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HIGHER-LEVEL $\mathfrak{sl}_2$ CONFORMAL BLOCKS DIVISORS ON $\bar{M}_{0,n}$

VALERY ALEXEEV[1], ANGELA GIBNEY[1] AND DAVID SWINARSKI[2]

[1] *Department of Mathematics, The University of Georgia, Athens, GA 30602, USA* (valery@math.uga.edu; agibney@math.uga.edu)

[2] *Department of Mathematics, Lincoln Center Campus, Fordham University, New York, NY 10023, USA* (dswinarski@fordham.edu)

Abstract We study a family of semi-ample divisors on the moduli space of n-pointed genus 0 curves $\bar{M}_{0,n}$ given by higher-level $\mathfrak{sl}_2$ conformal blocks. We derive formulae for their intersections with a basis of 1-cycles, show that they form a basis for the S_n-invariant Picard group, and generate a full-dimensional subcone of the S_n-invariant nef cone. We find their position in the nef cone and study their associated morphisms.

Keywords: moduli space; vector bundles; conformal field theory

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1. Introduction

The moduli stacks $\bar{\mathcal{M}}_{g,n}$ parametrize isomorphism classes of stable n-pointed curves of genus g. The geometry of $\bar{\mathcal{M}}_{g,n}$ is important for understanding the behaviour of families of curves, and also because our intuition as to how moduli spaces of higher-dimensional objects should behave is based to some extent on our evolving understanding of these spaces. A central problem is to describe the morphisms admitted by $\bar{\mathcal{M}}_{g,n}$.

There is a well-known family of vector bundles on $\bar{\mathcal{M}}_{g,n}$, the so-called vector bundles of conformal blocks, that arise from conformal field theory. Each is specified by a Lie algebra $\mathfrak{g}$, a positive integer ℓ, called the level, and an n-tuple of dominant integral weights $\boldsymbol{\lambda}$. These were first constructed in the 1980s by Ueno *et al.* [**33**]; more recently, Fakhruddin [**8**] observed that they are globally generated on $\bar{\mathcal{M}}_{0,n}$. Thus, their first Chern classes $\mathbb{D}^{\mathfrak{g}}_{\ell,\boldsymbol{\lambda}}$, which we call conformal blocks divisors, are globally generated; it is natural to explore the morphisms on $\bar{\mathcal{M}}_{0,n}$ associated with suitably large multiples of these divisors.

Starting with Fakhruddin's 2008 preprint, much inquiry has centred on conformal blocks for $\mathfrak{sl}_k$ and level 1, and their associated morphisms, which are now well understood [**2**, **8**, **10**, **13**, **14**]. Less is known about the morphisms associated with higher-level conformal blocks divisors for any $\mathfrak{g}$. Fakhruddin relates *critical level* $\mathfrak{sl}_2$ conformal blocks (those for which $\sum\lambda_i = 2\ell + 2$) to the geometric invariant theory (GIT) quotients

$(\mathbb{P}^1)^n/\!/_{\boldsymbol{\lambda}}\,\mathrm{SL}_2$, but we know of no other results on the morphisms associated with general $\mathfrak{sl}_2$ conformal blocks.

In this work, we study the higher-level $\mathfrak{sl}_2$ conformal blocks divisors $\mathbb{D}^{\mathfrak{sl}_2}_{\ell,(1,\dots,1)}$ for $1 \leqslant \ell \leqslant g$ on $\bar{\mathcal{M}}_{0,n}$, where $n = 2g+2$, deriving simple formulae for their intersection with a basis of 1-cycles, as well as formulae for their divisor classes. We show that our family forms a basis for the S_n-invariant Picard group, which implies that it generates a full-dimensional subcone of semi-ample divisors in the S_n-invariant nef cone of $\bar{\mathcal{M}}_{0,n}$. We prove that divisors in our family lie on the boundary of the S_n-invariant nef cone, and are generally not log canonical. We analyse the morphisms defined by the linear series $|m\mathbb{D}^{\mathfrak{sl}_2}_{\ell,(1,\dots,1)}|$ for $m \gg 0$. We show that all of the associated morphisms are birational divisorial contractions on $\bar{\mathcal{M}}_{0,n}/S_n$, and we use these divisors $\mathbb{D}^{\mathfrak{sl}_2}_{\ell,(1,\dots,1)}$ to obtain nef divisors on $\bar{\mathcal{M}}_{2g+2}$.

Note that our family contains all the non-trivial $\mathfrak{sl}_2$ divisors with weights $\boldsymbol{\lambda} = (1,\dots,1)$. Indeed, $\mathbb{D}^{\mathfrak{sl}_2}_{\ell,\boldsymbol{\lambda}}$ is trivial for $2\ell \geqslant \sum_i \lambda_i$, by [**8**], and for odd $\sum \lambda_i$, by Proposition 3.4. As $\bar{\mathcal{M}}_{0,n}$ is a fine moduli space, we hereafter work on the smooth projective variety $\bar{M}_{0,n}$ that represents it.

1.1. Associated morphisms

Fakhruddin proved that the morphisms given by $\mathfrak{sl}_2$ conformal blocks divisors factor through Hassett's weighted spaces (see [**8**, **21**]). We show that those divisors in our family that span extremal rays of the S_n-invariant nef cone define morphisms to varieties that may be constructed as GIT quotients and have interpretations as moduli spaces (see § 6). For example, the following hold.

- $\mathbb{D}^{\mathfrak{sl}_2}_{1,(1,\dots,1)}$ defines a morphism from $\bar{M}_{0,n}$ to Satake's compactification of A_g (see Theorem 6.2). Its image has Picard rank 1 [**2**], and can be constructed as a GIT quotient [**13**].

- $\mathbb{D}^{\mathfrak{sl}_2}_{2,(1,\dots,1)}$ is the pullback from the hyperelliptic locus in $\bar{M}_g$ of $12\lambda - \delta_0$, and its image under the linear system $|12\lambda - \delta_0|$ is expected to have a modular interpretation.

- $\mathbb{D}^{\mathfrak{sl}_2}_{g-1,(1,\dots,1)}$ defines the morphism from $\bar{M}_{0,n}$ to the GIT quotient $\mathrm{Con}(n)/\!/_{(\gamma,\boldsymbol{c})}\,\mathrm{SL}_3$, where $\gamma = (g-2)/g$ and $\boldsymbol{c} = (1/g,\dots,1/g)$. Here, $\mathrm{Con}(n)$ is the space of n-pointed conics in $\mathbb{P}^2$ [**15**].

- The divisor $\mathbb{D}^{\mathfrak{sl}_2}_{g,(1,\dots,1)}$ defines the morphism from $\bar{M}_{0,n}$ to the GIT quotient $(\mathbb{P}^1)^n/\!/\,\mathrm{SL}_2$ with the symmetric linearization.

We show the surprising fact that, for $\ell \geqslant \frac{1}{3}n - 1$, the divisor $\mathbb{D}^{\mathfrak{sl}_2}_{\ell,(1,\dots,1)}$ lies in the cone generated by conformal blocks divisors that give maps to the GIT quotients $\mathrm{Con}(n)/\!/_{(\gamma,\boldsymbol{c})}\,\mathrm{SL}_3$. Giansiracusa [**15**] has exhibited $\mathfrak{sl}_k$ level 1 conformal blocks divisors that give rise to maps to the GIT quotients $\mathrm{Con}(n)/\!/_{(\gamma,\boldsymbol{c})}\,\mathrm{SL}_3$ when $\gamma = 0$. This raises the following natural question.

Question 1.1. Is there a more general correspondence between higher-level divisors and these GIT quotients when $\gamma \neq 0$?

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1.2. Outline of the paper

In § 2 we give definitions and references for divisors and curves on $\bar{M}_{g,n}$. In § 3 we give several fundamental results about $\mathfrak{sl}_2$ conformal blocks. In § 4, we give the intersection formulae for the S_n-invariant divisors $\mathbb{D}^{\mathfrak{sl}_2}_{\ell,(1,\dots,1)}$ with a basis for the 1-cycles on $\bar{M}_{0,n}/S_n$, our main tool for studying the morphisms associated with the divisors $\mathbb{D}^{\mathfrak{sl}_2}_{\ell,(1,\dots,1)}$. In § 4.1 we give formulae for the classes of four of the divisors $\mathbb{D}^{\mathfrak{sl}_2}_{\ell,(1,\dots,1)}$, and in § 5 we show that the divisors $\mathbb{D}^{\mathfrak{sl}_2}_{\ell,(1,\dots,1)}$ lie on the boundary of the symmetric nef cone, and that they reside in the part of the cone that was not previously well understood. In § 6, we study morphisms associated with the divisors $\mathbb{D}^{\mathfrak{sl}_2}_{\ell,(1,\dots,1)}$. In § 7, we explain how conformal blocks on $\bar{M}_{0,n}$ can be used to produce nef divisors on $\bar{M}_n$. Finally, in § 8, we prove the independence of two families of curves on $\bar{M}_{0,n}$.

1.3. Software

A number of results in this paper were first explored through computer calculations. The third author has written a package, ConformalBlocks, which can be used to compute ranks, divisor classes and intersection numbers of conformal block bundles and divisors in Macaulay2 (see [**20**,**31**]). We also used the software NefWiz and polymake to explore which of the divisors $\mathbb{D}^{\mathfrak{sl}_2}_{\ell,(1,\dots,1)}$ are extremal in the symmetric nef cone, and which of the divisors $\mathbb{D}^{\mathfrak{sl}_2}_{\ell,(1,\dots,1)}$ are log canonical (see [**12**,**18**]).

2. Divisors and curves on the moduli stack

2.1. Divisor classes

As is standard in the literature, λ denotes the first Chern class of the Hodge bundle. For $1 \leqslant i \leqslant n$, we denote by σ_i the n sections of the universal family $\pi\colon \bar{\mathcal{M}}_{g,n+1} \to \bar{\mathcal{M}}_{g,n}$. For ω_π, the rank 1 relative dualizing sheaf, one then has the ψ-classes $\psi_i = c_1(\sigma_i^*(\omega_\pi))$. We refer to the sum $\Psi = \sum_{i=1}^n \psi_i$ as the total Ψ-class. The divisor $\kappa = \kappa_1 = \pi_*(\omega_\pi^2)$ is ample.

We write δ_0 for the class of the boundary component Δ_0, the divisor whose general element has a single non-separating node. For $0 \leqslant i \leqslant \lfloor \frac{1}{2}g \rfloor$, and $J \subset \{1,\dots,n\}$, let $\delta_{i,J}$ be the class of the boundary divisor $\Delta_{i,J}$. The general element of $\Delta_{i,J}$ has a single separating node that breaks the curve into two components, one of which is a curve of genus i and has $|J|+1$ marked points consisting of an attaching point together with points labelled by the set J. If $n = 0$, then it is customary to write δ_i instead of $\delta_{i,\emptyset}$, and if $g = 0$, then one must have that $2 \leqslant |J| \leqslant n-2$, and it is customary to write δ_J rather than $\delta_{0,J}$. By Δ we mean the sum of all the boundary divisors.

By [**28**, Theorem 1] the class of the canonical divisor is given by

$$K_{\bar{M}_{0,n}} = \Psi - 2\Delta = \sum_{i=2}^{\lfloor n/2 \rfloor} \left(\frac{i(n-i)}{(n-1)} - 2 \right) B_i, \quad \text{where } B_j = \sum_{\substack{J \subset \{1,\dots,n\}, \\ |J| = j}} \delta_J. \tag{2.1}$$

In [**26**, Theorem 1.3], the B_j were shown to generate the extremal rays of the cone of effective divisors of the quotient $\bar{M}_{0,n}/S_n$. The set $\{B_j\}_{j=2}^{\lfloor n/2 \rfloor}$ is a basis for $\operatorname{Pic}(\bar{M}_{0,n})^{S_n}$.

2.2. F-curves on $\bar{M}_{0,n}$

An F-curve on $\bar{M}_{0,n}$ is any curve that is numerically equivalent to a one-dimensional component of the boundary. A point in such a stratum parametrizes a nodal genus 0 curve that contains a unique $\mathbb{P}^1$ with four special points (nodes or marked points). By varying the cross ratio of these four points, we obtain a rational curve inside $\bar{M}_{0,n}$. Its class depends only on the partition of the n marked points into four non-empty sets. In order to intersect F-curves with symmetric divisor classes on $\bar{M}_{0,n}$, we need only know the size of the cells of the partition. In other words, a partition $a+b+c+d=n$ of the integer n into four positive integers determines an F-curve class, up to S_n symmetry. We denote such a curve by $F_{a,b,c,n-(a+b+c)}$, or, even more briefly, by $F_{a,b,c}$ (see [**19**, Theorem 2.2, Figure 2.3]).

We next define three families of independent F-curves on $\bar{M}_{0,n}$, used to prove Theorem 4.12.

Theorem 2.1. *Let $n \geqslant 6$ and define g by $n = 2g+2$ or $n = 2g+3$. Each of the following three sets then consists of independent curves.*

(1) $\mathscr{C}_1 = \{F_{1,1,i} \colon 1 \leqslant i \leqslant g\}$.

(2) $\mathscr{C}_2 = \{F_{2,2,i} \colon 1 \leqslant i \leqslant g-1\}$.

(3) $\mathscr{C}_3 = \begin{cases} \{F_{3,3,2i+1} \colon 0 \leqslant i \leqslant k-2\} \cup \{F_{1,1,2i+1} \colon 0 \leqslant i \leqslant k-1\} & \textit{if } g = 2k, \\ \{F_{3,3,2i+1} \colon 0 \leqslant i \leqslant k-3\} \cup \{F_{1,1,2i+1} \colon 0 \leqslant i \leqslant k-1\} & \textit{if } g = 2k-1. \end{cases}$

Theorem 2.1, proved in § 8, leads to the following well-known corollary.

Corollary 2.2. *$\mathscr{C}_1$ is a basis for $N_1(\bar{M}_{0,n}/S_n, \mathbb{Q})$.*

3. $\mathfrak{sl}_2$ conformal blocks

The main reference for the construction of vector bundles of conformal blocks is Ueno's recent monograph [**33**] (see also [**3**, **8**, **27**]).

In this section we recall the factorization and fusion rules for $\mathfrak{sl}_2$ conformal block bundles (CB-bundles), used to prove Proposition 3.4. We work with $\mathfrak{g} = \mathfrak{sl}_2$ and an arbitrary (but fixed) level ℓ. In this case, the root system may be identified with $\mathbb{Z}$, and the dominant integral weights λ_i of level less than or equal to ℓ are simply the non-negative integers $0 \leqslant \lambda_i \leqslant \ell$. Let $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n)$ be a vector of dominant integral weights of level ℓ.

Notation. We write

$$r_{\boldsymbol{\lambda}} := \operatorname{rank} \mathbb{V}(\mathfrak{sl}_2, \ell, \boldsymbol{\lambda}). \tag{3.1}$$

The rank of the vector bundle $\mathbb{V}(\mathfrak{sl}_2, \ell, \boldsymbol{\lambda})$ is given by the Verlinde formula [**33**, Example 5.23], but it is often more efficient to compute ranks by using the factorization rules. These may be stated for any simple Lie algebra $\mathfrak{g}$, but we only work with $\mathfrak{g} = \mathfrak{sl}_2$.

Proposition 3.1 (propagation for CB-bundles). *Let $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n)$, and suppose that $\lambda_n = 0$. Then $\mathbb{V}(\mathfrak{g}, \ell, \boldsymbol{\lambda}) = \pi_n^* \mathbb{V}(\mathfrak{g}, \ell, \hat{\lambda})$, where $\hat{\lambda} = (\lambda_1, \dots, \lambda_{n-1})$ and $\pi_n \colon \bar{M}_{g,n} \to \bar{M}_{g,n-1}$ is the map forgetting the nth marked point. In particular, $r_{\boldsymbol{\lambda}} = r_{\hat{\boldsymbol{\lambda}}}$.*

Proposition 3.2 (factorization for $\mathfrak{sl}_2$ CB-bundles). *Let $\boldsymbol{\mu} \cup \boldsymbol{\nu}$ be a partition of the vector $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n)$ into two vectors each of length at least 2. Then,*

$$r_{\boldsymbol{\lambda}} = \sum_{\alpha=0}^{\ell} r_{\boldsymbol{\mu} \cup \alpha} r_{\boldsymbol{\nu} \cup \alpha}.$$

Factorization can be used to compute $r_{\boldsymbol{\lambda}}$ in terms of ranks of conformal block bundles on $\bar{M}_{0,3} = pt$; these numbers are known as the fusion rules. For $\mathfrak{g} = \mathfrak{sl}_2$, these are given as follows.

Proposition 3.3 (fusion rules for $\mathfrak{sl}_2$ [3, Lemma 4.2, Corollary 4.4]). *Let $n = 3$. Then,*

$$r_{(\lambda_1,\lambda_2,\lambda_3)} = \begin{cases} 1 & \text{if } \sum_{i=1}^{3} \lambda_i \equiv 0 \mod 2,\ \sum_{i=1}^{3} \lambda_i \leqslant 2\ell \text{ and } \lambda_i \leqslant \frac{1}{2} \sum_{i=1}^{3} \lambda_i, \\ 0 & \text{otherwise.} \end{cases}$$

The next proposition contains some easy technical results that are used throughout the paper.

Proposition 3.4. *Let $\mathfrak{g} = \mathfrak{sl}_2$.*

(1) (*Odd sum rule.*) *If $\sum_{i=1}^{n} \lambda_i$ is odd, then $r_{\boldsymbol{\lambda}} = 0$.*

(2) (*Generalized triangle inequality.*) *If there exists $i \in \{1, \dots, n\}$ such that $\lambda_i > \sum_{j \neq i} \lambda_j$, then $r_{\boldsymbol{\lambda}} = \mathrm{rk}(\mathbb{V}(\mathfrak{sl}_2, \ell, \boldsymbol{\lambda})) = 0$.*

(3) (*Some special four-point ranks.*)

$$\mathrm{rk}(\mathbb{V}(\mathfrak{sl}_2, \ell, (\mu_1, \mu_2, 1, 1))) = \begin{cases} 2 & \text{if } \mu_1 = \mu_2 \text{ and } \mu_1 \notin \{0, \ell\}, \\ 1 & \text{if } \mu_1 = \mu_2 \text{ and } \mu_1 \in \{0, \ell\}, \\ 1 & \text{if } \mu_2 = \mu_1 \pm 2, \\ 0 & \text{otherwise.} \end{cases} \tag{3.2}$$

(4) (*Some special four-point degrees.*)

$$\deg \mathbb{V}(\mathfrak{sl}_2, \ell, (\mu_1, \mu_2, 1, 1)) = \begin{cases} 0 & \text{if } \boldsymbol{\mu} \neq (\ell, \ell, 1, 1), \\ 1 & \text{if } \boldsymbol{\mu} = (\ell, \ell, 1, 1). \end{cases} \tag{3.3}$$

Proof. The odd sum rule is a special case of the more general fact that $r_{\boldsymbol{\lambda}} = 0$ if $\sum_{i=1}^{n} \lambda_i$ is not in the root lattice (see, for example, [**11**, p. 13]). It can also be proved using factorization and induction, with $n = 3$ as the base case.

The generalized triangle inequality can be proved using factorization and induction, with $n = 3$ as the base case.

For (3), we first consider $r_{\boldsymbol{\mu}}$. By factorization applied to the partition $\boldsymbol{\mu} = (\mu_1, \mu_2) \cup (1, 1)$, we obtain that

$$r_{\boldsymbol{\mu}} = \sum_{\alpha=0}^{\ell} r_{(\mu_1,\mu_2,\alpha)} r_{(1,1,\alpha)}. \tag{3.4}$$

By the fusion rules for $\mathfrak{sl}_2$, we have that $r_{(1,1,\alpha)} = 0$ unless $\alpha \in \{0, 2\}$. Thus,

$$r_{\boldsymbol{\mu}} = r_{(\mu_1,\mu_2,0)} r_{(1,1,0)} + r_{(\mu_1,\mu_2,2)} r_{(1,1,2)} = r_{(\mu_1,\mu_2)} + r_{(\mu_1,\mu_2,2)}.$$

By the fusion rules for $\mathfrak{sl}_2$, $r_{(\mu_1,\mu_2)} = 0$ unless $\mu_1 = \mu_2$, and then this rank is 1. We also have that $r_{(\mu_1,\mu_2,2)} = 0$ unless $\mu_2 \in \{\mu_1 - 2, \mu_1, \mu_1 + 2\}$. Also, $r_{(0,0,2)} = 0$, and $r_{(\ell,\ell,2)} = \mathrm{rk}(\mathbb{V}(\mathfrak{sl}_2, \ell, (\ell, \ell, 2))) = 0$. The result follows.

Part (4) follows from [**8**, Proposition 4.2]. □

4. Intersecting the divisors $\mathbb{D}^{\mathfrak{sl}_2}_{\boldsymbol{\ell},(\mathbf{1},\dots,\mathbf{1})}$ with F-curves

In Theorem 4.2, we give a simple formula for the intersection of the divisors $\mathbb{D}^{\mathfrak{sl}_2}_{\ell,(1,\dots,1)}$ with a basis of 1-cycles given by the first family of curves defined in Theorem 2.1 (1). This result is one of our most powerful tools for studying the morphisms associated with the divisors $\mathbb{D}^{\mathfrak{sl}_2}_{\ell,(1,\dots,1)}$.

Definition 4.1. Suppose that n is even. We define

$$r_\ell(j, t) := \mathrm{rank}\, \mathbb{V}(\mathfrak{sl}_2, \ell, (\underbrace{1, \dots, 1}_{j \text{ times}}, t)).$$

Theorem 4.2. *We have that* $\mathbb{D}^{\mathfrak{sl}_2}_{\ell,(1,\dots,1)} \cdot F_{n-i-2,i1,1} = r_\ell(i, \ell) r_\ell(n - i - 2, \ell)$.

Proof. We write $P_\ell = \{0, 1, \dots, \ell\}$, and write $\boldsymbol{\mu} = (\mu_1, \mu_2, \mu_3, \mu_4)$ for elements of P_ℓ^4. We use [**8**, Proposition 2.7] applied to $\mathbb{V}(\mathfrak{sl}_2, \ell, (1, \dots, 1))$ and the symmetric F-curve $F = F_{j_1,j_2,j_3,j_4}$, given by a partition $n = j_1 + j_2 + j_3 + j_4$.

Using the notation from Definition 4.1, this states that

$$\deg(\mathbb{V}(\mathfrak{sl}_2, \ell, (1, \dots, 1))|_F) = \sum_{\boldsymbol{\mu} \in P_\ell^4} \deg \mathbb{V}(\mathfrak{sl}_2, \ell, \boldsymbol{\mu}) \prod_{k=1}^{4} r_\ell(j_k, \mu_k). \tag{4.1}$$

The fusion rules for $\mathfrak{sl}_2$ imply that $r_{(a,b)} = 0$ unless $a = b$, in which case $r_{(a,b)} = 1$. Since our F-curves have two 1s on the spine, the only non-zero summands in (4.1) occur when $\mu_3 = \mu_4 = 1$. By Proposition 3.4, we have that $\deg \mathbb{V}(\mathfrak{sl}_2, \ell, \boldsymbol{\mu}) = 0$ if $\boldsymbol{\mu} \neq (\ell, \ell, 1, 1)$, and $\deg \mathbb{V}(\mathfrak{sl}_2, \ell, \boldsymbol{\mu}) = 1$ otherwise. The formula follows. □

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The following example suggests several corollaries to Theorem 4.2. Consider the matrix of intersection numbers $\mathbb{D}^{\mathfrak{sl}_2}_{\ell,(1,\dots,1)} \cdot F_{n-i-2,i,1,1}$ for $n = 16$, where in the matrix we put $\mathbb{D}^{\mathfrak{sl}_2}_{\ell}$ for $\mathbb{D}^{\mathfrak{sl}_2}_{\ell,(1,\dots,1)}$:

	$\mathbb{D}^{\mathfrak{sl}_2}_1$	$\mathbb{D}^{\mathfrak{sl}_2}_2$	$\mathbb{D}^{\mathfrak{sl}_2}_3$	$\mathbb{D}^{\mathfrak{sl}_2}_4$	$\mathbb{D}^{\mathfrak{sl}_2}_5$	$\mathbb{D}^{\mathfrak{sl}_2}_6$	$\mathbb{D}^{\mathfrak{sl}_2}_7$
$F_{1,1,1}$	1	0	0	0	0	0	0
$F_{1,1,2}$	0	32	0	0	0	0	0
$F_{1,1,3}$	1	0	55	0	0	0	0
$F_{1,1,4}$	0	32	0	40	0	0	0
$F_{1,1,5}$	1	0	63	0	19	0	0
$F_{1,1,6}$	0	32	0	52	0	6	0
$F_{1,1,7}$	1	0	64	0	25	0	1

Note that this matrix has full rank. This shows that the divisors are independent. Moreover, since in all of the columns there are curves that intersect the CB divisors in degree 0, this also shows that the divisors lie on the boundary of the nef cone.

We next give seven corollaries to Theorem 4.2, reflecting that the pattern displayed by the zero/non-zero entries of the matrix above holds in general.

Corollary 4.3.

(1) *If* $i < \ell$, *then* $\mathbb{D}^{\mathfrak{sl}_2}_{\ell,(1,\dots,1)} \cdot F_{n-i-2,i,1,1} = 0$.

(2) *If* $i \not\equiv \ell \mod 2$, *then* $\mathbb{D}^{\mathfrak{sl}_2}_{\ell,(1,\dots,1)} \cdot F_{n-i-2,i,1,1} = 0$.

(3) $\mathbb{D}^{\mathfrak{sl}_2}_{\ell,(1,\dots,1)}$ *intersects at least* $\lceil (\ell + g - 2)/2 \rceil$ *independent curves in degree 0.*

Proof. For the first statement, use the generalized triangle inequality (see Proposition 3.4). For the second statement, use the odd sum rule (see Proposition 3.4). For the third statement, there are $\ell - 1$ curves with $i < \ell$, and $\lceil (g - \ell)/2 \rceil$ curves with $i > \ell$ and $i \not\equiv \ell \mod 2$. □

Of course, the curve count given in Corollary 4.3 (3) is not always sharp, since there could be other curves not of the form $F_{1,1,i}$ that intersect $\mathbb{D}^{\mathfrak{sl}_2}_{\ell,(1,\dots,1)}$ in degree 0.

Lemma 4.4.

(1) $r_\ell(k, k) = 1$ *for all* $1 \leqslant k \leqslant \ell$.

(2) $r_\ell(k, k - 2) = k - 1$ *for all* $2 \leqslant k \leqslant \ell + 1$.

(3) $r_\ell(\ell + 2, \ell) = \ell$.

(4) $r_\ell(\ell + 2p, \ell) \neq 0$ *for* $p \in \mathbb{Z}$, $p \geqslant 0$.

Proof. For (1) we use induction on k, with the base case $r_\ell(1, 1) = 1$, which holds by the fusion rules. Assume that $r_\ell(j, j) = 1$ for $j < k$ and apply factorization with the partition $1^{k-1}|(1, k)$ to get that

$$r_\ell(k, k) = \sum_{0 \leqslant \mu \leqslant \ell} r_\ell(k - 1, \mu) r_{(1,k,\mu)}.$$

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The fusion rules give that $r_{(1,k,\mu)} = 0$ if $\mu < k-1$, $\mu > k+1$ or $\mu = k$, and $r_{(1,k,\mu)} = 1$ if $\mu = k-1$ or $\mu = k+1$. However, if $\mu = k+1$, then $r_\ell(k-1, k+1) = 0$ by the generalized triangle inequality (see Proposition 3.4). So the only non-zero summand in $r_\ell(k,k)$ is $r_\ell(k-1,k-1)$, which is 1 by the induction hypothesis, and so we are done.

For (2) use induction on k with the base case $r_\ell(2,0) = \operatorname{rank} \mathbb{V}(\mathfrak{sl}_2, \ell, (1,1,0)) = 1$, which holds by propagation and the fusion rules. Assume that $r_\ell(j, j-2) = j-1$ for $2 \leqslant j \leqslant k-1$, and apply factorization to $1^{k-1} \cup (1, k-2)$:

$$r_\ell(k, k-2) = \sum_{0 \leqslant \mu \leqslant \ell} r_\ell(k-1, \mu) r_{(1,k-2,\mu)}.$$

By the fusion rules, we have that $r_{(1,k-2,\mu)}$ if $\mu < k-3$, $\mu > k-1$ or $\mu = k-2$. Also, $r_{(1,k-2,\mu)} = 1$ if $\mu = k-3$ or $\mu = k-1$. Thus, there exist only two non-zero summands in $r_\ell(k, k-2)$:

$$r_\ell(k, k-2) = r_\ell(k-1, k-3) + r_\ell(k-1, k-1).$$

By the induction hypothesis, we have that $r_\ell(k-1, k-3) = k-2$, and, by the first statement of this lemma, we have that $r_\ell(k-1, k-1) = 1$. Thus, $r_\ell(k, k-2) = k-1$, as claimed.

For (3) apply factorization using the partition $1^{\ell+1} \cup (1, \ell)$:

$$r_\ell(\ell+2, \ell) = \sum_{0 \leqslant \mu \leqslant \ell} r_\ell(\ell+1, \mu) r_{(1,\ell,\mu)}.$$

We argue, as above, that there exists only one non-zero summand, and it occurs for $\mu = \ell - 1$. Thus, $r_\ell(\ell+2, \ell) = r_\ell(\ell+1, \ell-1)$, and by (2) this is ℓ.

For (4) we use induction on p, with the base case $p = 0$ given by (1). Consider $r_\ell(\ell + 2p, \ell)$ and apply factorization using the partition $1^2 \cup (1^{\ell+2p-2}, \ell)$:

$$r_\ell(\ell+2p, \ell) = \sum_{0 \leqslant \mu \leqslant \ell} r_\ell(2, \mu) \operatorname{rank}(\mathfrak{sl}_2, \ell, (1^{\ell+2p-2}, \ell, \mu)).$$

When $\mu = 0$, by the fusion rules, $r_\ell(2, \mu) = 1$. By induction, $\operatorname{rank}(\mathfrak{sl}_2, \ell, (1^{\ell+2p-2}, \ell, 0)) = \operatorname{rank}(\mathfrak{sl}_2, \ell, (1^{\ell+2p-2}, \ell)) = r_\ell(\ell+2p-2, \ell) \neq 0$. Therefore, $r_\ell(\ell+2p, \ell) \neq 0$. $\square$

Corollary 4.5. *$\mathbb{D}^{\mathfrak{sl}_2}_{\ell,(1,\dots,1)} \cdot F_{1,1,\ell+2q,n-\ell-2-2q} \neq 0$ for $1 \leqslant \ell \leqslant g$ and $q \in \mathbb{Z}$, $0 \leqslant q \leqslant (g-\ell)/2$.*

Proof. By Theorem 4.2, we have that

$$\mathbb{D}^{\mathfrak{sl}_2}_{\ell,(1,\dots,1)} \cdot F_{1,1,\ell+2q,n-\ell-2-2q} = r_\ell(\ell+2q, \ell) r_\ell(n-\ell-2-2q, \ell).$$

By Lemma 4.4 (iv), we have that $r_\ell(\ell+2q, \ell) \neq 0$. Also, since $q \leqslant (g-\ell)/2$, we have that $n - \ell - 2 - 2q = \ell + 2p$ for some $p \geqslant 0$. Thus, by Lemma 4.4 (iv), we have that $r_\ell(n-\ell-2-2q, \ell) \neq 0$ as well. $\square$

Corollary 4.6. *$\{\mathbb{D}^{\mathfrak{sl}_2}_{\ell,(1,\dots,1)} \colon 1 \leqslant \ell \leqslant g\}$ is a basis for $\mathrm{Pic}(\bar{M}_{0,2g+2}/S_{2g+2})$.*

Proof. The matrix of intersection numbers between these divisors and the F-curves $F_{n-i-2,i,1,1}$ is lower triangular with non-zero entries on the diagonal, and so the divisors are linearly independent. To see this, note that by the generalized triangle inequality, Proposition 3.4 (2), one has that $r_\ell(i,\ell) = 0$ if $i < l$. Thus, the entries above the diagonal are all zero. By Corollary 4.5, the diagonal entries are non-zero. □

The following lemma is needed to give formulae for intersection numbers of the extremal $\mathbb{D}^{\mathfrak{sl}_2}_{\ell,(1,\dots,1)}$.

Lemma 4.7.

(1) *Suppose that $\ell = 1$. For $k \in \mathbb{Z}$, $k \geqslant 0$, we then have that $r_1(2k+1,1) = 1$.*

(2) *Suppose that $\ell = 2$. For $k \in \mathbb{Z}$, $k \geqslant 1$, we then have that $r_2(2k,2) = 2^{k-1}$.*

(3) *Suppose that $\ell = 2$. For $k \in \mathbb{Z}$, $k \geqslant 0$, we then have that $r_2(2k+1,1) = 2^k$.*

Proof. We use induction on k and factorization.

For the first formula, by the fusion rules, $r_1(1,1) = 1$. So, suppose that the formula is true up to $k-1$. Factorization and applying the odd sum lemma yield that

$$r_1(2k+1,1) = r_1(2k,0)r_1(2,0) + r_1(2k,1)r_1(2,1) = r_1(2(k-1)+1,1) = 1.$$

For the second two formulae, we may check that $r_2(2,2) = 1$ and $r_1(1,1) = 1$. So, suppose that these two formulae work up to $k-1$. Factorization and applying the odd sum lemma yields that

$$r_2(2k,2) = \sum_{\mu=0}^{2} r_2(2k-1,\mu) r_{(1,2,\mu)} = r_2(2k-1,1) = r_2(2(k-1)+1,1) = 2^{k-1}$$

and

$$r_2(2k+1,1) = r_2(2(k-1)+1,1) r_{(1,1,0)} + r_2(2k,2) r_{(1,1,2)} = 2^{k-1} + 2^{k-1} = 2^k.$$

□

Corollary 4.8. *We have that*

$$\mathbb{D}^{\mathfrak{sl}_2}_{1,(1,\dots,1)} \cdot F_{a,b,c,d} = \begin{cases} 1, & abcd \text{ odd}, \\ 0, & abcd \text{ even}. \end{cases} \tag{4.2}$$

Proof. If $abcd$ is even, then at least one of the four integers, say a, is even. When we apply (4.1) to compute $\mathbb{D}^{\mathfrak{sl}_2}_{1,(1,\dots,1)} \cdot F_{a,b,c,d}$, to get a non-zero summand, we must have $\mu_1 = 0$ to have $r_\ell(a,\mu_1) \neq 0$. (By the odd sum lemma, we need μ_1 even, but $P_\ell = \{0,1\}$ since $\ell = 1$.) Since $\mu_1 = 0$, by propagation, we know that $\mathbb{V}(\mathfrak{sl}_2,\ell,\boldsymbol{\mu})$ is a pullback from $\bar{M}_{0,3} = pt$. Hence, $\deg \mathbb{V}(\mathfrak{sl}_2,1,\boldsymbol{\mu}) = 0$. For $abcd$ odd, the only non-zero summand in (4.1) occurs when $\boldsymbol{\mu} = (1,1,1,1)$. We can compute $\deg \mathbb{V}(\mathfrak{sl}_2,1,(1,1,1,1)) = 1$, and, by Lemma 4.7, $r_1(a,1)r_1(b,1)r_1(c,1)r_1(d,1) = 1$. □

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Corollary 4.9. *We have that*

$$\mathbb{D}^{\mathfrak{sl}_2}_{2,(1,\dots,1)} \cdot F_{a,b,c,d} = \begin{cases} 0, & abcd \text{ odd}, \\ 2^{g-2}, & abcd \text{ even}. \end{cases} \tag{4.3}$$

Proof. If $abcd$ is odd, then if any μ_i is even, $r_\ell(a,\mu_i) = 0$ by the odd sum lemma. We have that $P_\ell = \{0,1,2\}$ since $\ell = 2$, so we only possibly get a non-zero summand in (4.1) when $\boldsymbol{\mu} = (1,1,1,1)$. We can compute $\deg \mathbb{V}(\mathfrak{sl}_2, 2, (1,1,1,1)) = 0$, so this summand is 0. If a and b are even, while c and d are odd, then to get a non-zero summand in (4.1) we must have μ_1 and μ_2 even and μ_3 and μ_4 odd. However, if μ_1 or μ_2 is 0, then, by propagation, we know that $\mathbb{V}(\mathfrak{sl}_2, \ell, \boldsymbol{\mu})$ is a pullback from $\bar{M}_{0,3} = pt$, and hence $\deg \mathbb{V}(\mathfrak{sl}_2, 1, \boldsymbol{\mu}) = 0$. Thus, we only get a non-zero summand in (4.1) when $\boldsymbol{\mu} = (2,2,1,1)$. We compute $\deg \mathbb{V}(\mathfrak{sl}_2, 2, (2,2,1,1)) = 1$, and use Lemma 4.7 to show that

$$r_2(a,2)r_2(b,2)r_2(c,2)r_2(d,2) = 2^{a/2-1}2^{b/2-1}2^{(c-1)/2}2^{(d-1)/2} = 2^{(a+b+c+d)/2-3} = 2^{g-2}.$$

If a, b, c and d are all even, then, as above, we only get a non-zero summand in (4.1) when $\boldsymbol{\mu} = (2,2,2,2)$, and then $\deg \mathbb{V}(\mathfrak{sl}_2, 2, (2,2,2,2)) = 2$. We use Lemma 4.7 (ii) to show that

$$2r_2(a,2)r_2(b,2)r_2(c,2)r_2(d,2) = 2 \cdot 2^{a/2-1}2^{b/2-1}2^{c/2-1}2^{d/2-1} = 2^{g-2}.$$

□

Corollary 4.10. *We have that*

$$\mathbb{D}^{\mathfrak{sl}_2}_{g-1,(1,\dots,1)} \cdot F_{n-i-2,i,1,1} = \begin{cases} 0, & i \neq g-1, \\ g-1, & i = g-1. \end{cases} \tag{4.4}$$

Proof. If $i \leqslant g-2$, by Corollary 4.3, $\mathbb{D}^{\mathfrak{sl}_2}_{g-1,(1,\dots,1)} \cdot F_{n-i-2,i,1,1} = 0$. For $i = g$, by Theorem 4.2,

$$\mathbb{D}^{\mathfrak{sl}_2}_{g-1,(1,\dots,1)} \cdot F_{g,g,1,1} = r_{g-1}(g,g-1) \cdot r_{g-1}(g,g-1).$$

As $2g-1$ is odd, by Proposition 3.4, $r_{g-1}(g,g-1) = 0$. To get $i = g-1$, use Theorem 4.2:

$$\mathbb{D}^{\mathfrak{sl}_2}_{g-1,(1,\dots,1)} \cdot F_{g+1,g-1,1,1} = r_{g-1}(g-1,g-1) \cdot r_{g-1}(g+1,g-1).$$

Lemma 4.4 (1) implies that $r_{g-1}(g-1,g-1) = 1$. Lemma 4.4 (3) implies that $r_{g-1}(g+1,g-1) = g-1$. □

Corollary 4.11. *We have that*

$$\mathbb{D}^{\mathfrak{sl}_2}_{g,(1,\dots,1)} \cdot F_{n-i-2,i,1,1} = \begin{cases} 0, & i \leqslant g-1, \\ 1, & i = g. \end{cases} \tag{4.5}$$

Proof. If $i \leqslant g-1$, Corollary 4.3 implies that $\mathbb{D}^{\mathfrak{sl}_2}_{g,(1,\dots,1)} \cdot F_{n-i-2,i,1,1} = 0$. For $i = g$, Theorem 4.2 gives $\mathbb{D}^{\mathfrak{sl}_2}_{g,(1,\dots,1)} \cdot F_{g,g,1,1} = r_g(g,g) \cdot r_g(g,g)$. Lemma 4.4 (1) implies that $r_g(g,g) = 1$. □

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4.1. Classes of extremal divisors in the family

In § 5.1 we show that for $\ell \in \{1, 2, g-1, g\}$ the divisors $\mathbb{D}^{\mathfrak{sl}_2}_{\ell,(1,\dots,1)}$ generate extremal rays of the symmetric nef cone. These expressions are used in § 5.2 to show that $\mathbb{D}^{\mathfrak{sl}_2}_{\ell,(1,\dots,1)}$ is not log canonical if $\ell \in \{1, g-1, g\}$ and n is sufficiently large.

Theorem 4.12. *We have that*

$$\mathbb{D}^{\mathfrak{sl}_2}_{1,(1,\dots,1)} = \sum_{2\leqslant k\leqslant g+1,\, k \text{ even}} \frac{k(n-k)}{4(n-1)} B_k + \sum_{2\leqslant k\leqslant g+1,\, k \text{ odd}} \frac{(k-1)(n-k-1)}{4(n-1)} B_k,$$

$$\mathbb{D}^{\mathfrak{sl}_2}_{2,(1,\dots,1)} = 3\cdot 2^{g-1}\Bigg(\sum_{2\leqslant k\leqslant g+1,\, k \text{ even}} \left(\frac{k(n-k)}{8(n-1)} - \frac{1}{6}\right) B_k + \sum_{2\leqslant k\leqslant g+1,\, k \text{ odd}} \frac{(k-1)(n-k-1)}{8(n-1)} B_k \Bigg),$$

$$\mathbb{D}^{\mathfrak{sl}_2}_{g-1,(1,\dots,1)} = (g-1)\left(\sum_{k=2}^{g} \frac{(k-1)k}{(n-1)} B_k + \left(\frac{g^2-g-1}{(n-1)}\right) B_{g+1}\right),$$

$$\mathbb{D}^{\mathfrak{sl}_2}_{g,(1,\dots,1)} = \frac{1}{2}\sum_{k=2}^{g+1} \frac{(k-1)k}{(n-1)} B_k.$$

Proof. We produce the classes from the intersection numbers $\{D\cdot F_{1,1,j}\}_{j=1}^{g}$ using formulae given in [**2**, Lemma 4.2]. □

5. Position of the divisors $\mathbb{D}^{\mathfrak{sl}_2}_{\ell,(1,\dots,1)}$ in the nef cone

The $\mathbb{D}^{\mathfrak{sl}_2}_{\ell,(1,\dots,1)}$ are semi-ample, and so give rise to morphisms on $\bar{M}_{0,n}$. Any S_n-invariant divisor lies in the interior of the cone of effective divisors (see [**16, 26**]), and since the $\mathbb{D}^{\mathfrak{sl}_2}_{\ell,(1,\dots,1)}$ are symmetric, the morphisms they define are birational. As the divisors are nef, we can tell more about the morphisms they define by finding their location in the nef cone. Divisors in the interior of the nef cone are ample, and hence suitably large multiples of them define embeddings; base-point free divisors on the boundary define contractions.

5.1. The divisors $\mathbb{D}^{\mathfrak{sl}_2}_{\ell,(1,\dots,1)}$ define birational contractions

We show that the morphisms on $\bar{M}_{0,n}/S_n$ given by the divisors $\mathbb{D}^{\mathfrak{sl}_2}_{\ell,(1,\dots,1)}$ are birational contractions by proving the following.

Theorem 5.1.

(1) *The divisors $\mathbb{D}^{\mathfrak{sl}_2}_{\ell,(1,\dots,1)}$, for $1\leqslant \ell\leqslant g$, lie on the boundary of* $\mathrm{Nef}(\bar{M}_{0,n}/S_n)$. *In particular, the divisors $\mathbb{D}^{\mathfrak{sl}_2}_{\ell,(1,\dots,1)}$, for $1\leqslant \ell\leqslant g$, define birational contractions on $\bar{M}_{0,n}/S_n$.*

(2) *For each $1\leqslant \ell\leqslant g$, the divisor $\mathbb{D}^{\mathfrak{sl}_2}_{\ell,(1,\dots,1)}$ lies on a face of codimension at least $\lceil(\ell+g-2)/2\rceil$.*

(3) *For $\ell \in \{1, 2, g-1, g\}$, the divisor $\mathbb{D}^{\mathfrak{sl}_2}_{\ell,(1,\dots,1)}$ spans an extremal ray of the S_n-invariant nef cone.*

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Proof. By basic linear algebra, a symmetric nef divisor lies on a codimension k face of the symmetric nef cone if it intersects k independent symmetric curves in degree 0. Recall that the space $\mathrm{Pic}(\bar{M}_{0,n}/S_n)\otimes\mathbb{R}$ is g dimensional. Thus, if a symmetric nef divisor intersects $g-1$ independent symmetric F-curves in degree 0, then it spans an extremal ray of the symmetric nef cone.

For (1) we note that, by Corollary 4.3, each $\mathbb{D}^{\mathfrak{sl}_2}_{\ell,(1,\dots,1)}$ intersects at least one F-curve in degree 0, and hence lies on the boundary of the symmetric nef cone.

For (2), write $n=2(g+1)$, and apply Corollary 4.3 (3).

For (3), for $\ell\in\{g-1,g\}$, Corollary 4.3 (3) states that $\mathbb{D}^{\mathfrak{sl}_2}_{\ell,(1,\dots,1)}$ intersects $g-1$ independent curves in degree 0. This means that these two divisors are extremal in the symmetric nef cone. To show that $\mathbb{D}^{\mathfrak{sl}_2}_{1,(1,\dots,1)}$ and $\mathbb{D}^{\mathfrak{sl}_2}_{2,(1,\dots,1)}$ also generate extremal rays, we use the two other sets of independent curves $\mathscr{C}_2$ and $\mathscr{C}_3$ from Theorem 2.1. First, by [**8**, Proposition 5.2], which we can apply since $\ell=1$, one has that $\mathbb{D}^{\mathfrak{sl}_2}_{1,(1,\dots,1)}\cdot F_{a,b,c,d}=0$ if $abcd\equiv 0 \mod 2$. In particular, $\mathbb{D}^{\mathfrak{sl}_2}_{1,(1,\dots,1)}\cdot F_{2,2,i,n-4-i}=0$ for all $1\leqslant i\leqslant g-1$. Since these are the curves from $\mathscr{C}_2$, and since $\mathbb{D}^{\mathfrak{sl}_2}_{1,(1,\dots,1)}$ is non-trivial, this divisor spans an extremal ray of the symmetric nef cone. Next, we show that $\mathbb{D}^{\mathfrak{sl}_2}_{2,(1,\dots,1)}\cdot F_{a,b,c,d}=0$ if $abcd\equiv 1 \mod 2$. This implies that a, b, c, d are all odd. This is proved using [**8**, Proposition 2.7]:

$$\mathbb{D}^{\mathfrak{sl}_2}_{2,(1,\dots,1)}\cdot F_{a,b,c,d}=\sum_{\substack{\boldsymbol{\mu}=\{\mu_1,\dots,\mu_4\},\\ 0\leqslant\mu_i\leqslant 2}}\deg(\mathbb{V}(\mathfrak{sl}_2,2,\boldsymbol{\mu}))r_2(a,\mu_1)r_2(b,\mu_2)r_2(c,\mu_3)r_2(d,\mu_4).$$

Since the level is 2, each μ_i can only be 0, 1 or 2. If μ_i is even for any i, then the corresponding rank is 0, e.g. if μ_1 is even, then $r_{a\mu_1}=0$. It only remains to consider the case where $\mu_i=1$ for $i=1,\dots,4$. But an explicit calculation using [**8**, Proposition 4.2] shows that $\deg(\mathbb{V}(\mathfrak{sl}_2,2,(1,1,1,1)))=0$. Thus, there are no non-zero contributions to [**8**, (4.1)]. We show in Theorem 2.1 (3) that, for $g=2k$ or $g=2k-1$, the set $\mathscr{C}_3=\{F_{3,3,2i+1}\colon 0\leqslant i\leqslant k-2\}\cup\{F_{1,1,2i+1}\colon 0\leqslant i\leqslant k-1\}$ consists of independent curves. Therefore, $\mathbb{D}^{\mathfrak{sl}_2}_{2,(1,\dots,1)}$ spans an extremal ray of the symmetric nef cone. □

5.2. The divisors $\mathbb{D}^{\mathfrak{sl}_2}_{\ell,(1,\dots,1)}$ are not all log canonical

The Ray theorem (see [**26**, Theorem 1.2]; [**9**, Theorem 4]) gives us a tool for detecting nef divisors in what we call the log canonical part of the cone.

Definition 5.2. We say a divisor D on $\bar{M}_{0,n}$ is *log canonical* if D may be expressed as an effective combination

$$D=c\Big(K_{\bar{M}_{0,n}}+\sum c_I\delta_I\Big),$$

where c and the c_I are any non-negative rational numbers such that $0\leqslant c_I\leqslant 1$ for all I. (Here, the summation is over the standard set of boundary divisors: $2\leqslant|I|\leqslant n/2$ and $1\in I$ if $|I|=n/2$.)

If D is linearly equivalent to a log canonical divisor and intersects every F-curve non-negatively, then D is nef by the Ray theorem, and is semi-ample by [**4**].

If D is S_n-invariant and linearly equivalent to a log canonical divisor, then, by averaging its log canonical expression over S_n, we obtain

$$D = c\bigg(K_{\bar{M}_{0,n}} + \sum_{i=2}^{g+1} b_i B_i\bigg),$$

where $c > 0$ and $0 \leqslant b_i \leqslant 1$ for all i. We call a divisor of this form *symmetrically log canonical*.

We interpret the failure of a divisor to be linearly equivalent to a log canonical divisor to mean that it is outside the part of $\mathrm{Nef}(\bar{M}_{0,n})$ that can be understood combinatorially. This motivated us to study the geometry behind such divisors in more detail.

Proposition 5.3.

(1) $\mathbb{D}^{\mathfrak{sl}_2}_{1,(1,\ldots,1)}$ *is log canonical for* $n = 6, 8, 10$, *but not for* $n \geqslant 12$.

(2) $\mathbb{D}^{\mathfrak{sl}_2}_{2,(1,\ldots,1)}$ *is log canonical for all* $n \geqslant 6$.

(3) $\mathbb{D}^{\mathfrak{sl}_2}_{g-1,(1,\ldots,1)}$ *is log canonical for* $n = 10, 12, 14$, *but not for* $n \geqslant 16$.

(4) $\mathbb{D}^{\mathfrak{sl}_2}_{g,(1,\ldots,1)}$ *is log canonical for* $n = 8, 10, 12$, *but not for* $n \geqslant 14$.

Proof. By the above remarks, it is enough to test whether these divisors are symmetrically log canonical. The claims made for small values of n may be verified by direct calculation. The following identity shows that $\mathbb{D}^{\mathfrak{sl}_2}_{2,(1,\ldots,1)}$ is symmetrically log canonical for all n:

$$\frac{8}{3\cdot 2^{g-1}}\mathbb{D}^{\mathfrak{sl}_2}_{2,(1,\ldots,1)} = K_{\bar{M}_{0,n}} + \frac{2}{3}\sum_{i \text{ even}} B_i + \sum_{i \text{ odd}} B_i.$$

We prove (1) when g is odd; the proofs of the other parts are similar. Suppose, for the purposes of contradiction, that

$$c^{-1}\mathbb{D}^{\mathfrak{sl}_2}_{1,(1,\ldots,1)} = K_{\bar{M}_{0,n}} + \sum_{i=2}^{g+1} b_i B_i, \tag{5.1}$$

with $0 \leqslant b_i \leqslant 1$ for all i. By using (2.1) and the formula for $\mathbb{D}^{\mathfrak{sl}_2}_{1,(1,\ldots,1)}$ from Theorem 4.12 and extracting the coefficients of B_2 and B_{g+1} on each side of (5.1), we obtain the two equations

$$c^{-1}\frac{2(n-2)}{4(n-1)} = \frac{-2}{n-1} + b_2,$$
$$c^{-1}\frac{(g-1)(n-g-1)}{4(n-1)} = \frac{g(n-g)}{n-1} - 2 + b_g.$$

We apply the inequalities $b_2 \leqslant 1$ and $b_g \geqslant 0$ and substitute $n = 2g+2$ to obtain that

$$\frac{4(g^2-2g-2)}{g^2-1} \leqslant c^{-1} \leqslant \frac{2g-1}{g},$$

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But one can easily show that, for $g \geqslant 5$, we have that

$$\frac{4(g^2-2g-2)}{g^2-1} > \frac{2g-1}{g}.$$

Hence, there exists no triple (c, b_2, b_g) making $\mathbb{D}^{\mathfrak{sl}_2}_{1,(1,\dots,1)}$ symmetrically log canonical if g is odd and $g \geqslant 5$.

(1) for g even and (3) and (4) may be established by similar analyses of the triples (c, b_2, b_{g+1}), (c, b_3, b_g) and (c, b_2, b_{g+1}), respectively. □

6. Morphisms defined by the divisors $\mathbb{D}^{\mathfrak{sl}_2}_{\ell,(1,\dots,1)}$

We next consider morphisms defined by the extremal divisors $\mathbb{D}^{\mathfrak{sl}_2}_{\ell,(1,\dots,1)}$ for $\ell \in \{1, 2, g-1, g\}$, and relate the divisors $\mathbb{D}^{\mathfrak{sl}_2}_{\ell,(1,\dots,1)}$ for $\frac{1}{3}n - 1 \leqslant \ell$ to the GIT quotients of n-pointed conics studied in [**15**].

6.1. Levels 1 and 2 and extended Torelli maps

Let $h\colon \bar{M}_{0,2(g+1)} \to \bar{M}_g$ be the morphism defined by taking $(C, \boldsymbol{p}) \in \bar{M}_{0,2(g+1)}$ to the stable curve of genus g obtained by taking a double cover of C branched at the set marked points $\boldsymbol{p} = \{p_1, \dots, p_n\}$. In this section we show that the divisor $\mathbb{D}^{\mathfrak{sl}_2}_{1,(1,\dots,1)}$ defines a morphism that factors through h.

We use the following formula for the pullback of a divisor on $\bar{M}_g$ along the map h.

Lemma 6.1. *Let $h\colon \bar{M}_{0,2(g+1)} \to \bar{H}_g \subset \bar{M}_g$ be the isomorphism onto the hyperelliptic locus in $\bar{M}_g$, and let*

$$D = a\lambda - \sum_{i=0}^{\lfloor g/2 \rfloor} b_i \delta_i$$

be a divisor on $\bar{M}_g$. Then,

$$h^*(D) = \sum_{\substack{2 \leqslant k \leqslant \lfloor g/2 \rfloor,\\ k \text{ even}}} \left(\frac{ak(n-k)}{8(n-1)} - 2b_0 \right) B_k + \sum_{\substack{2 \leqslant k \leqslant \lfloor g/2 \rfloor,\\ k \text{ odd}}} \left(\frac{a(k-1)(n-k-1)}{8(n-1)} - \frac{b_i}{2} \right) B_k.$$

Proof. This follows from [**5**, pp. 468–470 and Proposition 4.7]. □

The classical Torelli map that takes a smooth curve X of genus g to its Jacobian extends to a regular map $t^{\mathrm{Sat}}\colon \bar{M}_g \to \bar{A}_g^{\mathrm{Sat}}$, where $\bar{A}_g^{\mathrm{Sat}}$ is the Satake compactification of A_g. Moreover, $\lambda = (t^{\mathrm{Sat}})^*(\Theta)$, where Θ is the ample divisor of weight one modular forms on $\bar{A}_g^{\mathrm{Sat}}$ [**1**].

Theorem 6.2. *The divisor $\mathbb{D}^{\mathfrak{sl}_2}_{1,(1,\dots,1)}$ defines the composition*

$$\bar{M}_{0,2g+2}/S_{2g+2} \xrightarrow{h} \bar{M}_g \xrightarrow{\bar{t}^{\mathrm{Sat}}} \bar{A}_g^{\mathrm{Sat}}.$$

Proof. To prove this, we use Lemma 6.1 and Theorem 4.12 to show that $\mathbb{D}^{\mathfrak{sl}_2}_{1,(1,\dots,1)} = 2h^*(\lambda)$. Because λ is the semi-ample divisor that defines the morphism

$$\bar{M}_g \xrightarrow{\bar{t}^{\mathrm{Sat}}} \bar{A}_g^{\mathrm{Sat}},$$

the result follows. □

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The next result shows that, at least for $g \leqslant 11$, the morphism given by $\mathbb{D}^{\mathfrak{sl}_2}_{2,(1,\ldots,1)}$ also factors through the map to the hyperelliptic locus.

Theorem 6.3. *We have that*

$$\mathbb{D}^{\mathfrak{sl}_2}_{2,(1,\ldots,1)} = \tfrac{1}{2}h^*(12\lambda - \delta_0), \quad \textit{where } h\colon \bar{M}_{0,2g+2}/S_{2g+2} \to \bar{H}_g \hookrightarrow \bar{M}_g.$$

Proof. Use Lemma 6.1 and Theorem 4.12. □

Question 6.4. Is $12\lambda - \delta_0$ semi-ample on $\bar{M}_g$ for all $g \geqslant 4$?

It is known that $12\lambda - \delta_0$ is nef on $\bar{M}_g$ for all g (see [**7**, Proposition 3.3]). By [**25**], $12\lambda - \delta_0$ is semi-ample on $\bar{M}_g$ when defined over a finite field. Over $\mathbb{C}$, $12\lambda - \delta_0$ is semi-ample for $g = 3$ by [**29**], and for $g \leqslant 11$ by [**17**]. In these cases, Theorem 6.3 implies that the morphism associated with $\mathbb{D}^{\mathfrak{sl}_2}_{2,(1,\ldots,1)}$ factors through h. Let X denote the image of the linear system $|12\lambda - \delta_0|$. For $g = 2$, $X = \bar{M}^{ps}_2$; this follows from [**22**, Propositions 2.7 and 4.2]. But, for $3 \leqslant g \leqslant 11$, we do not know a description for X. It seems a reasonable guess that there might be morphisms from $\bar{A}^{\mathrm{Vor}(2)}_g$ and $\bar{M}^{ps}_g$ to X that are small modifications. Here $\bar{M}^{ps}_g$ stands for the moduli space of pseudo-stable curves (see [**23, 24, 30**]).

6.2. $\mathbb{D}^{\mathfrak{sl}_2}_{\ell,(1,\ldots,1)}$ is in the SL_3 GIT cone if $\ell \geqslant n/3 - 1$

In [**15**], Giansiracusa and Simpson study the GIT quotients of n-pointed conics. In this section, we relate the divisors $\mathbb{D}^{\mathfrak{sl}_2}_{\ell,(1,\ldots,1)}$ for $\ell \geqslant n/3 - 1$ to these quotients.

We recall the notation of [**15**]. Conics in $\mathbb{P}^2$ are parametrized by $(\mathbb{P}^5)^\vee$. Let $\mathrm{Con}(n)$ be the incidence locus in $(\mathbb{P}^5)^\vee \times \prod_{i=1}^n \mathbb{P}^2$. We consider GIT quotients of the form $\mathrm{Con}(n)/\!/_{\gamma,\boldsymbol{c}}\,\mathrm{SL}_3$. Giansiracusa and Simpson construct a morphism

$$\bar{M}_{0,n} \to \mathrm{Con}(n)/\!/_{(\gamma,\boldsymbol{c})}\,\mathrm{SL}_3$$

for each linearization $(\gamma, \boldsymbol{c})$. Moreover, each GIT quotient $\mathrm{Con}(n)/\!/_{(\gamma,\boldsymbol{c})}\,\mathrm{SL}_3$ has a distinguished polarization descending from the linearization.

Definition 6.5. Let $\mathrm{GS}(\gamma, \boldsymbol{c})$ denote the divisor on $\bar{M}_{0,n}$ that is the pullback of the distinguished line bundle on $\mathrm{Con}(n)/\!/_{(\gamma,\boldsymbol{c})}\,\mathrm{SL}_3$ along the Giansiracusa–Simpson morphism $\bar{M}_{0,n} \to \mathrm{Con}(n)/\!/_{(\gamma,\boldsymbol{c})}\,\mathrm{SL}_3$. We refer to these divisors as 'SL_3 GIT divisors'.

We next define the divisors R_j that, for $\frac{1}{3}n-1 \leqslant j$, span extremal rays of the symmetric nef cone.

Definition 6.6. For each j from 1 to $\lfloor n/2 \rfloor - 1$, let R_j be the divisor in $\mathrm{Pic}\,(\bar{M}_{0,n})^{S_n}$ such that

$$R_j \cdot F_{n-i-2,i,1,1} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

(We avoided using a Kronecker delta in the formula above to avoid confusion with the divisor class $\delta_{ij} \in \mathrm{Pic}(\bar{M}_{0,n})$.)

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Proposition 6.7.

(1) *$R_j = \sum_{k=2}^{\lfloor n/2 \rfloor} b_k B_k$, where*

$$b_k = \begin{cases} \dfrac{(k-1)k}{n-1} - k + 1 + j & \text{if } j < n/2 - 1,\ j < k - 1, \\ \dfrac{(k-1)k}{n-1} & \text{if } j < n/2 - 1,\ j \geqslant k - 1, \\ \dfrac{(k-1)k}{2(n-1)} & \text{if } j = n/2 - 1. \end{cases}$$

(2) *If $\frac{1}{3}n - 1 \leqslant j \leqslant \frac{1}{2}n - 1$, then R_j is a multiple of $\mathrm{GS}(\gamma, \boldsymbol{c})$, where $\gamma = 3 - n/(j+1)$, $\boldsymbol{c} = (1/(j+1), \ldots, 1/(j+1))$. Hence, R_j is semi-ample.*

(3) *If $j < \frac{1}{3}n - 1$, then R_j is not nef.*

Proof. For (1) we use the formulae given in [**2**, Lemma 4.2].

For (2) we use [**15**, Lemma 5.1], which, after a minor correction, states the following. Let F_{I_1,I_2,I_3,I_4} be an F-curve, and let $(\gamma, \boldsymbol{c})$ be a linearization such that $\gamma = 3 - \sum_{p=1}^{n} c_p$ satisfies $0 \leqslant \gamma \leqslant 1$. Let $x_q = \sum_{p \in I_q} c_p$. Assume that $x_1 \leqslant x_2 \leqslant x_3 \leqslant x_4$. Then, F_{I_1,I_2,I_3,I_4} is contracted under the morphism $\bar{M}_{0,n} \to \mathrm{Con}(n)/\!/_{\gamma,\boldsymbol{c}}\,\mathrm{SL}_3$ if and only if either $x_1 + x_2 + x_3 \leqslant 1$ or $x_3 \geqslant 1$. Thus, $\mathrm{GS}(\gamma, \boldsymbol{c}) \cdot F_{I_1,I_2,I_3,I_4} = 0$ if and only if either $x_1 + x_2 + x_3 \leqslant 1$ or $x_3 \geqslant 1$.

Suppose that $\frac{1}{3}n - 1 \leqslant j \leqslant \frac{1}{2}n - 1$. We set $\boldsymbol{c} = (1/(j+1), \ldots, 1/(j+1))$. Then $0 \leqslant \gamma \leqslant 1$, and so we can apply the results of the previous paragraph to compute the intersection numbers of $\mathrm{GS}(\gamma, \boldsymbol{c})$ with F-curves. Specifically, we intersect $\mathrm{GS}(\gamma, \boldsymbol{c})$ with curves of the form $F_{n-i-2,i,1,1}$, which form a basis of $H_2(\bar{M}_{0,n}, \mathbb{Q})^{S_n}$. When $i \leqslant j - 1$, we have that $x_1 = x_2 = 1/(j+1)$, $x_3 = i/(j+1)$, so $x_1 + x_2 + x_3 \leqslant 1$, and hence $\mathrm{GS}(\gamma, \boldsymbol{c}) \cdot F_{n-i-2,i,1,1} = 0$. When $j = i$, we have that $x_1 + x_2 + x_3 = (j+2)/(j+1) > 1$ and $x_3 = j/(j+1) < 1$, and hence $\mathrm{GS}(\gamma, \boldsymbol{c}) \cdot F_{n-i-2,i,1,1} > 0$. When $i \geqslant j + 1$, we have that $x_3 = i/(j+1) \geqslant 1$, and hence $\mathrm{GS}(\gamma, \boldsymbol{c}) \cdot F_{n-i-2,i,1,1} = 0$. We see that the vector of intersection numbers of $\mathrm{GS}(\gamma, \boldsymbol{c})$ is a multiple of the vector of intersection numbers defining R_j, and hence we conclude that R_j is a multiple of $\mathrm{GS}(\gamma, \boldsymbol{c})$.

For (3), if $j = 1$, then $R_1 \cdot F_{1,2,2,n-5} = -1$. If $j \geqslant 2$, we have that

$$R_j \cdot F_{\lceil (n-2j)/2 \rceil, \lfloor (n-2j)/2 \rfloor, j, j} = \begin{cases} -j + 1 & \text{if } j < \dfrac{n-1}{4} \ (n \text{ odd}) \text{ or } \dfrac{n-2}{4} \ (n \text{ even}), \\ -n + 3j + 3 & \text{if } j \geqslant \dfrac{n-1}{4} \ (n \text{ odd}) \text{ or } \dfrac{n-2}{4} \ (n \text{ even}). \end{cases}$$

□

Proposition 6.8.

(1) *The divisor $\mathbb{D}^{\mathfrak{sl}_2}_{g,(1,\ldots,1)}$ is extremal in the symmetric nef cone and defines morphisms to $(\mathbb{P}^1)^n /\!/ \mathrm{SL}_2$ with the symmetric linearization, or to $\mathrm{Con}(n)/\!/_{\gamma,\boldsymbol{c}}\,\mathrm{SL}_3$, where $\gamma = 1$, $\boldsymbol{c} = (1/(g+1), \ldots, 1/(g+1))$.*

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(2) *The divisor $\mathbb{D}^{\mathfrak{sl}_2}_{g-1,(1,\ldots,1)}$ is extremal in the symmetric nef cone and defines a morphism to* $\mathrm{Con}(n)/\!/_{\gamma,\boldsymbol{c}}\mathrm{SL}_3$*, where $\gamma = (g-2)/g$ and $\boldsymbol{c} = (1/g,\ldots,1/g)$.*

(3) *If $\frac{1}{3}n - 1 \leqslant \ell < g-1$, the divisor $\mathbb{D}^{\mathfrak{sl}_2}_{\ell,(1,\ldots,1)}$ has the following properties:*

  (a) *the divisor $\mathbb{D}^{\mathfrak{sl}_2}_{\ell,(1,\ldots,1)}$ lies on a face of the symmetric nef cone of codimension at least $\lceil(\ell+g-2)/2\rceil$,*

  (b) *the divisor $\mathbb{D}^{\mathfrak{sl}_2}_{\ell,(1,\ldots,1)}$ lies in the* SL_3 *GIT cone,*

  (c) *the divisor $\mathbb{D}^{\mathfrak{sl}_2}_{\ell,(1,\ldots,1)}$ does not span an extremal ray of the symmetric nef cone.*

Proof. For (1) note that $\mathbb{D}^{\mathfrak{sl}_2}_{g,(1,\ldots,1)}$ is at the critical level (that is, $\sum\lambda_i = 2\ell+2$), and, therefore, it follows by [**8**, Theorem 4.5] that this divisor defines a morphism to $(\mathbb{P}^1)^n/\!/\mathrm{SL}_2$ with symmetric linearization. For $\ell \in \{g-1, g\}$, matching the images of the morphism given by $\mathbb{D}^{\mathfrak{sl}_2}_{\ell,(1,\ldots,1)}$ with these SL_3 GIT quotients is a consequence of comparing the intersection numbers of $\mathbb{D}^{\mathfrak{sl}_2}_{\ell,(1,\ldots,1)} \cdot F_{1,1,i}$ given in Corollaries 4.10 and 4.11 with those defining R_j above, and then applying Proposition 6.7.

Part (3) (a) was observed earlier in Theorem 5.1.

For (3) (b), we apply Corollary 4.3; the only non-zero intersection numbers $\mathbb{D}^{\mathfrak{sl}_2}_{\ell,(1,\ldots,1)} \cdot F_{1,1,i}$ occur when $\frac{1}{3}n - 1 \leqslant i$, and we can use these numbers to write $\mathbb{D}^{\mathfrak{sl}_2}_{\ell,(1,\ldots,1)}$ as an effective combination of the divisors R_j with $\frac{1}{3}n - 1 \leqslant j$. But, by Proposition 6.7, the divisors R_j are multiples of SL_3 GIT divisors, yielding (3) (b).

For (3) (c), we showed in Corollary 4.5 that $\mathbb{D}^{\mathfrak{sl}_2}_{\ell,(1,\ldots,1)}$ has a positive intersection number with the F-curves $\{F_{1,1,\ell+2i} \colon \ell + 2i \leqslant g\}$. Thus, $\mathbb{D}^{\mathfrak{sl}_2}_{\ell,(1,\ldots,1)}$ is a positive combination of the nef divisors $\{R_{\ell+2i} \colon \ell+2i \leqslant g\}$, and hence $\mathbb{D}^{\mathfrak{sl}_2}_{\ell,(1,\ldots,1)}$ does not generate an extremal ray in the symmetric nef cone. □

Corollary 6.9. *The images of the morphisms associated with $\mathbb{D}^{\mathfrak{sl}_2}_{g-1,(1,\ldots,1)}$ and $\mathbb{D}^{\mathfrak{sl}_2}_{g,(1,\ldots,1)}$ are related by a (generalized) flip. The Hassett space $\bar{M}_{0,A}$, with weights $A = (1/g,\ldots,1/g)$, lies above both.*

Proof. The first statement is a consequence of the theory of variation of GIT [**6**,**32**]. That the morphism $\bar{M}_{0,n} \to (\mathbb{P}^1)/\!/\mathrm{SL}_2$ factors through $\bar{M}_{0,A}$ with $A = (1/g,\ldots,1/g)$ was observed by Hassett in [**21**]; Giansiracusa and Simpson proved in [**15**, Theorem 1.4] that the morphism $\bar{M}_{0,n} \to \mathrm{Con}(n)/\!/\mathrm{SL}_3$ with $\gamma = (g-2)/g$ and $\boldsymbol{c} = (1/g,\ldots,1/g)$ also factors through this $\bar{M}_{0,A}$. □

7. Finding nef divisors on $\bar{M}_{2(g+1)}$ using the flag morphism

As the conformal block vector bundles are globally generated, their first Chern classes $\mathbb{D}$ are semi-ample divisors, and so the linear series $|m\mathbb{D}|$, for $m \gg 0$, defines morphisms on $\bar{M}_{0,n}$. In contrast, while vector bundles of conformal blocks are defined on the moduli stacks of higher genus curves, in [**8**] Fakhruddin showed that these vector bundles are often not even effective, much less globally generated. However, using our conformal

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blocks divisors on $\bar{M}_{0,n}$, together with [**19**, Theorem 0.3], we produce new nef divisors on $\bar{M}_n$. We do this in Proposition 7.4.

Given any point $(E;p) \in \bar{M}_{1,1}$, let $f\colon \bar{M}_{0,n} \to \bar{M}_n$ be the morphism given by taking $(C;p_1,\dots,p_n) \in \bar{M}_{0,n}$ to the curve of genus n obtained by attaching n copies of E to C by identifying p and p_i. We call this the *flag morphism*. In [**19**, Theorem 2.2 and Figure 2.3], five types of F-curves on $\bar{M}_{g,n}$ are defined. We refer to these as F-curves of types (1)–(5). For the reader's convenience, we state a combinatorial definition of an F-divisor D on $\bar{M}_{2(g+1)}$ from [**19**], which we rewrite slightly to fit our situation exactly.

Theorem 7.1 (Gibney *et al.* [19, Theorem 2.1]). *Let $n = 2(g+1)$ and consider the divisor*

$$D = a\lambda - \sum_{i=0}^{g+1} b_i\delta_i$$

on $\bar{M}_{2(g+1)}$. Then, D is an F-divisor if and only if it satisfies the following inequalities:

(1) $a - 12b_0 + b_1 \geqslant 0$,

(2) $b_i \geqslant 0$,

(3) $2b_0 - b_i \geqslant 0$,

(4) $b_i + b_j \geqslant b_{i+j}$ *for all* $i, j \neq 0$,

(5) $b_i+b_j+b_k+b_\ell \geqslant b_{i+j}+b_{i+k}+b_{i+\ell}$ *for all* $i,j,k,\ell \neq 0$ *such that* $i+j+k+\ell = 2(g+1)$.

Each of the inequalities (1)–(5) of Theorem 7.1 is satisfied by a divisor D as long as D non-negatively intersects the corresponding F-curves of types (1)–(5). In [**19**, Theorem 4.7] a nef divisor, which we denote by $\mathcal{D}$, is defined with the property that $\mathcal{D}$ strictly positively intersects the F-curves of types (1)–(4), while it intersects the F-curves of types (5) in degree 0. In particular, it is shown that $f^*\mathcal{D}$ is trivial. We use this divisor $\mathcal{D}$ in Proposition 7.4, and so, for the reader's convenience, we recall its definition.

Definition 7.2. On $\bar{M}_{2(g+1)}$ we consider the divisor

$$\mathcal{D} = \alpha\lambda - \beta\delta_0 - \sum_{i=1}^{g+1} i(2(g+1)-i)\delta_i.$$

Theorem 7.3 (Gibney, Keel and Morrison [19, Theorem 4.7]). *Let $\mathcal{D}$ be the divisor from Definition 7.2. For any choice of α and β such that $\alpha > 12\beta - (2g+1)$, and $2\beta > (g+1)^2$,*

(1) *$\mathcal{D}$ is nef,*

(2) *$f^*(\mathcal{D}) = 0$ and*

(3) *$\mathcal{D}$ strictly positively intersects all the F-curves of type (1)–(4).*

Proposition 7.4. *For $\ell \in \{1, 2, g-1, g\}$, there exists a positive constant c_ℓ and a non-negative constant d_ℓ such that*

$$\mathbb{D}^{\mathfrak{sl}_2}_{\ell,(1,\dots,1)} = f^*(c_\ell D^\ell_{a,b} + d_\ell \mathcal{D}),$$

and $c_\ell D^\ell_{a,b} + d_\ell \mathcal{D}$ is a nef divisor on $\bar{M}_{2(g+1)}$. Here,

(1) *$c_1 D^1_{a,b} = a\lambda - b\delta_0 - \sum_{i=0}^{\lfloor g/2 \rfloor} \delta_{2i+1}$, where $c_1 = \frac{1}{4}$, $b \geqslant \frac{1}{2}$ and $a \geqslant 12b - 1$;*

(2) *$c_2 D^2_{a,b} = a\lambda - b\delta_0 - \sum_{i=0}^{\lfloor g/2 \rfloor} \delta_{2i+1} - \sum_{i=1}^{\lfloor (g+1)/2 \rfloor} \frac{4}{3}\delta_{2i}$, where $c_2 = \frac{4}{3}$, $b \geqslant \frac{8}{3}$ and $a \geqslant 12b - 1$;*

(3) *$c_{g-1} D^{g-1}_{a,b} = a\lambda - b\delta_0 - \sum_{i=1}^{g} i(n-2i+1)\delta_i/(n-1) - (3g+2)\delta_{g+1}/(n-1)$, where $c_{g-1} = 1/(g-1)$, $b \geqslant \frac{1}{2}\max\{i(n-2i+1)/(n-1)\}_{i=1}^{g}$ and $a \geqslant 12b - 1$;*

(4) *$c_g D^g_{a,b} = a\lambda - b\delta_0 - \sum_{i=1}^{g+1} i(n-2i+1)\delta_i/(n-1)$, where $c_g = 1$, $b \geqslant \frac{1}{2}\max\{i(n-2i+1)/(n-1)\}_{i=1}^{g+1}$ and $a \geqslant 12b - 1$.*

We may take $d_1, d_2 \geqslant 0$. For $\ell \in \{g-1, g\}$, we may choose any d_ℓ such that $(c_\ell D^\ell_{a,b} + d_\ell \mathcal{D}) \cdot \mathcal{C} \geqslant 0$, where $\mathcal{C}$ is any F*-curve of type* (4).

Proof. By [**16**, Lemma 2.4], the pullback to $\bar{M}_{0,2(g+1)}$ of $D = a\lambda - \sum_{i=0}^{g+1} b_i\delta_i$ along f is

$$f^*D = \sum_{j=2}^{g+1} \left(\frac{j(n-j)}{(n-1)} b_1 - b_j \right) B_j, \quad \text{where } B_j = \sum_{J \subset \{1,\dots,n\},\, |J|=j} \delta_J,$$

where $n = 2(g+1)$. Using this and the fact that $f^*\mathcal{D} = 0$, one can check that, for $\ell \in \{1, 2, g-1, g\}$, the divisors $c_\ell D^\ell_{ab} + d_\ell \mathcal{D}$ on $\bar{M}_{2(g+1)}$ pull back to $\mathbb{D}^{\mathfrak{sl}_2}_{\ell,(1,\dots,1)}$. So, it remains to check that, for each $\ell \in \{1, 2, g-1, g\}$, the divisor $c_\ell D^\ell_{ab} + d_\ell \mathcal{D}$ is nef. Our main tool for proving that divisors on $\bar{M}_{2(g+1)}$ are nef is to check the conditions of Theorem 7.1 and apply [**19**, Theorem 0.3]. We first analyse the cases $\ell = 1$ and $\ell = 2$. It is easy to check that Theorem 7.1 (1)–(3) hold for D^ℓ_{ab} for all $\ell \in \{1, 2\}$, since we chose a and b to make this happen. Condition (5) is just the combinatorial formulation that $(c_\ell D^\ell_{ab}) \cdot F^{2(g+1)}_{i,j,k,\ell} \geqslant 0$, where $F^{2(g+1)}_{i,j,k,\ell}$ on $\bar{M}_n$ is the image of the F-curve $F_{i,j,k,\ell}$ on $\bar{M}_{0,n}$ under the flag map. In other words, this is equivalent to

$$f^*(c_\ell D^\ell_{ab}) \cdot F_{i,j,k,\ell} = \mathbb{D}^{\mathfrak{sl}_2}_{\ell,(1,\dots,1)} \cdot F_{i,j,k,\ell} \geqslant 0,$$

which holds since $\mathbb{D}^{\mathfrak{sl}_2}_{\ell,(1,\dots,1)}$ is nef. We check (4) for D^1_{ab}. Since b_k only depends on the parity of k, we need only consider two cases. If i and j have the same parity, then the equality reads $2 \geqslant 0$. If i and j have opposite parity, then the inequality reads $1 \geqslant 1$. So we may conclude that $D^1_{a,b}$ is nef. By [**19**, Theorem 4.7] $\mathcal{D}$ is also nef, and hence, for any non-negative c_1, d_1, the divisor $c_1 D^1_{a,b} + d_1\mathcal{D}$ is nef. We next check (4) for D^2_{ab}. Note that $b_i + b_j \geqslant 2$, while $b_{i+j} \leqslant \frac{4}{3}$, so (4) holds. Thus, D^2_{ab} is nef, and hence, for any $c_2, d_2 \geqslant 0$, the divisor $c_2 D^2_{a,b} + d_2\mathcal{D}$ is nef. Next, we analyse the cases $\ell = g-1$ and $\ell = g$. The two divisors D^{g-1}_{ab} and D^g_{ab} are not nef by themselves, as (4) does not always hold. In particular, it is necessary to choose a sufficiently large d_{g-1} and d_g. The

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hypotheses on a, b, α and β ensure that (1)–(3) are satisfied. Condition (5) follows since we know that $c_\ell D^\ell_{a,b} + d_\ell \mathcal{D}$ pulls back to a nef divisor on $\bar{M}_{0,n}$. This leaves (4). We know that $\mathcal{D}$ has positive intersection with F-curves of type (4). Therefore, we simply need to choose d_ℓ sufficiently large that $(c_\ell D^\ell_{a,b} + d_\ell \mathcal{D}) \cdot \mathcal{C} \geqslant 0$ for all F-curves of type (4). There are only finitely many such curves to check (or, only finitely many inequalities of type $b_i + b_j \geqslant b_{i+j}$), so this can be arranged. Then, $c_\ell D^\ell_{a,b} + d_\ell \mathcal{D}$ is F-nef, and hence nef by [**19**, Theorem 0.3]. $\square$

8. Three collections of independent curves

In this section we prove Theorem 2.1. For a proof that the first family $\mathscr{C}_1$ is independent, see [**2**, Proposition 4.1]. For the families $\mathscr{C}_2$ and $\mathscr{C}_3$, it is convenient to assume that $n \geqslant 16$. (This is because, for instance, for smaller n, the formulae for $R \cdot B_i$ used in the proof below are slightly different.) We can check the independence of the families $\mathscr{C}_2$ and $\mathscr{C}_3$ for $6 \leqslant n \leqslant 15$ by direct calculation. The strategy used to prove the independence of $\mathscr{C}_2$ is completely different from the strategy used to prove the independence $\mathscr{C}_3$.

8.1. The family $\mathscr{C}_2$ is independent

Let $n \geqslant 16$. Suppose that $R = \sum_{i=1}^{g-1} b_i F_{2,2,i}$ is numerically equivalent to 0. To show that $b_i = 0$ for all i, we show the following:

(1) $b_i = 0$ for i odd,

(2) $b_i = -cB$ for i even, where $0 < c \in \mathbb{Q}$, and $B = \sum_{i=1}^{g-1} b_i$.

These two steps together show that $b_i = 0$ for all $1 \leqslant i \leqslant g-1$.

We show by induction that $b_i = 0$ for i odd. Since R is equivalent to 0, $R \cdot \Psi = b_1 = 0$, and $R \cdot B_3 = 2b_1 - b_3 = 0$, so $b_3 = b_1 = 0$, establishing our base case. Now let any odd index $i \in \{5, \ldots, g-1\}$ be given, and, for the induction hypothesis, we assume that $b_{i-2k} = 0$ for $1 \leqslant k \leqslant (i-1)/2$. In particular, $b_{i-2} = b_{i-4} = 0$. In the case $i \in \{5, \ldots, g-2\}$, one has that $R \cdot B_i = 2b_{i-2} - b_i - b_{i-4} = 0$, and so $b_i = 2b_{i-2} - b_{i-4} = 0$. In the remaining case, when $g-1$ is odd, both $g-3$ and $g-5$ are odd too, so, by induction, $b_{g-3} = b_{g-5} = 0$. Now $R \cdot B_{g-1} = 2b_{g-3} - 2b_{g-1} - b_{g-5} = 0$, and so $b_{g-1} = b_{g-3} - \frac{1}{2} b_{g-5} = 0$. The first claim is proved.

Now we prove the second claim. Set $B = \sum_{j=1}^{g-1} b_j$. We show by induction that for $i \in \{1, \ldots, \lfloor (g-1)/2 \rfloor\}$ one has $b_{2i} = -cB$, where c is a positive constant depending on i, as follows:

(1) $b_{2i} = -(i+1)B$

  (a) for $n = 2(g+1)$ and $2 \leqslant 2i \leqslant g-2$,

  (b) for $n = 2(g+1) + 1$ and $2 \leqslant 2i \leqslant g-1$,

(2) $b_{2i} = -(i+1)B/2$ for $n = 2(g+1)$ in the case $2i = g-1$.

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For the base case we show that the assertion holds for $i = 1$ and $i = 2$. We have that

$$0 = R \cdot B_2 = -2 \sum_{1 \leqslant i \leqslant g-1} b_i - b_2 = -2B - b_2,$$

so $b_2 = -2B$. Also, $0 = R \cdot B_4 = -b_4 + B + 2b_2 = -b_4 + B - 4B$, so $b_4 = -3B$. Now let $2i \in \{6, \ldots, g-1\}$ be given, and, for induction, assume that $b_{2i-2k} = b_{2(i-k)} = -(i-k+1)B$ for $2 \leqslant 2k \leqslant 2i-2$. In particular, $b_{2(i-1)} = -iB$ and $b_{2(i-2)} = -(i-1)B$. In the case $2i \in \{6, \ldots, g-2\}$, $0 = R \cdot B_{2i} = 2b_{2(i-1)} - b_{2i} - b_{2(i-2)}$, so $b_{2i} = 2b_{2(i-1)} - b_{2(i-2)} = (-2i + (i-1))B = -(i+1)B$, as asserted. This also holds if $n = 2(g+1)+1$ and $2i = g-1$. Finally, assume that $n = 2(g+1)$, $2i = g-1$, and use $0 = R \cdot B_{g-1} = 2b_{g-3} - 2b_{g-1} - b_{g-5}$. By the induction hypothesis,

$$2b_{g-1} = \left(-2\left(\frac{g-3}{2} + 1\right) + \left(\frac{g-5}{2} + 1\right)\right)B = -\left(\frac{g+1}{2}\right)B.$$

In other words, $b_{g-1} = b_{2i} = -(g+1)B/4 = -(i+1)/2$, proving (2). By summing the b_i and the expressions for the b_i, we obtain an equation of the form $B = dB$, where $d > 1$. But then $B = 0$, and so $b_{2i} = 0$.

8.2. The family $\mathscr{C}_3$ is independent

We use techniques introduced in [**2**].

Lemma 8.1. *Let $\gamma_{i,j}$ be the coefficient defined by the equation $F_{3,3,2i+1,n-2i-7} = \sum_{j=1}^{g} \gamma_{i,j} F_{1,1,j,n-j-2}$.*

(1)

$$\gamma_{i,4+2i} = \begin{cases} 1 & \text{if } i = 0, \\ 3 & \text{if } n = 2g+3,\ g = 2k,\ i = k-2, \\ 2 & \text{otherwise.} \end{cases}$$

(2) $\gamma_{i,4+2p} = 0$ *if* $p > i$ *and* $4 + 2p \leqslant g$.

Proof. We use formulae for $\gamma_{i,j}$ and notation for $A(a,b,c,d)$ and $B(j,a,b,c,d)$ as given in [**2**, Proposition 4.3], with $a = b = 3$, $c = 2i+1$, $d = n-2i-7$. We next give details for the generic case. Suppose that $i > 0$ and $d > g+1$. We then have that $A(a,b,c,d) = 0$. In the formula for $B(j,a,b,c,d)$, we have that $f(a) = 3$, $f(b) = 3$, $f(c) = 2i+1$, $f(d) = n-d = a+b+c = 2i+7$, $f(a+b) = 6$, $f(a+c) = 2i+4$ and $f(a+d) = n-d-3 = a+b+c-3 = 2i+4$. If $j \geqslant 4+2i$, then the only non-zero term in $B(j,a,b,c,d)$ is $\max\{f(d)-1-j, 0\} = 2$ when $j = 4+2i$, and this term is 0 if $j = 4+2p$ with $p > i$. Thus, $\gamma_{i,4+2i} = 2$, and $\gamma_{i,4+2p} = 0$ if $p > i$. This takes care of most cases. When $i = 0$, we get an extra contribution -1 from $-\max\{a+b-1-j, 0\}$. We need to check the cases where $d \leqslant g+1$, involving five curves and six coefficients, which we compute directly. □

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We now prove that $\mathscr{C}_3$ consists of independent curves, whose ith member we denote by C_i. We examine the matrix $\Gamma = (\gamma_{i,j})$, whose ith row is obtained by writing the curve C_i in the basis $\{F_{1,1,j,n-j-2}\}_{j=1}^{g}$. That is, $\gamma_{i,j}$ is defined by

$$C_i = \sum_{j=1}^{n} \gamma_{i,j} F_{1,1,j,n-j-2}.$$

The rows of Γ that come from the $(1,1)$-curves in $\mathscr{C}_3$ look like rows of the identity matrix. In particular, since all the curves $F_{1,1,j,n-j-2}$ with j odd are in $\mathscr{C}_3$, this means that we can use these rows to echelonize all the columns with odd indices. Thus, it is enough to show that the submatrix M of Γ consisting of rows labelled by the $(3,3)$-curves and columns corresponding to $F_{1,1,j,n-j-2}$, with j even, has full rank. Apply Lemma 8.1. This shows that after dropping the leftmost column of M, corresponding to $F_{1,1,2,n-4}$, we obtain a submatrix N that is lower triangular with non-zero entries on the diagonal. Thus, N and M have rank $k-3$ if g is odd and $k-2$ if g is even, and Γ has full rank.

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