimits. Hence, every κ-compact presheaf X is a subobject of a κ-small colimit of κ-small presheaves and hence κ-small. Vice versa, every simplicial presheaf X is the colimit of its canonical diagram of generators $yC \otimes \Delta^n$; whenever X is κ-small, so is the associated canonical diagram by the Yoneda Lemma. Thus, X is a κ-small colimit of the generators $yC \otimes \Delta^n$. Closing the generators under finite colimits gives a description of X as a κ-small directed colimit of κ-compact objects, so X is κ-compact again by Adámek and Rosický (1994, Remark 2.15). Part 2 follows directly from Part 1 by definition. □

Lemma 3.4. *Let $\mathbb{C}$ and $\mathbb{D}$ be locally presentable categories and let*

$$F\colon \mathbb{C} \rightleftarrows \mathbb{D}\colon G$$

be an adjoint pair. Let κ be a regular cardinal such that

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1. both F and G preserve κ-compact objects,
2. κ-compact objects in $\mathbb{C}$ are closed under fiber products.

Then G preserves relatively κ-compact maps.

Proof. Straight-forward. □

Remark 3.5. Although the class of μ-compact objects in a locally κ-presentable category is not necessarily closed under fiber products for all $\mu \gg \kappa$, the class of such μ is unbounded in the class of regular cardinals sharply larger than κ (Shulman 2019*a*, Proposition 4.5). It nevertheless is a trivial property in the case of (simplicial) presheaf categories whenever μ is infinite by Lemma 3.3, since μ-smallness of sets is closed under finite limits.

Corollary 3.6. *Let* **C** *and* **D** *be small simplicial categories and let*

$$F\colon \mathrm{sPsh}(\mathbf{C}) \rightleftarrows \mathrm{sPsh}(\mathbf{D})\colon G$$

be an adjoint pair. Let $\kappa \gg \max(|\mathbf{C}|, |\mathbf{D}|)$ be regular and suppose both F and G preserve κ-small objects. Then G preserves κ-small maps.

Proof. Follows directly from Lemmas 3.3.2, 3.4 and Remark 3.5. □

The aim of this section is to compare this ordinary notion of compactness in a combinatorial model category $\mathbb{M}$ with the notion of compactness in its underlying $(\infty, 1)$-category $\mathrm{Ho}_\infty(\mathbb{M}, \mathcal{W})$ (Lurie 2009, Definitions 5.3.4.5, 6.1.6.4). The validity of this comparison was addressed in a question posted in Lurie (2012) by Shulman; for objects it is given in Proposition 3.11 in a special case, and in Corollary 3.16 in general. For maps it is given in Theorem 3.15 in case $\mathbb{M}$ is (the left Bousfield localization of) a simplicial presheaf category equipped with the projective model structure, and in Theorem 3.21 in case it is such equipped with the injective model structure. An argument for the objectwise statement was outlined by Lurie in the same post, which in one direction coincides with our proof given in Proposition 3.11. Before we state the theorems, we make the following ad hoc construction, give one auxiliary folkore lemma, and define a simple axiomatic framework of minimal fibrations in a general model category that will provide a convenient setup to state intermediate results in.

Given a λ-accessible quasi-category $\mathcal{C}$ with generating set A and a regular cardinal $\mu \geq \lambda$, define the full subcategory $J^\mu \subseteq \mathcal{C}$ recursively as follows. Let

$$J_0^{\mu,0} := A$$

and $J^{\mu,0}$ be the full subcategory of $\mathcal{C}$ generated by $J_0^{\mu,0}$. Whenever $\beta < \mu$ is a limit ordinal, let

$$J_0^{\mu,\beta} = \bigcup_{\alpha<\beta} J_0^{\mu,\alpha}$$

and $J^{\mu,\beta}$ the full subcategory generated by $J_0^{\mu,\beta}$. On successors, given $J^{\mu,\alpha}$, let

$$J_0^{\mu,\alpha+1} := \{\mathrm{colim} F \mid F\colon I \to J^{\mu,\alpha}, I \in \mathrm{QCat} \text{ is } \mu\text{-small and } \lambda\text{-filtered}\} \tag{6}$$

(so we choose a set of representatives $V_\mu^{\Delta^{op}}$ for μ-small simplicial sets) and $J^{\mu,\alpha+1}$ be the corresponding full subcategory. Eventually, we define the full subcategory J^μ of $\mathcal{C}$ to have the set of objects:

$$J_0^\mu := \bigcup_{\alpha<\mu} J_0^{\mu,\alpha}.$$

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The following lemma is noted in Lurie (2009, Section 5.4.2) and is a generalization of the corresponding 1-categorical statement for accessible categories (Adámek and Rosický 1994, Remark 2.15).

Lemma 3.7. *Let $\mathcal{C}$ and $\mathcal{D}$ be presentable quasi-categories.*

(1) Suppose $\mathcal{C}$ is λ-presentable. Then, for every regular $\mu \gg \lambda$, the μ-compact objects in $\mathcal{C}$ are, up to equivalence, exactly the retracts of objects in J^{μ}.
(2) Let $F\colon \mathcal{C} \to \mathcal{D}$ be an accessible functor. Then there is a cardinal μ such that F preserves κ-compact objects for all regular $\kappa \gg \mu$.

Proof. See Stenzel (2019, Lemma 8.3.4). □

Notation 3.8. The following group of statements will in each case claim that a certain comparison holds for all κ "sufficiently large" or "large enough." That means in each case there is a cardinal μ such that for all $\kappa \gg \mu$ the given statement holds true. As we are not interested in a precise formula for the lower bound μ, we generally will not make the cardinal μ explicit. Instead, we note that we will have to impose the condition on κ to be "large enough" only finitely often and eventually take the corresponding supremum.

Definition 3.9. *Let $\mathbb{M}$ be a model category. Say $\mathbb{M}$ has a* theory of minimal fibrations *if there is a pullback stable class $\mathcal{F}^{\min}_{\mathbb{M}}$ of fibrations in $\mathbb{M}$ – the class of* minimal fibrations *– such that the following hold.*

(1) Let $p\colon X \twoheadrightarrow Y$ and $q\colon X' \twoheadrightarrow Y$ be minimal fibrations. Then every weak equivalence between X and X' over Y is an isomorphism.
(2) For every fibration $p\colon X \twoheadrightarrow Y$ in $\mathbb{M}$, there is an acyclic cofibration $M \overset{\sim}{\hookrightarrow} X$ such that the restriction $M \to Y$ is a minimal fibration.

Lemma 3.10. *Let $\mathbb{M}$ be a model category with a theory of minimal fibrations. Let T be a class of maps in $\mathbb{M}$ such that the left Bousfield localization $\mathcal{L}_T\mathbb{M}$ exists. Then the model category $\mathcal{L}_T\mathbb{M}$ has a theory of minimal fibrations.*

Proof. Given a model category $\mathbb{M}$ and a class T of maps in $\mathbb{M}$ as stated, simply define the class $\mathcal{F}^{\min}_T$ of minimal fibrations in $\mathcal{L}_T\mathbb{M}$ to be the class of fibrations in $\mathcal{L}_T\mathbb{M}$ which are minimal fibrations in $\mathbb{M}$. Pullback stability of $\mathcal{F}^{\min}_T$ is immediate. Property 1 follows readily, as T-local weak equivalences between T-local fibrations are weak equivalences in $\mathbb{M}$ itself. For Property 2, let $p\colon X \twoheadrightarrow Y$ be a fibration in $\mathcal{L}_T\mathbb{M}$. By the assumption that $\mathbb{M}$ has a theory of minimal fibrations, there is an acyclic cofibration $M \overset{\sim}{\hookrightarrow} X$ in $\mathbb{M}$ such that the restriction $M \twoheadrightarrow Y$ is a minimal fibration in $\mathbb{M}$. But $M \to X$ is a weak equivalence from the fibration $M \twoheadrightarrow Y$ to the fibration $p\colon X \twoheadrightarrow Y$ over Y. The latter is a fibration in $\mathcal{L}_T\mathbb{M}$, and it hence follows by Hirschhorn (2003, Proposition 3.4.6) that $M \twoheadrightarrow Y$ is a fibration in $\mathcal{L}_T\mathbb{M}$, too. □

Proposition 3.11. *Let $\mathbb{M}$ be a combinatorial model category.*

(1) For all sufficiently large regular cardinals κ, an object C in $\mathrm{Ho}_\infty(\mathbb{M})$ is κ-compact if there is a κ-compact object $D \in \mathbb{M}$ such that $C \simeq D$ in $\mathrm{Ho}_\infty(\mathbb{M})$.
(2) Suppose $\mathbb{M}$ has a theory of minimal fibrations. Then the converse of Part 1 holds.

Proof. For Part 1 and κ large enough, κ-filtered colimits in $\mathbb{M}$ are homotopy colimits and the κ-compact objects in $\mathbb{M}$ are exactly the κ-compact objects in the quasi-category $N(\mathbb{M})$. So

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the localization $N(\mathbb{M}) \to \mathrm{Ho}_\infty(\mathbb{M})$ preserves κ-filtered colimits and hence is κ-accessible. The statement now follows from Lemma 3.7.

For Part 2, we note that by our assumption and by Dugger's presentation theorem for combinatorial model categories (Dugger, 2001*a*, Theorem 1.1), it suffices to show the statement for objects $C \in J^\kappa$ on the one hand and left Bousfield localizations of simplicial presheaf categories $\mathrm{sPsh}(\mathbb{C})$ on the other. Indeed, given a combinatorial model category $\mathbb{M}$ together with a category $\mathbb{C}$, a set $T \subset \mathrm{sPsh}(\mathbb{C})$ of arrows and a Quillen-equivalence of the form:

$$\mathcal{L}_T(\mathrm{sPsh}(\mathbb{C}))_{\mathrm{proj}} \underset{R}{\overset{L}{\rightleftarrows}} \mathbb{M},$$

suppose we have shown the statement for all $C \in J^\kappa$ and all κ large enough in the case of $\mathcal{L}_T(\mathrm{sPsh}(\mathbb{C}))_{\mathrm{proj}}$. Then, as both categories $\mathbb{M}$ and $\mathrm{sPsh}(\mathbb{C})$ are presentable, we find $\kappa \gg |\mathbb{C}|$ large enough such that the right adjoint R preserves κ-compact objects. Certainly, the derived functors $\mathbb{L}L$ and $\mathbb{R}R$ preserve κ-compactness in $\mathrm{Ho}_\infty(\mathbb{M})$, so whenever an object $C \in \mathrm{Ho}_\infty(\mathbb{M})$ is contained in J^κ, we may choose a κ-compact presheaf $D \in \mathrm{sPsh}(\mathbb{C})$ weakly equivalent to $\mathbb{R}RX$. Without loss of generality D is cofibrant by Dugger (2001*a*, Proposition 2.3.(iii)) and so $L(D)$ is κ-compact in $\mathbb{M}$ and presents C in $\mathrm{Ho}_\infty(\mathbb{M})$.

Now, every κ-compact object $A \in \mathrm{Ho}_\infty(\mathbb{M})$ is the retract of an object $C \in J^\kappa$ by Lemma 3.7.1. We thus may present A by a bifibrant object $B \in \mathbb{M}$, and C by a κ-compact bifibrant object $D \in \mathbb{M}$ again via Dugger (2001*a*, Proposition 2.3.(iii)) and by the above. This yields a map $j\colon B \to D$ with homotopy retract $r\colon D \to B$ in $\mathbb{M}$. Pick a minimal fibrant object $\iota\colon M \hookrightarrow B$. Since every acyclic cofibration between fibrant objects allows a retract ρ itself, we see that M is a homotopy retract of D. Hence, the composition $(\rho r)(j\iota)$ is homotopic to the identity 1_M and thus a homotopy equivalence (Hovey 1999, Theorem 1.2.10.(iv)). It follows that it is an isomorphism in virtue of minimality of M. Thus, M is a retract of D and as such κ-compact in $\mathbb{M}$ itself.

Therefore, assume $\mathbb{M} = \mathcal{L}_T(\mathrm{sPsh}(\mathbb{C}))_{\mathrm{proj}}$, and suppose $C \in \mathrm{Ho}_\infty(\mathbb{M})$ is contained in J^κ. The representatives for the colimits in the construction of $(J^{\kappa,\alpha} | \alpha < \kappa)$ can be chosen to be homotopy colimits of strict diagrams $F\colon I \to \mathbb{M}$ for κ-small categories I by Lurie (2009, Proposition 4.2.3.14) and Lurie (2017, Proposition 1.3.4.25). Hence, they can be computed according to the Bousfield–Kan formula:

$$\mathrm{hocolim} F = \mathrm{coeq}\left(\coprod_{i \to j} F(i) \otimes N(j/I)^{\mathrm{op}} \rightrightarrows \coprod_{i} F(i) \otimes N(i/I)^{\mathrm{op}} \right)$$

because $\mathbb{M} = \mathcal{L}_T \mathrm{sPsh}(\mathbb{C})_{\mathrm{proj}}$ is a simplicial model category (Hirschhorn 2003, Example 18.3.6). But this representative of the homotopy colimit is κ-compact whenever I is κ-small and furthermore each $F(i)$ for $i \in I$ in $\mathbb{M}$ is κ-compact. Hence, by induction, every object $C \in J^\kappa$ is presented by a κ-compact object D in $\mathrm{sPsh}(\mathbb{C})$. □

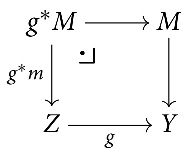
In the following, we generalize Proposition 3.11 to relatively κ-compact maps.

Proposition 3.12. *Let $\mathbb{M}$ be a combinatorial model category.*

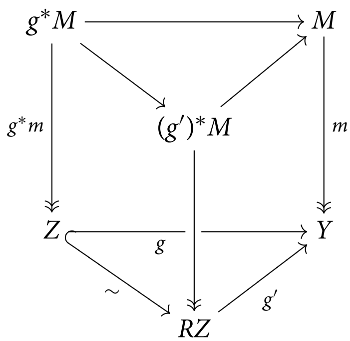
(1) Suppose the converse of Proposition 3.11.1 holds in $\mathbb{M}$. Then for all sufficiently large regular cardinals κ, a morphism $f\colon C \to D$ in $\mathrm{Ho}_\infty(\mathbb{M})$ is relatively κ-compact if there is a relatively κ-compact fibration $p \in \mathbb{M}$ between fibrant objects such that $p \simeq f$ in $\mathrm{Ho}_\infty(\mathbb{M})$.
(2) Suppose $\mathbb{M}$ has a theory of minimal fibrations and κ-compact objects in $\mathbb{M}$ are closed under fiber products. Then the converse of Part 1. holds.

Proof. For Part 1, let $p\colon X \twoheadrightarrow Y$ be a relatively κ-compact fibration between fibrant objects in $\mathbb{M}$. Let $g\colon A \to Y$ be a map in $\mathrm{Ho}_\infty(\mathbb{M})$ with κ-compact domain. In order to show that the pullback of X along g is κ-compact in $\mathrm{Ho}_\infty(\mathbb{M})$, by assumption we can present A by a κ-compact object A' in $\mathbb{M}$. Without loss of generality A' is bifibrant by Dugger (2001*a*, Proposition 2.3.(iii)), so we obtain a map $g'\colon A' \to Y$ presenting g. Also the pullback $(g')^*X$ is a homotopy pullback and it is κ-compact in $\mathbb{M}$ by assumption. Hence, it is κ-compact in $\mathrm{Ho}_\infty(\mathbb{M})$ again by Proposition 3.11. This shows that p is relatively κ-compact in $\mathrm{Ho}_\infty(\mathbb{M})$.

For Part 2, assume that $f\colon C \to D$ is relatively κ-compact in $\mathrm{Ho}_\infty(\mathbb{M})$ and $p\colon X \twoheadrightarrow Y$ is a fibration in $\mathbb{M}$ such that Y is fibrant in $\mathbb{M}$ and $p \simeq f$ in $\mathrm{Ho}_\infty(\mathbb{M})$. By assumption, there is an acyclic cofibration $M \overset{\sim}{\hookrightarrow} X$ such that the restriction $m\colon M \twoheadrightarrow Y$ of p is a minimal fibration. As m and p are homotopy equivalent over Y, the fibration m is relatively κ-compact in $\mathrm{Ho}_\infty(\mathbb{M})$, too. We want to show that m is a relatively κ-compact fibration in $\mathbb{M}$. Therefore, let $g\colon Z \to Y$ be a map for some κ-compact object $Z \in \mathbb{M}$; we have to show that the pullback:



is a κ-compact object in $\mathbb{M}$ as well. By Dugger (2001*a*, Proposition 2.3.(iii)) there is a κ-compact fibrant replacement RZ of Z. Since the object Y itself is fibrant, we obtain an extension $g'\colon RZ \to Y$ of g along the acylic cofibration $Z \overset{\sim}{\hookrightarrow} RZ$ and hence a factorization of the following form.



All three faces of the diagram are pullback squares, and by assumption κ-compact objects in $\mathbb{M}$ are closed under fiber products. Hence, in order to show that the object $g^*M \in \mathbb{M}$ is κ-compact, it suffices to show that the object $(g')^*M \in \mathbb{M}$ is κ-compact.

As RZ is κ-compact in $\mathbb{M}$, it also is κ-compact in the underlying quasi-category $\mathrm{Ho}_\infty(\mathbb{M})$ by Proposition 3.11, and hence so is the (homotopy-)pullback $(g')^*M$ by our assumption on the morphism f that we started with.

By Proposition 3.11 and Dugger (2001*a*, Proposition 2.3.(iii)), we find a cofibrant κ-compact object X together with a weak equivalence $e\colon X \to (g')^*M$. The composition $(g')^*m \circ e\colon X \to RZ$ is a map between κ-compact objects in $\mathbb{M}$, and so we find a factorization $X \overset{\sim}{\hookrightarrow} RX \twoheadrightarrow RZ$ such that RX is κ-compact as well again by Dugger (2001*a*, Proposition 2.3.(iii)). We obtain a weak equivalence $RX \to (g')^*M$ between the respective fibrations over RZ as a lift to the resulting square.

Since $\mathbb{M}$ has a theory of minimal fibrations, there is an acyclic cofibration $j\colon N \overset{\sim}{\hookrightarrow} RX$ such that the restriction $n\colon N \twoheadrightarrow RZ$ of $RX \twoheadrightarrow RZ$ is a minimal fibration. Since j is an acyclic cofibration between fibrations, it has a retraction, and so N is still κ-compact in $\mathbb{M}$. But the composition of weak equivalences $N \simeq (g')^*M$ over RZ is a weak equivalence between minimal fibrations and hence is an isomorphism. Thus, $(g')^*M$ is κ-compact in $\mathbb{M}$. □

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Corollary 3.13. *Let $\mathbb{P}$ be an Eilenberg–Zilber category in the sense of Cisinski (2014, Section 2.1) and $\mathbb{M} = \mathrm{sPsh}(\mathbb{P})_{\mathrm{inj}}$ be the category of simplicial presheaves on $\mathbb{P}$ equipped with the injective model structure. Then for all sufficiently large regular cardinals κ, a morphism $f \in \mathrm{Ho}_\infty(\mathbb{M})$ is relatively κ-compact if and only if there is a κ-small fibration $p \in \mathbb{M}$ between fibrant objects such that $p \simeq f$ in $\mathrm{Ho}_\infty(\mathbb{M})$.*

Proof. The model category $\mathbb{M}$ supports a theory of minimal fibrations as shown in Cisinski (2014, 2.13–2.16), and κ-small objects in (simplicial) presheaf categories are closed under fiber products for infinite κ. Thus, the statement follows from Proposition 3.12 for $\mathbb{M} = \mathrm{sPsh}(\mathbb{P})_{\mathrm{inj}}$. □

3.1 The projective case

We now make use of the observations in Section 2 to generalize Corollary 3.13 to the category of simplicial presheaves over arbitrary small simplicial categories.

Therefore, we want to make use of Dugger (2001*b*, Proposition 5.10, Corollary 6.5) which shows that any zig-zag $\mathbb{M}_1 \leftarrow \cdots \rightarrow \mathbb{M}_n$ of Quillen-equivalences between combinatorial model categories can be reduced to a single Quillen-equivalence whenever either $\mathbb{M}_1$ or $\mathbb{M}_n$ is a "standard presentation" of the form $\mathcal{L}_T\mathrm{sPsh}(\mathbb{C})_{\mathrm{proj}}$ for some small category $\mathbb{C}$ and some set of maps $T \subset \mathrm{sPsh}(\mathbb{C})$. In our case however, we we wish to start with model categories of the form $\mathcal{L}_T\mathbf{sPsh}(\mathbf{C})_{\mathrm{proj}}$ for general small *simplicial* categories $\mathbf{C}$ instead. Therefore, we show a simplicially enriched version of Dugger (2001*b*, Proposition 5.10) first.

Proposition 3.14. *Let $\mathbf{C}$ be a small simplicial category, and $\mathbb{M}$, $\mathbb{N}$ be simplicial model categories together with a simplicial Quillen-equivalence $(L, R)\colon \mathbb{N} \xrightarrow{\sim} \mathbb{M}$. Let T be a class of arrows in $\mathrm{sPsh}(\mathbf{C})$ such that its left Bousfield localization exists, and let $(F, G)\colon \mathcal{L}_T\mathbf{sPsh}(\mathbf{C})_{\mathrm{proj}} \to \mathbb{M}$ be a simplicial Quillen pair. Then there is a simplicial Quillen pair $(F', G')\colon \mathcal{L}_T\mathbf{sPsh}(\mathbf{C})_{\mathrm{proj}} \to \mathbb{N}$ such that the functors $L \circ F'$ and F are Quillen-homotopic in the sense of Dugger (2001b, Definition 5.9). In other words, simplicial Quillen pairs with domain $\mathcal{L}_T\mathbf{sPsh}(\mathbf{C})_{\mathrm{proj}}$ can be lifted up to homotopy along Quillen-equivalences.*

Proof. Let λ and ρ denote cofibrant and fibrant replacements, respectively, and $\mathbb{L} = \lambda^*$, $\mathbb{R} = \rho^*$ denote their associated left and right derivations of functors. Let the composition

$$\lambda\mathbb{R}(R)Fy\colon \mathbf{C} \to \mathbb{N}$$

be denoted by p. Note that the left and right derivation $\mathbb{L}$ and $\mathbb{R}$ of simplicial functors may be chosen to be simplicial again by Riehl (2014, Corollary 13.2.4); thus, p is a simplicial functor and we can consider the simplicially enriched left Kan extension:

$$\begin{array}{ccc} \mathbf{C} & \xrightarrow{\;p\;} & \mathbb{N}. \\ {\scriptstyle y}\big\downarrow & \nearrow_{\mathrm{Lan}_y p} & \\ \mathbf{sPsh}(\mathbf{C}) & & \end{array}$$

We claim that $F' := \mathrm{Lan}_y p$ is the left Quillen functor we are looking for. First, let us construct the Quillen homotopies connecting $L \circ F'$ and F.

Recall that, as explained for instance in Kelly (2005, 4.31), for every presheaf $X \in \mathbf{sPsh}(\mathbf{C})$ the object $\mathrm{Lan}_y p(X)$ is the colimit of p weighted by X, that is,

$$F' = \_ \star p.$$

The left Quillen functor $L\colon \mathbb{N} \to \mathbb{M}$ is a left adjoint and hence preserves weighted colimits; thus, we have that $L \circ F' \cong \_ \star Lp$. Furthermore, by Gambino (2010, Theorem 3.3), the weighted

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colimit functor

$$\_ \star \_ : \mathbf{sPsh}(\mathbf{C})_{\mathrm{proj}} \times [\mathbf{C}, \mathbb{M}]_{\mathrm{inj}} \to \mathbb{M}$$

is a left Quillen bifunctor. In particular, for cofibrant presheaves $X \in \mathbf{sPsh}(\mathbf{C})_{\mathrm{proj}}$ the X-weighted colimit

$$X \star \_ : [\mathbf{C}, \mathbb{M}]_{\mathrm{inj}} \to \mathbb{M}$$

is a left Quillen functor. But both Fy and $Lp \cong \mathbb{L}(L)\mathbb{R}(R)Fy$ are cofibrant objects in $[\mathbf{C}, \mathbb{M}]_{\mathrm{inj}}$: the former because representables are projectively cofibrant and F preserves cofibrant objects, and the latter because L preserves cofibrant objects. Thus, if $\rho_{Fy} \colon Fy \to r(Fy)$ denotes an injective fibrant replacement of Fy, the counit $\varepsilon_{r(Fy)} \colon Lp \Rightarrow r(Fy)$ of the Quillen-equivalence (L, R) induces a span of natural weak equivalences between the cofibrant objects Lp, $r(Fy)$, and Fy. Thus, for cofibrant presheaves $X \in (\mathrm{sPsh}(\mathbf{C}))_{\mathrm{proj}}$, we obtain a zig-zag of natural weak equivalences between $X \star Lp$ and $X \star Fy$. But $\_ \star Fy$ is just F (by Kelly 2005, 4.51), so we have constructed a span of Quillen homotopies between $L \circ F'$ and F.

Second, the fact that $F' \colon \mathbf{sPsh}(\mathbf{C})_{\mathrm{proj}} \to \mathbb{N}$ is a left Quillen functor with right adjoint $G'(N) = \mathbb{N}(p\ \_\, , N)$ was basically already shown above (following for instance, as it were, from Gambino 2010, Theorem 3.3.).

We are left to show that, third, the Quillen pair

$$(F', G') \colon \mathbf{sPsh}(\mathbf{C})_{\mathrm{proj}} \to \mathbb{N}$$

descends to the localization at T whenever F does so. That is, we have to show that every arrow $f \in T$ is mapped to a weak equivalence by F' in $\mathbb{N}$ assuming every such arrow is mapped to a weak equivalence by F in $\mathbb{M}$. Without the loss of generality, all arrows $f \in T$ have cofibrant domain and codomain. Then, given $f \in T$, the arrow $F(f)$ is a weak equivalence in $\mathbb{M}$, and so is $LF'(f) \in \mathbb{M}$ since F and LF' are Quillen-homotopic. Thus, $\mathbb{R}(R)(LF'(f))$ is a weak equivalence in $\mathbb{N}$, but this arrow is weakly equivalent to $F'(f)$ since (L, R) is a Quillen-equivalence. It follows that $F'(f)$ is a weak equivalence in $\mathbb{N}$ itself. □

Theorem 3.15. *Let* $\mathbf{C}$ *be a small simplicial category,* $T \subset \mathrm{sPsh}(\mathbf{C})$ *be a set of maps and* $\mathbb{M} = \mathcal{L}_T(\mathbf{sPsh}(\mathbf{C}))_{\mathrm{proj}}$. *Then for all sufficiently large regular cardinals* κ*, a morphism* $f \in \mathrm{Ho}_\infty(\mathbb{M})$ *is relatively* κ*-compact if and only if there is a* κ*-small fibration* $p \in \mathbb{M}$ *between fibrant objects such that* $p \simeq f$ *in* $\mathrm{Ho}_\infty(\mathbb{M})$.

Proof. Let $\mathbf{C}$ be a small simplicial category and $T \subset \mathrm{sPsh}(\mathbf{C})$ be a set of maps. By Proposition 2.5, we obtain a relative poset $(\mathbb{P}, V)$ of degree at most ω and a zig-zag of Quillen-equivalences of the form:

$$\mathcal{L}_{y[V]}\mathbf{sPsh}(\mathbb{P})_{\mathrm{inj}} \underset{\mathrm{id}}{\overset{\mathrm{id}}{\leftrightarrows}} \mathcal{L}_{y[V]}\mathbf{sPsh}(\mathbb{P})_{\mathrm{proj}} \underset{\varphi^*}{\overset{\varphi_!}{\leftrightarrows}} \mathcal{L}_{y[F_\Delta(V)]}\mathbf{sPsh}(F_\Delta\mathbb{P})_{\mathrm{proj}}$$

$$\mathcal{L}_{y[F_\Delta(V)]}\mathbf{sPsh}(F_\Delta\mathbb{P})_{\mathrm{proj}} \underset{j^*}{\overset{j_!}{\rightleftarrows}} \mathbf{sPsh}(\mathcal{L}_\Delta(\mathbb{P}, V))_{\mathrm{proj}} \underset{f_n^*}{\overset{(f_n)_!}{\leftrightarrows}} \dots \underset{f_1^*}{\overset{(f_1)_!}{\leftrightarrows}} \mathbf{sPsh}(\mathbf{C})_{\mathrm{proj}}.$$

This yields a zig-zag of Quillen-equivalences:

$$\mathcal{L}_{(y[V]\cup\bar{T})}\mathbf{sPsh}(\mathbb{P})_{\mathrm{inj}} \rightleftarrows \dots \rightleftarrows \mathcal{L}_T\mathbf{sPsh}(\mathbf{C})_{\mathrm{proj}}$$

where $\bar{T} \subset \mathrm{sPsh}(\mathbb{P})$ is obtained from $T \subset \mathrm{sPsh}(\mathbf{C})$ by transferring T along the finitely many Quillen-equivalences successively. We denote the union $y[V] \cup \bar{T} \subset \mathrm{sPsh}(\mathbb{P})$ short-handedly by U.

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By Proposition 3.14, this chain of Quillen-equivalences induces a single Quillen-equivalence:

$$\mathcal{L}_T\mathbf{sPsh}(\mathbf{C})_{\text{proj}} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{L}_U\mathbf{sPsh}(\mathbb{P})_{\text{inj}}. \tag{7}$$

The left Bousfield localization $\mathcal{L}_U\mathbf{sPsh}(\mathbb{P})_{\text{inj}}$ has a theory of minimal fibrations by Lemma 3.10 and Corollary 3.13. Thus, let $\kappa \gg |\mathbf{C}|, |\mathbb{P}|$ be regular and, first, large enough such that Corollary 3.13 applies to $\mathbb{P}$, second, large enough such that Proposition 3.12 applies to $\mathcal{L}_U\mathbf{sPsh}(\mathbb{P})_{\text{inj}}$, and third, large enough such that both F and G preserves κ-compact objects (via Lemma 3.7 or its ordinary categorical analogon as right adjoints between locally presentable categories are accessible again).

Now, let $f \in \text{Ho}_\infty(\mathbb{M})$ be relatively κ-compact. Since the pair (7) is a Quillen-equivalence, the quasi-category $\text{Ho}_\infty(\mathbb{M})$ is equivalent to the underlying quasi-category of $\mathcal{L}_U\mathbf{sPsh}(\mathbb{P})_{\text{inj}}$. Then, by Proposition 3.12, there is a κ-small fibration $p\colon X \twoheadrightarrow Y$ between fibrant objects in $\mathcal{L}_U\mathbf{sPsh}(\mathbb{P})_{\text{inj}}$ presenting f in $\text{Ho}_\infty(\mathbb{M})$. By assumption, both adjoints preserve κ-compact objects, so by Corollary 3.6 the right Quillen functor G preserves κ-small maps. Thus, $Gp\colon GX \twoheadrightarrow GY$ is a κ-small fibration between fibrant objects presenting f in $\text{Ho}_\infty(\mathbb{M})$.

In particular, the converse of Proposition 3.11.1 holds in $\mathbb{M}$ for every such κ, and so the other direction follows directly from Proposition 3.12.1. □

Corollary 3.16. *Let $\mathbb{M}$ be a combinatorial model category.*

(1) For all sufficiently large regular cardinals κ, an object C in $\text{Ho}_\infty(\mathbb{M})$ is κ-compact if and only if there is a κ-compact object $D \in \mathbb{M}$ such that $C \simeq D$ in $\text{Ho}_\infty(\mathbb{M})$.

(2) Let $\mathcal{L}_T(\text{sPsh}(\mathbb{C}))_{\text{proj}}$ be the presentation of $\mathbb{M}$ from Dugger's representation theorem for combinatorial model categories in Dugger (2001a). Then for all sufficiently large regular cardinals κ, a morphism $f \in \text{Ho}_\infty(\mathbb{M})$ is relatively κ-compact if and only if there is a κ-small fibration $p \in \mathcal{L}_T(\text{sPsh}(\mathbb{C}))_{\text{proj}}$ between fibrant objects such that $p \simeq f$ in $\text{Ho}_\infty(\mathbb{M})$.

Proof. For Part 1, one direction is exactly Proposition 3.11.1. For the other direction, let

$$\mathcal{L}_T\text{sPsh}(\mathbb{C})_{\text{proj}} \underset{R}{\overset{L}{\rightleftarrows}} \mathbb{M}$$

be the Quillen-equivalence from Dugger's representation theorem. Then for κ large enough and every κ-compact object $C \in \text{Ho}_\infty(\mathbb{M})$, we obtain a κ-compact object $D \in \mathcal{L}_T\text{sPsh}(\mathbb{C})_{\text{proj}}$ from Theorem 3.15 which presents C. As the left adjoint L preserves κ-compact objects, we find a κ-compact fibrant replacement of $L(D)$ in $\mathbb{M}$ which presents C.

Part 2 is just a special case of Theorem 3.15. □

Remark 3.17. The reason why in Corollary 3.16.2 we do not obtain the comparison result for $\mathbb{M}$ itself is that there is no obvious reason why the Quillen-equivalence:

$$\mathcal{L}_T\text{sPsh}(\mathbb{C})_{\text{proj}} \underset{R}{\overset{L}{\rightleftarrows}} \mathbb{M}$$

given by Dugger's presentation theorem should preserve relatively κ-compact maps. While the right adjoint certainly does preserve such maps, the left adjoint does not seem to exhibit any properties with that respect.

3.2 The injective case

In this section, we prove an analogous result for the injective model structure and get rid of the condition on fibrancy of the bases whenever the localization is left exact. We will make use of Shulman's results (Shulman, 2019a) in two ways. Therefore, applied to the special case relevant

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for this paper, recall the forgetful functor:

$$U\colon \mathrm{sPsh}(\mathbf{C}) \to \mathbf{S}^{\mathrm{Ob}(\mathbf{C})} \tag{8}$$

with right adjoint:

$$G\colon \mathbf{S}^{\mathrm{Ob}(\mathbf{C})} \to \mathrm{sPsh}(\mathbf{C}).$$

The functor G takes objects $W \in \mathbf{S}^{\mathrm{Ob}(\mathbf{C})}$ to the presheaf evaluating an object $C \in \mathbf{C}$ at

$$G(W)(C) := \prod_{C' \in \mathbf{C}} W(C')^{\mathbf{C}(C',C)} \in \mathbf{S}.$$

The adjoint pair (U, G) gives rise to a comonad on $\mathrm{sPsh}(\mathbf{C})$ with standard resolution:

$$\mathsf{C}_\bullet(G, UG, U\ \_\ )\colon \mathrm{sPsh}(\mathbf{C}) \to \mathrm{sPsh}(\mathbf{C})^{\Delta}.$$

The associated cobar construction $\mathsf{C}(G, UG, U\ \_\ )\colon \mathrm{sPsh}(\mathbf{C}) \to \mathrm{sPsh}(\mathbf{C})$ is then defined as the pointwise totalization:

$$\mathrm{Tot}(\mathsf{C}_\bullet(G, UG, U\ \_\ )) = \int_{[n]\in\Delta} (\mathsf{C}_n(G, UG, U\ \_\ ))^{\Delta^n}.$$

A crucial observation of Shulman is that the cobar construction takes (acyclic) projective fibrations to pointwise weakly equivalent (acyclic) injective fibrations. More precisely, the natural coaugmentation $\eta\colon \mathrm{id} \Rightarrow \mathsf{C}(G, UG, U\ \_\ )$ is a pointwise weak equivalence, and the arrow $\mathsf{C}(G, UG, Up)$ is an (acyclic) injective fibration whenever p is an (acyclic) projective fibration. All this is covered in Shulman (2019*a*, Section 8) in much greater generality. It is not hard to see that the cobar construction preserves κ-smallness (for κ large enough).

Lemma 3.18. *Let* $\mathbf{C}$ *be a small simplicial category and* $f\colon X \to Y$ *be a* κ*-small map in* $\mathrm{sPsh}(\mathbf{C})$ *for* κ *large enough. Then* $\mathsf{C}(G, UG, Uf)$ *is* κ*-small, too.*

Proof. The forgetful functor (8) preserves κ-smallness of both objects and maps by Remark 3.1. Hence, by Lemma 3.4, the right adjoint G preserves κ-smallness of maps, too. It follows that for every κ-small map $f\colon X \to Y$ in $\mathrm{sPsh}(\mathbf{C})$, the map $\mathsf{C}_\bullet(G, UG, Uf)$ of cosimplicial objects is levelwise κ-small. Thus, we are only left to show that totalization preserves κ-smallness of cosimplicial objects. But, being a subobject of a countable product of κ-small simplicial sets, the statement follows. □

Therefore, we directly obtain an analogue of Theorem 3.15 for the injective model structure as follows.

Proposition 3.19. *Let* $\mathbf{C}$ *be a small simplicial category,* $T \subset \mathrm{sPsh}(\mathbf{C})$ *be a set of maps and* $\mathbb{M} = \mathcal{L}_T\mathbf{sPsh}(\mathbf{C})_{\mathrm{inj}}$. *Then for all sufficiently large regular cardinals* κ, *a morphism* $f \in \mathrm{Ho}_\infty(\mathbb{M})$ *is relatively* κ*-compact if and only if there is a* κ*-small fibration* $p \in \mathbb{M}$ *between fibrant objects such that* $p \simeq f$ *in* $\mathrm{Ho}_\infty(\mathbb{M})$.

Proof. Let f be relatively κ-compact in $\mathrm{Ho}_\infty(\mathbb{M})$. By Theorem 3.15, there is a κ-small fibration $p\colon X \twoheadrightarrow Y$ between fibrant objects in $\mathcal{L}_T\mathrm{sPsh}(\mathbf{C})_{\mathrm{proj}}$ such that $p \simeq f$ in the underlying quasi-category. Hence, by Lemma 3.18 and Shulman (2019*a*, Section 8), the map

$$\mathsf{C}(G, UG, Up)\colon \mathsf{C}(G, UG, UX) \twoheadrightarrow \mathsf{C}(G, UG, UY)$$

is a κ-small injective fibration between injectively fibrant objects. But the coaugmentations η_X and η_Y are pointwise weak equivalences in $\mathcal{L}_T\mathbf{sPsh}(\mathbf{C})_{\mathrm{inj}}$, and so the objects $\mathsf{C}(G, UG, UX)$ and $\mathsf{C}(G, UG, UY)$ are T-local and thus fibrant in $\mathbb{M}$. Hence, the map $\mathsf{C}(G, UG, Up)$ is a fibration between fibrant objects in $\mathbb{M}$.

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The other direction follows immediately from Theorem 3.15, since every injective fibration is a projective fibration. □

Remark 3.20. Whenever $\mathbb{M}$ satisfies the fibration extension property for relatively κ-compact maps (Stenzel 2019, Definition 2.2.1), we can get rid of the fibrancy condition on the bases of maps in Proposition 3.19. That is, because in that case every relatively κ-compact fibration is weakly equivalent to a relatively κ-compact fibration with fibrant base. For example, every left exact left Bousfield localization of $\mathrm{sPsh}(\mathbf{C})_{\mathrm{inj}}$ is a type-theoretic model topos by Shulman (2019*a*, Corollary 8.31, Theorem 10.5) and hence has univalent universes (with fibrant base) for κ-small fibrations for every regular κ large enough (Shulman 2019*a*, Theorem 5.22). It follows that the class S_κ of κ-small maps does satisfy the fibration extension property in every left exact left Bousfield localization of $\mathrm{sPsh}(\mathbf{C})_{\mathrm{inj}}$ (Stenzel 2019, Lemma 2.2.2).

Theorem 3.21. *Let* $\mathbf{C}$ *be a small simplicial category, and let* $T \subset \mathrm{sPsh}(\mathbf{C})$ *be a set of maps such that the localization* $\mathbb{M} = \mathcal{L}_T\mathbf{sPsh}(\mathbf{C})_{\mathrm{inj}}$ *is left exact. Then for all sufficiently large regular cardinals* κ*, a morphism* $f \in \mathrm{Ho}_\infty(\mathbb{M})$ *is relatively* κ*-compact if and only if there is a* κ*-small fibration* $p \in \mathbb{M}$ *such that* $p \simeq f$ *in* $\mathrm{Ho}_\infty(\mathbb{M})$.

Proof. Immediate by Proposition 3.19 and Remark 3.20. □

4. Object Classifiers and Weak Tarski Universes

We conclude with a comment on the relevance of these results for the $(\infty, 1)$-categorical semantics of homotopy type theory. Let $\mathcal{M}$ be an ∞-topos and let $\mathbf{C}$ be a small simplicial category with a set T of arrows in $\mathrm{sPsh}(\mathbf{C})$ such that the localization $\mathbf{sPsh}(\mathbf{C})_{\mathrm{inj}} \to \mathcal{L}_T\mathbf{sPsh}(\mathbf{C})_{\mathrm{inj}}$ is left exact and presents $\mathcal{M}$. Then, $\mathbb{M} := \mathcal{L}_T\mathbf{sPsh}(\mathbf{C})_{\mathrm{inj}}$ is a type-theoretic model category as shown in Gepner and Kock (2012, Section 7). Shulman recently has shown in Shulman (2019*a*) (among other results) that this presentation $\mathbb{M}$ in fact can be enhanced to a type-theoretic model *topos* and hence exhibits an infinite sequence of categorical models of univalent strict Tarski universes. Furthermore, as for example stated in the Introduction of Gepner and Kock (2012), it is somewhat folklore to assume that these categorical models of universes are object classifiers in $\mathcal{M}$, and that more generally the object classifiers in $\mathcal{M}$ correspond to categorical models for univalent weak Tarski universes in $\mathbb{M}$. Here, by a categorical model of a weak Tarski universe, we understand a regular (or potentially more specific) cardinal κ together with a fibration that is weakly universal for the class of κ-small fibrations. Weak universality of a fibration $p\colon E \twoheadrightarrow B$ for a class S of fibrations in turn means that p is contained in S, that it is univalent, and that for all fibrations $q\colon X \twoheadrightarrow Y$ in S there is a map $w\colon X \to B$ such that q is the homotopy pullback of p along w. Clearly, every univalent strictly universal fibration is a weakly universal fibration for the same class of maps whenever the model category $\mathbb{M}$ is right proper.

Since all fibrant objects in $\mathbb{M} = \mathcal{L}_T\mathbf{sPsh}(\mathbf{C})_{\mathrm{inj}}$ are cofibrant and $\mathbb{M}$ is right proper indeed, it is easy to see that a univalent weakly universal fibration for a pullback stable class S of fibrations in $\mathbb{M}$ yields a classifying object for the class $\mathrm{Ho}_\infty[S]$ of morphisms in $\mathcal{M}$ and that, vice versa, every classifying object for a pullback stable class T of morphisms in $\mathcal{M}$ yields a univalent weakly universal fibration for the class:

$$\bar{T} := \{f \in \mathcal{F}_{\mathbb{M}} \mid f \in \mathrm{Ho}_\infty(\mathbb{M}) \text{ is in } T\}$$

of maps in $\mathbb{M}$. Here, the higher categorical notion of univalence that characterizes object classifiers corresponds to the model categorical – and hence to the syntactical – notion of univalence by Gepner and Kock (2012, Proposition 7.12).

There is one such pair of classes (S, T) of maps in each case which is relevant for the construction of strict Tarski universes in the internal language of $\mathbb{M}$ on the one hand, and the definition of object classifiers in $\mathcal{M}$ on the other. That is, given a sufficiently large regular cardinal κ, the class S_κ of κ-small fibrations in sPsh(**C**) and the class T_κ of relatively κ-compact maps in $\mathcal{M}$. In the former case, the common constructions of univalent universal fibrations $\pi_\kappa : \tilde{U}_\kappa \twoheadrightarrow U_\kappa$ use various functorial closure properties of S_κ and the fact that an infinite sequence of inaccessible cardinals yields a cumulative hierarchy of universal fibrations which are closed under all standard-type formers in this way. In the latter case, Lurie (2009, Theorem 6.1.6.8) characterizes ∞-toposes in terms of classifying objects $p_\kappa : \tilde{V}_\kappa \to V_\kappa$ for T_κ for all sufficiently large regular cardinals κ.

While the classifying map $p_\kappa : \tilde{V}_\kappa \to V_\kappa$ lifts to a fibration in $\mathbb{M}$ which is weakly universal for $\bar{T}_\kappa$, and π_κ descends to a classifying object for the class $\mathrm{Ho}_\infty[S_\kappa]$, it is a priori unclear whether $S_\kappa \subseteq \bar{T}_\kappa$ and $T_\kappa \subseteq \mathrm{Ho}_\infty[S_\kappa]$ hold. In other words, without a comparison of relative compactness notions as considered in Section 3, it is not clear whether the categorical construction of (either weak or strict) universal κ-small fibrations in $\mathbb{M}$ – which models Tarski universes in the associated type theory – also models universes in the underlying quasi-category. Theorem 3.21 however does show $T_\kappa = \mathrm{Ho}_\infty[S_\kappa]$. In other words, we obtain the following corollary.

Corollary 4.1. *Let $\mathbb{M} = \mathcal{L}_T\mathbf{sPsh}(\mathbf{C})_{\mathrm{inj}}$ be a model topos, and let κ be a sufficiently large regular cardinal. Then a relatively κ-compact map $p \in \mathrm{Ho}_\infty(\mathbb{M})$ is a classifying map for all relatively κ-compact maps in $\mathrm{Ho}_\infty(\mathbb{M})$ if and only if there is a univalent κ-small fibration $\pi \in \mathbb{M}$ which is weakly universal for all κ-small fibrations in $\mathbb{M}$ such that $p \simeq \pi$ in $\mathrm{Ho}_\infty(\mathbb{M})$.* □

Remark 4.2. Let us finish with a note on the closure under standard-type formers of a given Tarski universe, with regard to the existence of "sufficiently large" regular cardinals that has been a standing assumption along the way (Notation 3.8). Given a model topos of the form $\mathbb{M} = \mathcal{L}_T\mathbf{sPsh}(\mathbf{C})_{\mathrm{inj}}$, for a regular cardinal κ to be sufficiently large means to be contained in the class $\mathrm{Shl}(\lambda)$ of regular cardinals sharply larger than a specified cardinal λ associated with the small simplicial category **C** – or to the ∞-topos $\mathcal{M}$ that is.[2] For a universal fibration $\pi_\kappa \in \mathbb{M}$ as in Corollary 4.1 to be closed under the standard-type formers in an appropriate sense (Shulman 2019*a*, Section 6), the cardinal κ has to be assumed to be strongly inaccessible. Thus, if we start with an ∞-topos $\mathcal{M}$ and wish to show that its type-theoretic presentation $\mathbb{M} := \mathcal{L}_T\mathbf{sPsh}(\mathbf{C})_{\mathrm{inj}}$ exhibits a universal fibration for κ-small fibrations that is closed under all standard-type formers, we need an object classifier V_κ in $\mathcal{M}$ classifying relatively κ-compact maps for a strongly inaccessible κ *in* $\mathrm{Shl}(\lambda)$. The same holds if we want to show that a given cumulative hierarchy of universal fibrations associated to strong inaccessibles κ_i in $\mathbb{M}$ yields a corresponding hierarchy of object classifiers in $\mathcal{M}$. Thus, the translation of the categorical structure together with a universe (or even a cumulative infinite hierarchy of such) between homotopy type theory and higher topos theory requires the existence of inaccessibles within any such given class of sharply larger cardinals. Fortunately, for strongly inaccessible cardinals κ, we have $\lambda \ll \kappa$ if and only if $\lambda < \kappa$. We hence do not need to make any large cardinal assumptions beyond the existence of the inaccessibles themselves.

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Notes

1 Shulman in fact argues for a presentation by inverse posets. But since localization commutes with taking opposite categories, this amounts to the same statement.
2 See Shulman (2019*b*) for an illuminating discussion on directly related issues.

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