On a Group of Parabolas associated with the Triangle.

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§ 1. The general theorem underlying the subject of this paper is as follows:—

If through $c_1, c_2, c_3, \ldots$ points on AB, a side of $\triangle ABC$, rays be drawn from two vertices O_1, O_2, the former meeting AC in $b_1, b_2, b_3, \ldots$, and the latter meeting BC in $a_1, a_2, a_3, \ldots$, then the lines $a_1 b_1$, $a_2 b_2$, ... envelope a conic touching the sides AC and BC. This follows since the ranges $a_1 a_2 a_3 \ldots$, $b_1 b_2 b_3 \ldots$, are homographic.

If O_1 and O_2 are points at infinity in the direction of BC, and in the direction perpendicular to BC respectively, a_1b_1, a_2b_2, ... will be the diagonals of rectangles inscribed in ABC and having one side on BC.

It is clear from the method of generation that in this case the line at infinity is a tangent to the conic, which accordingly is a parabola.

The following facts are also to be noted:—

(1) To C, looked on as a point on range AC, corresponds B on range BC; hence BC is a tangent to the conic at B.

(2) If a line be drawn through C perpendicular to BC meeting BA in P, and a parallel to CB be drawn through P meeting CA in X, we find that to C, looked on as a point on the range BC, corresponds X on AC; hence the conic touches CA at X.

(3) The perpendicular AD drawn from A to BC is a limiting case of the rectangles, and hence is also a tangent to the conic.

It is obvious that the set a_1b_1, ..., etc., generate one parabola, and the set of second diagonals $a_n c_1 \ldots$ of the same rectangles generate another parabola.

Consequently, from the three sets of rectangles that can be inscribed in any triangle, we get six parabolas. There are certain

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other groups of parabolas associated with the triangle, *e.g.* the parabolas of Artzt and the parabolas of Brocard, which have had their properties carefully investigated, but, so far as I know, the present group has not hitherto been discussed. The conics have some very interesting and important properties.

§ 2. From the method of generation we can construct geometrically their chief elements.

We can, for instance, find the direction of the axis of each of them. Take Parabola I. which touches BC at B, AD, CA, and the line at infinity (ι_∞). Since we know that B is the point of contact of BC, we know two consecutive tangents CB, CB, and we now apply Brianchon's Theorem to the hexagon

$$CB,\ CB,\ DA,\ CA,\ \iota_\infty,\ \iota_\infty.$$

The point of intersection of the two consecutive tangents $\iota_\infty\, \iota_\infty$ being the point where the conic touches ι_∞, gives the direction of axis.

This theorem then gives that

$$\left[\binom{CB}{CB}\binom{CA}{\iota_\infty}\right]\quad\left[\binom{CB}{DA}\binom{\iota_\infty}{\iota_\infty}\right]\quad\left[\binom{DA}{CA}\binom{\iota_\infty}{CB}\right]$$

are concurrent where $\binom{CB}{DA}$ means intersection of CB and DA,

i.e. $\left.\begin{array}{l} B \text{ and point at infinity on } CA, \\ D \text{ and point at infinity on axis}, \\ A \text{ and point at infinity on } CB, \end{array}\right\}$ are concurrent.

Hence the construction. (Fig. 1.)

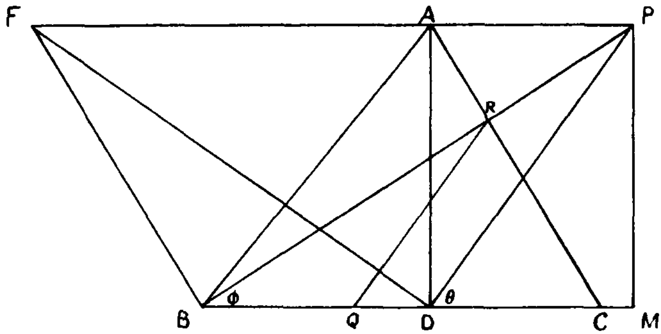

Figure 1

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Through B draw a parallel to CA. Through A draw a parallel to BC. Let these intersect at F. Then DF is the direction of the axis.

Also, since D is the point of intersection of two rectangular tangents, it must lie on the directrix. Hence a line drawn through D perpendicular to DF will be the directrix of this parabola.

We can also find the tangent at the vertex, for it is parallel to the directrix.

Applying Brianchon's Theorem to ι_∞, CB, CB, AD, CA, t, where t is the unknown tangent, we get the following construction :—

Through A draw a line parallel to CB, and through D a line parallel to direction of tangent, *i.e.* parallel to the directrix DP.

Let these cut at P. Join BP and let it cut CA at R. The line RQ drawn through R parallel to directrix is the tangent at the vertex.

The point of contact of this tangent can easily be found by another application of Brianchon's Theorem. This being constructed, the focus can then be found. The coordinates of the focus are discussed in a later section of this paper.

By drawing a tangent at an angle of 45° to BC, we can solve the problem of inscribing a square in a given triangle.

§ 3. This construction enables us also to calculate the length intercepted on the tangent at the vertex by CB and CA.

We have $\qquad \angle PDC = \angle ADF = \theta$, say,

and let $\qquad \angle PBC = \phi$.

Then we have

$$\frac{RQ}{PD} = \frac{BR}{BP}.$$

From $\triangle ABR$ we get $BR = \dfrac{c \sin A}{\sin \overline{C+\phi}}$,

and $\qquad BP = h_a \operatorname{cosec} \phi$, where h_a is altitude from A.

Substitution of these values gives us

$$RQ = \frac{c \sin A \sin \phi}{\sin\theta \sin \overline{C+\phi}} = \frac{c \sin A}{\sin\theta\,(\sin C \cot \phi + \cos C)}.$$

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But $\tan\theta = \frac{a}{h_a} = \frac{a}{b\sin C}$

and $\tan\phi = \frac{h_a}{BD+DM} = \frac{h_a}{h_a\cot B + h_a\cot\theta} = \frac{1}{\cot B + \cot\theta}$

$$= \frac{1}{\frac{\cos B}{\sin B} + \frac{b\sin C}{a}} = \frac{a\sin B}{a\cos B + b\sin C\sin B}.$$

$$\therefore \quad RQ = \frac{\sin A}{\sin\theta\left\{\sin C\,\frac{a\cos B + b\sin C\sin B}{a\sin B} + \cos C\right\}}$$

which, after reduction, gives

$$RQ = \frac{ac\sin A\sin B}{\sin\theta\{a\sin A + b\sin^2 C\sin B\}} = \frac{a^2c\sin B}{\sin\theta\,(a^2 + b^2\sin^2 C)}.$$

Now $\qquad \tan\theta = \frac{a}{b\sin C}$

$$\therefore \quad \sin^2\theta = \frac{1}{1+\cot^2\theta} = \frac{a^2}{a^2 + b^2\sin^2 C}$$

$$\therefore \quad RQ = c\sin B\sin\theta.$$

By carrying out the same geometrical construction with reference to Parabola II., *i.e.* the conic touching BC at C, AB, AD, it is easy to prove that its directrix is inclined to BC at the same angle as QR, but in the opposite sense. Hence the same is true of the tangent at the vertex. Now, since these tangents are the diagonals of inscribed rectangles, it follows that they must be diagonals of the same rectangle, and therefore equal in length.

The length of the tangents at the vertices of Parabolas III. and and IV. is

$$a\sin C\sin\theta' \text{ where } \tan\theta' = \frac{b}{c\sin A}.$$

Similarly, for Parabolas V. and VI., the length of the tangents at the vertices is

$$b\sin A\sin\theta'' \text{ where } \tan\theta'' = \frac{c}{a\sin B}.$$

(See page 82.)

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§ 4. The trilinear equation of these parabolas can be found as follows :—

A conic touching the sides of the triangle $\alpha=0$, $\beta=0$, $\gamma=0$ can have its equation in normal coordinates in either of the two equivalent forms :—

$$l^2\alpha^2+m^2\beta^2+n^2\gamma^2-2lm\,\alpha\beta-2mn\,\beta\gamma-2ln\,\alpha\gamma=0,$$

or
$$\sqrt{l\alpha}+\sqrt{m\beta}+\sqrt{n\gamma}=0.$$

In our case, if we are finding the equation of the parabola touching BC, AD, CA, we must write for $\gamma=0$, $\beta\cos B-\gamma\cos C=0$, which is the equation of AD. The equation now becomes

$$l^2\alpha^2+m^2\beta^2+n^2(\beta\cos B-\gamma\cos C)^2-2lm\,\alpha\beta$$
$$-2mn\,\beta(\beta\cos B-\gamma\cos C)-2ln\,\alpha(\beta\cos B-\gamma\cos C)=0.$$

We have now to determine the values of l, m, n, or rather their ratios.

Since the conic touches BC at B, the equation resulting from substituting $\alpha=0$ must give two zero values of γ. The equation giving $\dfrac{\beta}{\gamma}$, when $\alpha=0$, is

$$\beta^2(m-n\cos B)^2-2n\cos C\,(m-n\cos B)\,\beta\gamma+\gamma^2n^2\cos^2 C=0.$$

In order to have two zero values of γ, we must have

$$m-n\cos B=0,\ \text{ or }\ \frac{m}{n}=\cos B.$$

Utilising the fact that the line at infinity $a\alpha+b\beta+c\gamma=0$ is a tangent to the conic, we can get the value of $\dfrac{l}{n}$. For this purpose it is convenient to write the equation of the conic in the form

$$n(\beta\cos B-\gamma\cos C)=\{-(\sqrt{l\alpha}+\sqrt{m\beta})\}^2=l\alpha+m\beta+2\sqrt{lm\,\alpha\beta}.$$

Substituting for γ its value $\gamma=-\dfrac{a\alpha+b\beta}{c}$, and arranging the resulting equation, we get

$$\alpha(an\cos C-lc)+\beta(na-mc)=2c\sqrt{lm\,\alpha\beta},$$

or
$$\alpha^2(an\cos C-lc)^2+\beta^2(na-mc)^2$$
$$+2\,\alpha\beta\cdot\{(an\cos C-lc)(na-mc)-2c^2lm\}=0.$$

Expressing the condition that, in order that $a\alpha+b\beta+c\gamma=0$ may

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touch the conic, the quadratic must have equal roots, we get, after reduction, that

$$a \cos C - \frac{l}{n} c - c \cos B \cos C = 0.$$

$$\therefore \quad \frac{l}{n} = \frac{\cos C}{c} (a - c \cos B),$$

and, since $a = b \cos C + c \cos B$, we have that

$$\frac{l}{n} = \frac{b \cos^2 C}{c}.$$

By substituting these values just found for $\frac{m}{n}$, $\frac{l}{n}$, and rationalising we get as the equation to this conic, which I call Parabola I.,

$$(b \cos C\alpha + c\gamma)^2 = 4bc \cos B\, \alpha\beta.$$

Equation of Parabola

II. touching BA, BC at C, AD is $(c \cos B\, \alpha + b\, \beta)^2 = 4bc \cos C\, \alpha\gamma$.

III. " BA at A, BC, BE " $(a \cos C\, \beta + c\, \gamma)^2 = 4ac \cos A\, \alpha\beta$.

IV. " AB, AC at C, BE " $(c \cos A\, \beta + a\, \alpha)^2 = 4ac \cos C\, \beta\gamma$.

V. " BA at A, BC, CF " $(a \cos B\, \gamma + b\, B)^2 = 4ab \cos A\, \alpha\gamma$.

VI. " AB at B, AC, CF " $(b \cos A\, \gamma + a\, \alpha)^2 = 4ab \cos B\, \beta\gamma$.

§ 5. *Common Tangents.*

This method of generating the conics leads readily to the investigation of their common tangents.

It is a well-known theorem in Elementary Geometry that the locus of the centres of the inscribed rectangles whose sides are parallel and perpendicular to BC is the line joining the mid point of BC to the mid point of the altitude AD.

Let this line $A'X$ cut AC in V, and BA in V'. Then obviously $A'V$ will be the diagonal of one of these inscribed rectangles, and is therefore a tangent to Parabola I. (See Fig. II.)

Also $A'V'$ will be the diagonal of an inscribed rectangle, and since it joins a point on BC to a point on BA, it will touch Parabola II.

Hence this line $A'VV'$ is a common tangent to these two conics.

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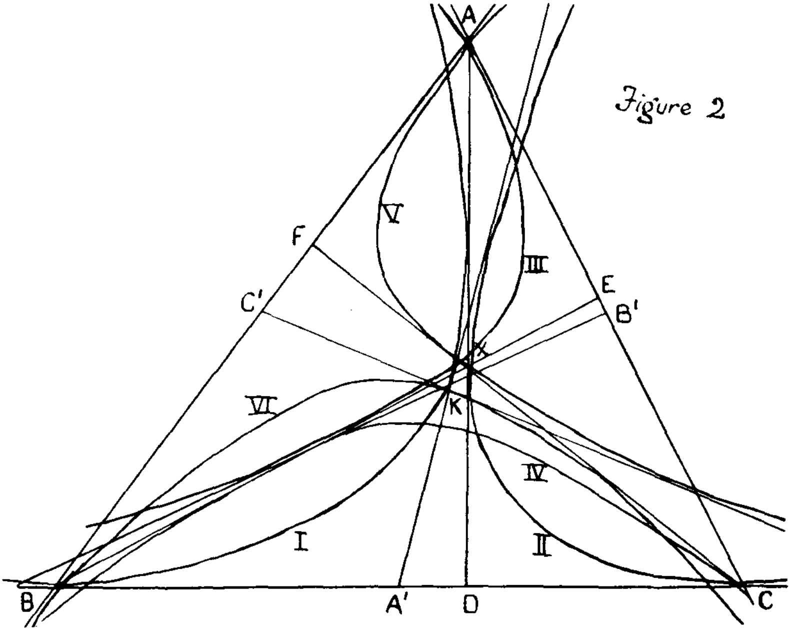

Figure 2

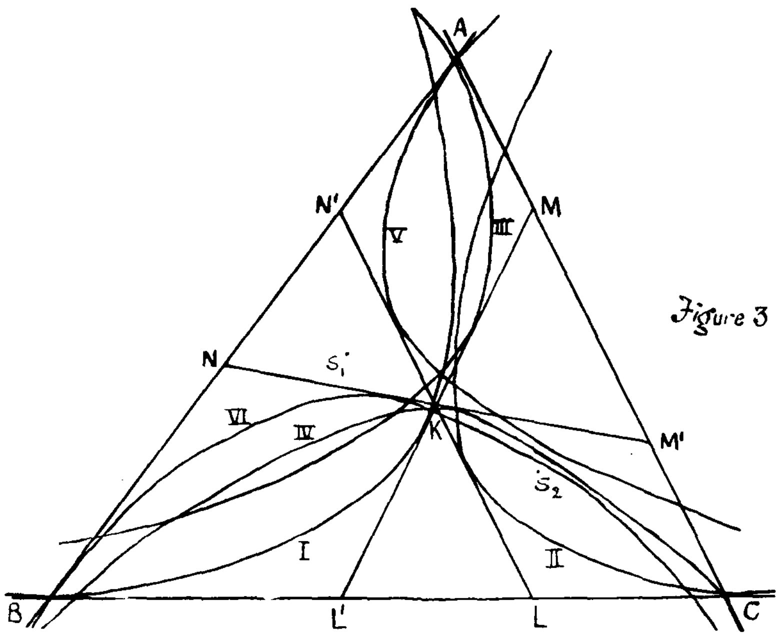

Figure 3

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Calling B', C' the mid-points of AC and AB; and Y, Z the mid-points of the altitudes BE, CF, we get that $B'Y$ is a common tangent to Parabolas III. and IV., and that $C'Z$ is a common tangent to Parabolas V. and VI.

Further, we know that these lines $A'X$, $B'Y$, $C'Z$ are concurrent and meet in K, the symmedian point of the triangle.

The pair of Parabolas I. and II. have thus four common tangents BC, AD, $A'X$, and the line at infinity; and this is the maximum number they can have.

Similarly for the Parabolas III. and IV., V. and VI.

We thus have the interesting property that these three pairs of conics have two sets of common tangents that are concurrent, viz., the altitudes AD, BE, CF, and the lines $A'X$, $B'Y$, $C'Z$, and the point of concurrency of the latter is the Symmedian point K.

Common Tangents to Parabolas belonging to different pairs.—Bearing in mind the locus problem previously quoted regarding the mid-point of the inscribed rectangle, we see that if two parabolas belonging to different groups—say Parabolas I. and III....have a common tangent other than AC, BC, and the line at infinity, that line must be the common diagonal of two rectangles which have K for the mid points.

Now it is obvious, if $L'M$ have K for its mid point, that we can draw two rectangles that have $L'M$ for a diagonal—

(1) one with sides parallel and perpendicular to BC;

(2) one with sides parallel and perpendicular to AC.

Hence $L'M$ is a common tangent to I. and III. Also LN' is a common diagonal of rectangles with sides parallel and perpendicular to BC, and parallel and perpendicular to AB respectively. Hence LN' is a common tangent to II. and V.

Similarly it can be shown that NM' is a common tangent to IV. and VI. (See Fig. III.)

Thus the remarkable fact emerges that the three common tangents to these three pairs of conics are concurrent in the Symmedian Point.

It may be remarked that the lines $L'M$, LN', NM', are the equal antiparallels to the sides of the triangle through K, and that L, L', M, M', N, N', lie on the Second Lemoine or Cosine Circle with its centre K.

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Thus (1) Six common tangents are concurrent in K.

(2) Three common tangents are concurrent in H, the orthocentre.

(3) Each side of the triangle is a tangent to four parabolas.

It can be shown by utilising the locus theorem previously employed, that these are all the possible pairs of common tangents other than the sides of the triangle.

We can show, for instance, that III. and VI. cannot have a common tangent other than AC, and the line at infinity, as follows:—

III. is the envelope of diagonals, *e.g.* ab, which have their extremities on BC and AC, while VI. is the envelope of diagonals, *e.g.* bc, which have their extremities on AC and AB. The mid point of the former set of diagonals lies on $B'Y$, and the mid point of the latter set on $C'Z$. Hence K must lie on the line bac, and must be the mid point of ba, and also of bc, which is impossible. Hence III. and VI. cannot have a common tangent other than AC and the line at infinity.

The points of contact of the common tangent $A'V$, $A'V'$ can very easily be found geometrically by applying Brianchon's Theorem to the hexagon

$$AX, \; XA', \; XA', \; A'B, \; i_x, \; AC.$$

We find in this way that the line joining X to point at infinity on CB, AA', and the line drawn from point of contact parallel to AC are concurrent.

Hence to find the point of contact of $A'V$. Draw through X a line parallel to BC. Let this meet AA' in O. The line through O parallel to AC meets $A'V$ in its point of contact with Parabola I.

Bearing in mind that X is mid-point of AD, we have that T is mid-point of $A'V$.

Similarly Parabola II. touches $A'V'$ in its middle point, and so on for the other pairs.

We proceed now to calculate the coordinates of the points of contact of Parabola I. with AD.

To do this, combine

$$(b \cos C\, \alpha + c\,\gamma)^2 = 4\, bc \cos B\, \alpha\beta \quad \text{and} \quad \beta \cos B - \gamma \cos C = 0.$$

We find that these give, by eliminating β,

$$(b \cos C\, \alpha - c\,\gamma)^2 = 0.$$

$$\therefore \quad b \cos C\, \alpha - c\,\gamma = 0 \text{ twice.}$$

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Calling the coordinates of the point of contact α_1, β_1, γ_1, we find that

$$\frac{\alpha_1}{c\cos B} = \frac{\beta_1}{b\cos^2 C} = \frac{\gamma_1}{b\cos B\cos C}. \quad \ldots\ldots\ldots\ldots\ldots\text{(A)}$$

Similarly if α_2, β_2, γ_2, are the coordinates of the point of contact of Parabola II. with AD, we find that

$$\frac{\alpha_2}{b\cos C} = \frac{\beta_2}{c\cos C\cos B} = \frac{\gamma_2}{c\cos^2 B}. \quad \ldots\ldots\ldots\ldots\ldots\text{(B)}$$

From (A) we get that each fraction

$$= \frac{a\,\alpha_1 + b\,\beta_1 + c\,\gamma_1}{ac\cos B + b^2\cos^2 C + bc\cos C\cos B}$$

$$= \frac{a\,\alpha_1 + b\,\beta_1 + c\,\gamma_1}{ac\cos B + b\cos C\,(b\cos C + c\cos B)} = \frac{a\,\alpha_1 + b\,\beta_1 + c\,\gamma_1}{a\,(c\cos B + b\cos C)} = \frac{2\,\Delta}{a^2}.$$

Similarly each fraction in $B = \dfrac{2\,\Delta}{a^2}$.

Hence we have the relations

$$\frac{\alpha_1 + \alpha_2}{c\cos B + b\cos C} = \frac{\beta_1 + \beta_2}{b\cos^2 C + c\cos C\cos B} = \frac{\gamma_1 + \gamma_2}{b\cos B\cos C + c\cos^2 B} = \frac{2\,\Delta}{a^2},$$

i.e. $$\frac{\alpha_1 + \alpha_2}{a} = \frac{\beta_1 + \beta_2}{a\cos C} = \frac{\gamma_1 + \gamma_2}{a\cos B} = \frac{2\,\Delta}{a^2}.$$

The α coordinate of the point midway between these points of contact is therefore given by $\dfrac{2\,\alpha}{a} = \dfrac{2\,\Delta}{a^2}$ or $\alpha = \dfrac{\Delta}{a}$,

$\therefore$ whence $\alpha = \dfrac{h_a}{2}$.

Thus the point of contact of Parabolas I. and II. with AD are symmetrically situated with respect to X, the mid point of AD.

A similar result holds for the other pairs III., IV., V., VI.

§ 6. *Common Chords.*

An investigation of the common chords also leads to an important property of the Symmedian point. A pair of common chords of Parabolas III. and V. is given by

$$\{(a\cos C\,\beta + c\gamma)^2 - 4ac\cos A\,\alpha\beta\} - \{(a\cos B\,\gamma + b\,\beta)^2 - 4ab\cos A\,\alpha\gamma\} = 0,$$

which reduces to

$$-\cos A\,(c\,\beta - b\,\gamma)\,\{\beta\,(a\cos C + b) + \gamma\,(c + a\cos B) + 4a\,\alpha\} = 0.$$

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One common chord, therefore, is $c\beta - b\gamma = 0$, which shows that it passes through K.

The line $4a\alpha + \beta(a \cos C + b) + \gamma(c + a \cos B)$ is an *ideal* common secant (using Poncelet's term), since it may be shown that it meets the curves in imaginary points.

We can show in the same way that BK is a common chord of I. and IV., and that CK is a common chord of II. and IV. Thus we have the additional interesting fact that three common secants, as well as six common tangents, pass through K.

If $A_1 B_1 C_1$ are points of intersection of III. V.; I. VI.; II. IV.; we have that triangles $A_1B_1C_1$, and ABC are homologous. K is the centre of homology, and the axis of homology is a line whose equation is

$$\frac{(-3a^2+b^2+c^2)^2}{a(b^2+c^2-a^2)}\alpha + \frac{(a^2-3b^2+c^2)^2}{b(a^2-b^2+c^2)}\beta + \frac{(a^2+b^2-3c^2)^2}{c(a^2+b^2-c^2)}\gamma = 0.$$

§7. *Foci.*

We have now to consider the foci of these parabolas. Their coordinates may be calculated by using the fact that the circum-circle of the triangle formed by any three tangents to a parabola passes through the focus.

If we wish to calculate the coordinates of S_1, the focus of Parabola I., we must first find equation to circle ADC, and also equation to circle $A'XD$.

The most convenient form of these equations is that which gives us the Radical Axis of the circles at a glance. We know that if

$$\beta\gamma \sin A + \gamma\alpha \sin B + \alpha\beta \sin C = 0$$

is equation to circle ABC, the equation of circle ADC can be put into the shape—

$$\beta(\alpha \sin A + \beta \sin B + \gamma \sin C) \\ + k(\beta\gamma \sin A + \gamma\alpha \sin B + \alpha\beta \sin C) = 0,$$

since $\beta = 0$ is the Radical Axis of these circles.

To determine k, we know that this circle passes through D, *i.e.* $\alpha = 0$; $\beta \cos B = \gamma \cos C$. Now, when $\alpha = 0$, we have

$$\beta(\beta \sin B + \gamma \sin C) + k\beta\gamma \sin A = 0.$$

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Neglecting $\beta=0$, which gives the other point of intersection C, we have

$$\beta \sin B+\gamma \sin C+k\gamma \sin A=0,$$

or
$$\frac{\beta}{\gamma}\sin B+\sin C+k\sin A=0.$$

Substituting $\dfrac{\cos B}{\cos C}$ for $\dfrac{\beta}{\gamma}$, we find that

$$\cos C\sin B+\cos B\sin C+k\sin A\cos B=0,$$

or
$$\sin A+k\sin A\cos B=0,$$

which, since $\sin A \neq 0$, gives $k=-\dfrac{1}{\cos B}$. The equation to circle ADC, therefore, is

$$-\beta\cos B\,(\alpha\sin A+\beta\sin B+\gamma\sin C)$$
$$+(\beta\gamma\sin A+\gamma\alpha\sin B+\alpha\beta\sin C)=0.$$

To find equation to circle $A'XD$, we can write the equation to any circle in the form—

$$(l\alpha+m\beta+n\gamma)(\alpha\sin A+\beta\sin B+\gamma\sin C)$$
$$+k(\beta\gamma\sin A+\gamma\alpha\sin B+\alpha\beta\sin C)=0.$$

This passes through D, *i.e.* $\alpha=0$ and $\beta\cos B=\gamma\cos C$.

Substituting these values of α and $\dfrac{\beta}{\gamma}$, we get

$$(m\cos C+n\cos B)(\cos C\sin B+\cos B\cos C)$$
$$+k\cos C\sin A\cos B=0,$$

or
$$(m\cos C+n\cos B)\sin A+k\cos C\sin A\cos B=0.$$

Since $\sin A \neq 0$, we have

$$m\cos C+n\cos B+k\cos C\cos B=0 \quad\ldots\ldots\ldots\ldots\ldots\ \text{I.}$$

The circle also passes through A', *i.e.* $\alpha=0$, and $\beta\sin B=\gamma\sin C$.

Substituting these values of α and $\dfrac{\beta}{\gamma}$ in the general equation, we get

$$(m\sin C+n\sin B)(2\sin B\sin C)+k\sin A\sin B\sin C=0.$$

Removing the factor $2\sin B\sin C$ we have

$$m\sin C+n\sin B+\frac{k}{2}\sin A=0.\ \ldots\ldots\ldots\ldots\ldots\ldots\ \text{II.}$$

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By eliminating m between I. and II. we get

$$n(\sin B\cos C-\sin C\cos B)+\frac{k}{2}\cos C(\sin A-2\cos B\sin C)=0,$$

which reduces to

$$n\sin\overline{B-C}+\frac{k}{2}\cos C.\sin\overline{B-C}=0.$$

Neglecting the factor $\sin\overline{B-C}$, we have

$$n+\frac{k}{2}\cos C=0$$

or $$\frac{n}{k}=-\frac{\cos C}{2}.$$

From the same two equations, by eliminating n, we get

$$\frac{m}{k}=-\frac{\cos B}{2}.$$

The equation to circle $A'XD$ may now be written

$$\left(\frac{l}{k}\alpha-\frac{\cos B}{2}\beta-\frac{\cos C}{2}\gamma\right)(\alpha\sin A+\beta\sin B-\gamma\sin C)$$
$$+(\beta\gamma\sin A+\ldots)=0.$$

To determine $\frac{l}{k}$, we know that the circle passes through X, whose coordinates are given by

$$\beta\cos B-\gamma\cos C=0$$

and $$\alpha=\frac{h_a}{2}=\frac{h_a a}{2a}=\frac{2\Delta}{2a}=\frac{a\alpha+b\beta+c\gamma}{2a}.$$

$$\therefore\quad -a\alpha+b\beta+c\gamma=0.$$

For the coordinates of X we now have

$$\frac{\alpha}{\gamma}=\frac{1}{\cos B}\text{ and }\frac{\beta}{\gamma}=\frac{\cos C}{\cos B}.$$

Substituting the values of $\frac{\alpha}{\gamma}$, $\frac{\beta}{\gamma}$ in the equation to the circle, we get

$$\left(\frac{l}{k}\frac{1}{\cos B}-\frac{\cos B}{2}\frac{\cos C}{\cos B}-\frac{\cos C}{2}\right)\left(\frac{\sin A}{\cos B}+\frac{\cos C}{\cos B}\sin B+\sin C\right)$$
$$+\left(\frac{\cos C}{\cos B}\sin A+\frac{\sin B}{\cos B}+\frac{1}{\cos B}\cdot\frac{\cos C}{\cos B}\sin C\right)=0,$$

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which reduces to

$$\left(\frac{l}{k}\,\frac{1}{\cos B}-\cos C\right)\left(\frac{2\sin A}{\cos B}\right)+\frac{\cos C\sin A}{\cos B}$$
$$+\frac{\sin B\cos B+\sin C\cos C}{\cos^2 B}=0.$$

The last term of the equation can be simplified; for it is equal to

$$\frac{\cos C\sin A}{\cos B}+\frac{\sin 2B+\sin 2C}{2\cos^2 B}=\frac{\cos C\sin A}{\cos B}+\frac{\sin\overline{B+C}\cos\overline{B-C}}{\cos^2 B}$$
$$=\sin A\left(\frac{\cos C}{\cos B}+\frac{\cos\overline{B-C}}{\cos^2 B}\right)$$
$$=\frac{\sin A}{\cos^2 B}(\cos C\cos B+\cos\overline{B-C})$$
$$=\frac{\sin A\cos C}{\cos B}+\frac{\sin A\cos\overline{B-C}}{\cos^2 B}.$$

After simplifying it, the equation in $\frac{l}{k}$ can be written as—

$$\frac{l}{k}\,\frac{2}{\cos B}-\cos C+\frac{\cos\overline{B-C}}{\cos B}=0,$$

or
$$\frac{l}{k}\,\frac{2}{\cos B}+\frac{\sin B\sin C}{\cos B}=0,$$

or
$$\frac{l}{k}=-\frac{\sin B\sin C}{2}.$$

The equation to circle $A'XD$ is therefore

$$\left(-\frac{\sin B\sin C}{2}\alpha-\frac{\cos B}{2}\beta-\frac{\cos C}{2}\gamma\right)(\alpha\sin A+\beta\sin B+\gamma\sin C)$$
$$+(\beta\gamma\sin A+\ldots)=0.$$

Hence, by examining the equations to circles $A'XD$ and ADC, we find that their Radical Axis is

$$-\frac{\sin B\sin C}{2}\alpha-\frac{\cos B}{2}\beta-\frac{\cos C}{2}\gamma=-\beta\cos B$$

or
$$\frac{\sin B\sin C}{2}\alpha-\beta\cos B+\gamma\cos C=0.$$

To find now the coordinates of S_1, the focus of Parabola I., we have to find the coordinates of the points of intersection of the Radical Axis of circle $A'XD$ and circle ADC with the circle $A'XD$.

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As we already know that D is one of the points of intersection, we simplify our working by eliminating α between the equation of the circle $A'XD$ and the equation to the Radical Axis. By doing so, and arranging the resulting equation in terms of $\frac{\beta}{\gamma}$, we get

$$\frac{\beta^2}{\gamma^2}\left(-\frac{\sin A\cos^2 B}{\sin B\sin C}-\sin B\cos B+\frac{\cos B}{\sin B}\right)$$

$$+\frac{\beta}{\gamma}\left(\frac{\cos B\cos C\sin A}{\sin B\sin C}-\sin C\cos B+\sin A+\frac{\cos B}{\sin C}-\frac{\cos C}{\sin B}\right)-\frac{\cos C}{\sin C}=0.$$

The coefficient of $\frac{\beta^2}{\gamma^2}=\frac{\cos B}{\sin B}\left(1-\frac{\sin A\cos B}{\sin C}\right)-\sin B\cos B$

$$=\frac{\cos B}{\sin B}\times\frac{\cos A\sin B}{\sin C}-\sin B\cos B$$

$$=\cos B\left(\frac{\cos A-\sin B\sin C}{\sin C}\right)$$

$$=-\frac{\cos^2 B\cos C}{\sin C}.$$

The coefficient of $\frac{\beta}{\gamma}$

$$=\frac{\cos C}{\sin B}\left(\frac{\cos B\sin A}{\sin C}-1\right)-\sin C\cos B+\sin\overline{B+C}+\frac{\cos B}{\sin C}.$$

$$=\frac{\cos C}{\sin B}\times-\frac{\cos A\sin B}{\sin C}+\sin B\cos C+\frac{\cos B}{\sin C}$$

$$=\cos C\left(\frac{\cos\overline{B+C}}{\sin C}+\sin B\right)+\frac{\cos B}{\sin C}$$

$$=\frac{\cos B\cos^2 C}{\sin C}+\frac{\cos B}{\sin C}=\frac{\cos B}{\sin C}(\cos^2 C+1).$$

The equation in $\frac{\beta}{\gamma}$ now becomes

$$\frac{\beta^2}{\gamma^2}-\frac{\cos^2 C+1}{\cos B\cos C}\,\frac{\beta}{\gamma}+\frac{\cos C}{\cos^2 B\cos C}=0.$$

$$\left(\frac{\beta}{\gamma}-\frac{\cos C}{\cos B}\right)\left(\frac{\beta}{\gamma}-\frac{1}{\cos C\cos B}\right)=0.$$

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$\therefore \quad \frac{\beta}{\gamma} = \frac{\cos C}{\cos B}$. This is point D, as we would expect; and $\frac{\beta}{\gamma} = \frac{1}{\cos C \cos B}$.

Knowing this and the equation to line DS_1, we get

$$\frac{\alpha}{\gamma} = \frac{\sin C}{\sin B \cos C}.$$

The coordinates of S_1 may therefore be written

$$\frac{\alpha}{\sin C \cos B} = \frac{\beta}{\sin B} = \frac{\gamma}{\sin B \cos B \cos C}.$$

In the same way by considering the equations of circles ADB and $A'DX$ and proceeding as above, we get for the coordinates of S_2 the focus of Parabola II.

$$\frac{\alpha}{\frac{1}{\sin C \cos B}} = \frac{\beta}{\frac{1}{\sin B}} = \frac{\gamma}{\frac{1}{\sin B \cos B \cos C}}.$$

We therefore have the interesting result that the foci S_1 and S_2 are inverse points with respect to $\triangle ABC$.

The coordinates of the remaining foci are

$$\begin{cases} S_3 : \dfrac{\alpha}{\sin C \cos C \cos A} = \dfrac{\beta}{\sin A \sin C} = \dfrac{\gamma}{\sin C} \\[2ex] S_4 : \dfrac{\alpha}{\frac{1}{\sin C \cos C \cos A}} = \dfrac{\beta}{\frac{1}{\sin A \cos C}} = \dfrac{\gamma}{\frac{1}{\sin C}} \end{cases}$$

$$\begin{cases} S_5 : \dfrac{\alpha}{\sin A} = \dfrac{\beta}{\sin A \cos A \cos B} = \dfrac{\gamma}{\sin B \sin A}. \\[2ex] S_6 : \dfrac{\alpha}{\frac{1}{\sin A}} = \dfrac{\beta}{\frac{1}{\sin A \cos A \cos B}} = \dfrac{\gamma}{\frac{1}{\sin B \sin A}}. \end{cases}$$

Inverse points with reference to a triangle possess many interesting properties. From one of these properties they are also called isogonally conjugate points.

For a discussion of these points see Rouché et Camberousse, Geométrie, Pt. I., Sixième Edition, pp. 437–439.

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