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RATIONAL POINTS ON THREE SUPERELLIPTIC CURVES

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Abstract

In this paper, we obtain all rational points (x, y) on the superelliptic curves

$$y^k = x(x+2),$$
$$y^k = x(x+2)(x+3),$$
$$y^k = x(x+1)(x+3).$$

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1. Introduction

In 1975, Erdős and Selfridge [4] proved that the superelliptic curve

$$y^k = (x+1)\cdots(x+l), \qquad x \geq 0,\ l \geq 2,\ k \geq 2, \tag{1.1}$$

has no integral solution. This put an end to the old question whether the product of consecutive positive integers could ever be a perfect power. In 1999, Sander [7] raised the following conjecture.

CONJECTURE. For $k \geq 2$ and $l \geq 2$, all rational points (x, y) on (1.1) are the trivial ones with $x = -j$ $(j = 1, \ldots, l)$ and $y = 0$, except for the case $k = l = 2$ where we have precisely those satisfying

$$x = \frac{2c_1^2 - c_2^2}{c_2^2 - c_1^2}, \qquad y = \frac{c_1 c_2}{c_2^2 - c_1^2},$$

with coprime integers $c_1 \neq \pm c_2$.

Sander [7] himself proved that the conjecture is true for $k \geq 2$ and $2 \leq l \leq 4$. Later, Lakhal and Sander [5] proved that it is true for $k \geq 2$ and $l = 5$.

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Meanwhile, let $x \geq 1$, $l \geq 3$, $k \geq 2$, and $0 \leq d_1 < d_2 < \cdots < d_{l-1} < l$ be integers, Erdős and Selfridge [4] also considered the superelliptic curve

$$y^k = (x + d_1) \cdots (x + d_{l-1}), \tag{1.2}$$

and conjectured that there are only solutions of (1.2) with $l \geq 4$ given by

$$\frac{6!}{5} = 12^2, \quad \frac{10!}{7} = 720^2, \quad \frac{4!}{3} = 2^3.$$

In 2003, Saradha and Shorey [8, 9] proved this result. One year later, Bennett [1] found all the 30 integral solutions of $by^k = (x + d_1) \cdots (x + d_{l-1})$ with $l = 3$ and $k \geq 3$, $l \in \{4, 5\}$ and $k \geq 2$, and $P(b) \leq l$, where $P(b)$ is the greatest prime divisor of b.

The purpose of this paper is to obtain all the rational points on superelliptic curves (1.2) for $k \geq 2$ and $l \in \{3, 4\}$.

Theorem 1.1. *For $k \geq 3$, the only rational points (x, y) on the curve*

$$y^k = x(x + 2) \tag{1.3}$$

are the trivial ones with $x = 0$ or -2, $y = 0$ for $k \geq 3$ and $x = -1$, $y = -1$ for all odd $k \geq 3$ and $x = -4$ or 2, $y = 2$ for $k = 3$. For $k = 2$, all rational solutions of (1.3) are given by

$$x = \frac{2c_1^2}{c_2^2 - c_1^2}, \quad y = \frac{2c_1c_2}{c_2^2 - c_1^2},$$

with coprime integers $c_1 \neq \pm c_2$.

Theorem 1.2. *For $k \geq 2$ and $k \neq 3$, the only rational points (x, y) on the curves*

$$y^k = x(x + 2)(x + 3), \tag{1.4}$$

$$y^k = x(x + 1)(x + 3) \tag{1.5}$$

are the trivial ones with $y = 0$.

Corollary 1.3. *If $X^3 + Y^3 = 6Z^3$ has finitely many solutions for certain pairwise coprime integers X, Y, Z, then there are finitely many rational points (x, y) on the curves (1.4) and (1.5) for $k = 3$.*

Corollary 1.4. *If $X^3 + 2Y^3 = 3Z^3$ has infinitely many solutions for pairwise coprime integers X, Y, Z, then there are infinitely many rational points (x, y) on the curves (1.4) and (1.5) for $k = 3$.*

2. Lemmas

Lemma 2.1 [2, 6]. *Let $p \geq 3$ be a prime. Then*

$$X^p + Y^p = 2Z^p$$

has only trivial solutions.

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Lemma 2.2 [3]. *The Diophantine equations*

$$X^4 \pm Y^4 = 2Z^2$$

have only trivial solutions.

Lemma 2.3 [6]. *Let $p \geq 3$ be a prime, $2 \leq \alpha < p$. Then*

$$X^p + Y^p = 2^\alpha Z^p$$

has only trivial solutions.

Lemma 2.4. *The Diophantine equation*

$$4X^4 + Y^2 = Z^4, \quad (Y, Z) = 1,$$

has only trivial solutions.

Proof. When $(Y, Z) = 1$, it is well known that

$$2X^2 = 2uv, \quad Y = u^2 - v^2, \quad Z^2 = u^2 + v^2,$$

for some coprime integers u, v satisfying $2|uv$. Hence $u = s^2, v = t^2$ or $u = -s^2, v = -t^2$ by $X^2 = uv$, $Z^2 = s^4 + t^4$, which has only trivial solutions. Therefore $4X^4 + Y^2 = Z^4$, $(Y, Z) = 1$ has only trivial solutions. □

Lemma 2.5 [3]. *If X, Y are integers and $XY \neq 0$. Then*

$$X^2 + 2Y^2, \quad X^2 + 3Y^2$$

and

$$X^2 - Y^2, \quad X^2 - 4Y^2$$

are not both squares.

Lemma 2.6 [3]. *A sufficient and necessary condition for integral solutions of $X^2 + Y^2 = mZ^2$, $XYZ \neq 0$, is that m be a sum of two squares.*

Lemma 2.7 [1]. *If s and t are coprime positive integers with $st = 2^\alpha 3^\beta$, where α and β are nonnegative integers such that either $\alpha = 0$ or $\beta = 0$ or $\alpha \geq 4$, then, if $n \geq 5$ is prime, the equation*

$$sX^n + tY^n = Z^n$$

has no solution in coprime nonzero integers (X, Y, Z) with $|XY| > 1$.

Lemma 2.8 [10]. *The Diophantine equations*

$$X^3 + Y^3 = 3Z^3,$$
$$X^3 + Y^3 = 18Z^3,$$
$$2X^3 + 9Y^3 = Z^3,$$
$$4X^3 + 9Y^3 = Z^3$$

have only trivial solutions.

Lemma 2.9 [10]. *A rational solution of the equation*

$$aX^3 + bY^3 + cZ^3 = 0, \quad abc \neq 0,$$

with $XYZ \neq 0$, leads to a rational solution of

$$X^3 + Y^3 = abcZ^3,$$

with $XYZ \neq 0$.

3. Proofs of the theorems

Let $x = a/b$ and $y = c/d$ for some integers a, c and positive integers b, d satisfying $(a, b) = (c, d) = 1$. Then (1.2) is equal to $c^k b^{l-1} = d^k(a + d_1 b) \cdots (a + d_{l-1} b)$. From $(a, b) = 1$, we get $(b, a + d_i b) = 1$, whence $b^{l-1} | d^k$. Obviously, $(c, d) = 1$ implies $(c^k, d^k) = 1$, so $d^k | b^{l-1}$. We conclude that $b^{l-1} = d^k$, so (1.2) is equal to

$$c^k = (a + d_1 b) \cdots (a + d_{l-1} b), \quad d^k = b^{l-1}.$$

Proof of Theorem 1.1. It is easy to see that (1.3) is equal to

$$c^k = a(a + 2b), \quad d^k = b^2.$$

Clearly, $(a, a + 2b) = (a, 2)$.

Case 1. $(a, 2) = 1$. If $k \geq 3$ is odd, then we have $a = c_1^k$ and $a + 2b = c_2^k$ for certain coprime integers c_1, c_2 satisfying $c_1 c_2 = c$. It clearly follows for a suitable integer b_1 that $b = b_1^k$, hence

$$c_1^k + 2b_1^k = c_2^k. \tag{3.1}$$

Since k has a prime factor $p \geq 3$, (3.1) is of type $X^p + 2Y^p = Z^p$, which has only trivial solutions by Lemma 2.1, hence $c = 0$ or $b = d = 1$, $a = c = -1$.

If k is even, we have $b = d^{k/2}$, $a = \pm c_1^k$ and $a + 2b = \pm c_2^k$ (where $+$ corresponds to $+$, $-$ corresponds to $-$) for some coprime nonnegative integers c_1, c_2 and $\pm c_1 c_2 = c$, hence

$$\pm c_1^k + 2d^{k/2} = \pm c_2^k. \tag{3.2}$$

If k has a prime factor $p \geq 3$, then (3.2) is of type $X^p + 2Y^p = Z^p$, which has only trivial solutions by Lemma 2.1, hence $c = 0$.

It remains to consider $k = 2^t$ for some $t \geq 1$. For $t \geq 2$, Equation (3.2) is of type $\pm c_1^4 + 2d^2 = \pm c_2^4$ and has only trivial solutions by Lemma 2.2.

We are left with the case $k = 2$. From the above, we have $b = d$, $a = \pm c_1^2$ and $a + 2b = \pm c_2^2$ with $(c_1, c_2) = 1$, hence $a = \pm c_1^2$ and $b = d = (\pm c_2^2 \mp c_1^2)/2$. Therefore

$$x = \frac{a}{b} = \frac{2c_1^2}{c_2^2 - c_1^2}, \quad y = \frac{c}{d} = \frac{2c_1 c_2}{c_2^2 - c_1^2},$$

with coprime integers $c_1 \neq \pm c_2$.

Case 2. $(a, 2) = 2$, $(b, 2) = 1$. If $k \geq 3$ is odd, we have $a = 2c_1^k$, $a + 2b = 2^{k-1} c_2^k$ or $a = 2^{k-1} c_1^k$, $a + 2b = 2c_2^k$ for $(c_1, c_2) = 1$ satisfying $2c_1 c_2 = c$. It clearly follows for a

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suitable integer b_1 that $b = b_1^k$, hence

$$c_1^k + b_1^k = 2^{k-2}c_2^k \tag{3.3}$$

or

$$2^{k-2}c_1^k + b_1^k = c_2^k. \tag{3.4}$$

Since k has a prime factor $p \geq 3$, both (3.3) and (3.4) are of type $X^p + Y^p = 2^{p-2}Z^p$, which has only trivial solutions by Lemmas 2.1 and 2.3, hence $c = 0$ for odd $k \geq 3$ and $b = d = 1$, $a = 2$ or -4, $c = 2$, for $k = 3$.

If k is even, we have $b = d^{k/2}$, $a = \pm 2c_1^k$ and $a + 2b = \pm 2^{k-1}c_2^k$ or $a = \pm 2^{k-1}c_1^k$, $a + 2b = \pm 2c_2^k$ for $(c_1, c_2) = 1$ satisfying $\pm 2c_1c_2 = c$, hence

$$\pm c_1^k + d^{k/2} = \pm 2^{k-2}c_2^k \tag{3.5}$$

or

$$\pm 2^{k-2}c_1^k + d^{k/2} = \pm c_2^k. \tag{3.6}$$

If k has a prime factor $p \geq 3$, then both (3.5) and (3.6) are of type $X^p + Y^p = 2^{p-2}Z^p$, which has only solutions satisfying $c_1c_2 = 0$ or $b_1|c_1c_2| = 1$ for $k = 3$ by Lemma 2.3, hence $c = 0$.

It remains to consider $k = 2^t$ for some $t \geq 1$. For $t \geq 2$, (3.5) is of type $\pm c_1^4 + d^2 = \pm 4c_2^4$, which is equal to $-c_1^4 + d^2 = -4c_2^4$ by taking the equation mod 4. Equation (3.6) is of type $\pm 4c_1^4 + d^2 = \pm c_2^4$, which is equal to $4c_1^4 + d^2 = c_2^4$ by taking the equation mod 4. Hence (3.5) and (3.6) have only trivial solutions by Lemma 2.4, hence $c = 0$.

We are left with the case $k = 2$. From the above, we have $b = d$, $a = \pm 2c_1^2$ and $a + 2b = \pm 2c_2^2$ with $(c_1, c_2) = 1$, hence $a = \pm 2c_1^2$ and $b = d = \pm c_2^2 \mp c_1^2$. Therefore,

$$x = \frac{a}{b} = \frac{2c_1^2}{c_2^2 - c_1^2}, \quad y = \frac{c}{d} = \frac{2c_1c_2}{c_2^2 - c_1^2},$$

with coprime integers $c_1 \neq \pm c_2$.

This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. For a positive integer k, let

$$k^* = \begin{cases} k, & 3 \nmid k, \\ \dfrac{k}{3}, & 3 \mid k. \end{cases}$$

It is easy to see that (1.4) is equal to

$$c^k = a(a + 2b)(a + 3b), \quad d^k = b^3.$$

Clearly, $(a + 2b, a + 3b) = 1$, $(a, a + 2b) = 1$ or 2, $(a, a + 3b) = 1$ or 3.

Case 1. $(a, a + 2b) = 1$, $(a, a + 3b) = 1$, a is odd. If k is odd, and k has a prime factor $p \geq 5$ or $9|k$,

$$a = c_1^k, \quad a + 2b = c_2^k, \quad a + 3b = c_3^k, \tag{3.7}$$

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for certain pairwise coprime integers c_1, c_2, c_3 satisfying $c_1c_2c_3 = c$. It clearly follows for a suitable integer b_1 that $b = b_1^{k^*}$. As above, this implies, with the second and third equations in (3.7),

$$b_1^{k^*} + c_2^k = c_3^k,$$

which is of type $X^p + Y^p = Z^p$, $p \geq 3$, which has only trivial solutions by Fermat's last theorem, hence $c = 0$.

If k is even, then k^* is even. We have (3.7) or

$$a = -c_1^k, \quad a + 2b = -c_2^k, \quad a + 3b = c_3^k, \tag{3.8}$$

for certain pairwise coprime integers c_1, c_2, c_3 satisfying $\pm c_1c_2c_3 = c$. Putting $b = b_1^{k^*}$ and the first equation in (3.7) and (3.8) into the second and third equations in (3.7) and (3.8), we obtain

$$c_1^k + 2b_1^{k^*} = c_2^k, \quad c_1^k + 3b_1^{k^*} = c_3^k, \tag{3.9}$$

$$-c_1^k + 2b_1^{k^*} = -c_2^k, \quad -c_1^k + 3b_1^{k^*} = c_3^k. \tag{3.10}$$

Equation (3.9) implies that both $X^2 + 2Y^2$ and $X^2 + 3Y^2$ are squares, because a is odd, while (3.10) is impossible by taking the equation mod 4. Hence $c = 0$ by Lemma 2.5.

Case 2. $(a, a + 2b) = 2$, $(a, a + 3b) = 1$, a is even, b is odd. If k is odd, and k has a prime factor $p \geq 5$ or $9|k$,

$$a = 2c_1^k, \quad a + 2b = 2^{k-1}c_2^k, \quad a + 3b = c_3^k, \quad 2 \nmid bc_1c_3 \tag{3.11}$$

or

$$a = 2^{k-1}c_1^k, \quad a + 2b = 2c_2^k, \quad a + 3b = c_3^k, \quad 2 \nmid bc_2c_3 \tag{3.12}$$

for certain pairwise coprime integers c_1, c_2, c_3 satisfying $2c_1c_2c_3 = c$. Putting $b = b_1^{k^*}$ and the first equation in (3.11) and (3.12) into the second equation in (3.11) and (3.12) respectively, we obtain $c_1^k + b_1^{k^*} = 2^{k-2}c_2^k$ and $2^{k-2}c_1^k + b_1^{k^*} = c_2^k$, which are of type $X^p + Y^p = 2^\alpha Z^p$, $p \geq 3$, $1 \leq \alpha \leq p - 2$, which has only trivial solutions by Lemmas 2.1 and 2.3. Hence $c = 0$. If k is even, then k^* is even. We have (3.11) or (3.12) or

$$a = -2c_1^k, \quad a + 2b = -2^{k-1}c_2^k, \quad a + 3b = c_3^k, \quad 2 \nmid bc_1c_3 \tag{3.13}$$

or

$$a = -2^{k-1}c_1^k, \quad a + 2b = -2c_2^k, \quad a + 3b = c_3^k, \quad 2 \nmid bc_2c_3 \tag{3.14}$$

for certain pairwise coprime integers c_1, c_2, c_3 satisfying $\pm 2c_1c_2c_3 = c$. In (3.11) and (3.12), we obtain $c_1^k + c_3^k = 2^{k-2}3c_2^k$ and $2^{k-2}c_1^k + c_3^k = 3c_2^k$ respectively since $a + 2(a + 3b) = 3(a + 2b)$, which are of type $X^2 + Y^2 = 3Z^2$. In (3.13) and (3.14), we obtain $c_3^k + 3 \cdot 2^{k-2}c_2^k = c_1^k$, $c_3^k + 2^{k-1}c_2^k = b_1^{k^*}$ and $c_3^k + 3c_2^k = 2^{k-2}c_1^k$, $c_3^k + 2c_2^k = b_1^{k^*}$ respectively by $2(a + 3b) - 3(a + 2b) = -a$, $(a + 3b) - (a + 2b) = b$, which implies that both $X^2 + 2Y^2$ and $X^2 + 3Y^2$ are squares. Hence $c = 0$ by Lemmas 2.6 and 2.5.

Case 3. $(a, a+2b) = 1$, $(a, a+3b) = 3$, a is odd. If k is odd, and k has a prime factor $p \geq 5$ or $9|k$,

$$a = 3c_1^k, \quad a+2b = c_2^k, \quad a+3b = 3^{k-1}c_3^k \tag{3.15}$$

or

$$a = 3^{k-1}c_1^k, \quad a+2b = c_2^k, \quad a+3b = 3c_3^k, \tag{3.16}$$

for certain pairwise coprime integers c_1, c_2, c_3 satisfying $3c_1c_2c_3 = c$. Putting $b = b_1^{k^*}$ and the first equation in (3.15) and (3.16) into the third equation in (3.15) and (3.16) respectively, we obtain $c_1^k + b_1^{k^*} = 3^{k-2}c_3^k$ and $3^{k-2}c_1^k + b_1^{k^*} = c_3^k$, which are of type $X^p + Y^p = 3^{p-2}Z^p$, $p \geq 3$, which has only solutions with $c = 0$ by Lemmas 2.7 and 2.8.

If k is even, then k^* is even. We have (3.15) or (3.16) or

$$a = -3c_1^k, \quad a+2b = -c_2^k, \quad a+3b = 3^{k-1}c_3^k \tag{3.17}$$

or

$$a = -3^{k-1}c_1^k, \quad a+2b = -c_2^k, \quad a+3b = 3c_3^k, \tag{3.18}$$

for certain pairwise coprime integers c_1, c_2, c_3 satisfying $\pm 3c_1c_2c_3 = c$. In (3.15) and (3.16), we obtain $c_2^k + b_1^{k^*} = 3^{k-1}c_3^k$ and $c_2^k + b_1^{k^*} = 3c_3^k$ respectively by $(a+2b)+b = a+3b$, which are of type $X^2 + Y^2 = 3Z^2$. In (3.17) and (3.18), we obtain $c_2^k + 2\cdot 3^{k-2}c_3^k = c_1^k$, $c_2^k + 3^{k-1}c_3^k = b_1^{k^*}$ and $c_2^k + 2c_3^k = 3^{k-2}c_1^k$, $c_2^k + 3c_3^k = b_1^{k^*}$ respectively by $2(a+3b) - 3(a+2b) = -a$, $(a+3b) - (a+2b) = b$, which implies that both $X^2 + 2Y^2$ and $X^2 + 3Y^2$ are squares. Hence $c = 0$ by Lemmas 2.6 and 2.5.

Case 4. $(a, a+2b) = 2$, $(a, a+3b) = 3$, a is even, b is odd. If k is odd, and k has a prime factor $p \geq 5$,

$$a = 6^{k-1}c_1^k, \quad a+2b = 2c_2^k, \quad a+3b = 3c_3^k, \quad 2 \nmid bc_2c_3 \tag{3.19}$$

or

$$a = 2^{k-1}\cdot 3c_1^k, \quad a+2b = 2c_2^k, \quad a+3b = 3^{k-1}c_3^k, \quad 2 \nmid bc_2c_3 \tag{3.20}$$

or

$$a = 2\cdot 3^{k-1}c_1^k, \quad a+2b = 2^{k-1}c_2^k, \quad a+3b = 3c_3^k, \quad 2 \nmid bc_1c_3 \tag{3.21}$$

or

$$a = 6c_1^k, \quad a+2b = 2^{k-1}c_2^k, \quad a+3b = 3^{k-1}c_3^k, \quad 2 \nmid bc_1c_3 \tag{3.22}$$

for certain pairwise coprime integers c_1, c_2, c_3 satisfying $6c_1c_2c_3 = c$. In the third equation of (3.19) and (3.20), we obtain $2^{k-1}3^{k-2}c_1^k + b_1^{k^*} = c_3^k$ and $2^{k-1}c_1^k + b_1^{k^*} = 3^{k-2}c_3^k$ respectively by the first equation and $b = b_1^{k^*}$. In (3.21) and (3.22), we obtain $2^{k-1}c_2^k + b_1^{k^*} = 3c_3^k$ and $2^{k-1}c_2^k + b_1^{k^*} = 3^{k-1}c_3^k$ respectively by $(a+2b)+b = a+3b$, which are all of type $sX^p + tY^p = Z^p$, $st = 2^\alpha 3^\beta$, $\alpha \geq 4$, $\beta \geq 0$, hence $c = 0$ by Lemma 2.7.

If k is odd, and $9|k$, we obtain $2^{k-2}3^{k-1}c_1^k + b_1^{k^*} = c_2^k$ by the second equation of (3.19), which is of type $X^3 + Y^3 = 18Z^3$. We have $2c_2^k + b_1^{k^*} = 3^{k-1}c_3^k$ in (3.20) by $(a+2b)+b = a+3b$, $3^{k-1}c_1^k + b_1^{k^*} = 2^{k-2}c_2^k$ by the second equation of (3.21), which

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are of type $2X^3 + 9Y^3 = Z^3$. We obtain $2^{k-1}c_2^k + b_1^{k^*} = 3^{k-1}c_3^k$ in (3.22) by $(a + 2b) + b = a + 3b$, which is of type $4X^3 + 9Y^3 = Z^3$. Hence $c = 0$ by Lemma 2.8.

If k is even, then k^* is even. We have (3.19) or (3.20) or (3.21) or (3.22) or

$$a = -6^{k-1}c_1^k, \quad a + 2b = -2c_2^k, \quad a + 3b = 3c_3^k, \quad 2 \nmid bc_2c_3 \tag{3.23}$$

or

$$a = -2 \cdot 3^{k-1}c_1^k, \quad a + 2b = -2^{k-1}c_2^k, \quad a + 3b = 3c_3^k, \quad 2 \nmid bc_1c_3 \tag{3.24}$$

or

$$a = -2^{k-1} \cdot 3c_1^k, \quad a + 2b = -2c_2^k, \quad a + 3b = 3^{k-1}c_3^k, \quad 2 \nmid bc_2c_3 \tag{3.25}$$

or

$$a = -6c_1^k, \quad a + 2b = -2^{k-1}c_2^k, \quad a + 3b = 3^{k-1}c_3^k, \quad 2 \nmid bc_1c_3 \tag{3.26}$$

for certain pairwise coprime integers c_1, c_2, c_3 satisfying $\pm 6c_1c_2c_3 = c$. We obtain, respectively,

$$\begin{aligned}
-2^{k-2}3^{k-1}c_1^k + b_1^{k^*} &= -c_2^k, \quad 2 \nmid b_1c_2,\\
-3^{k-1}c_1^k + b_1^{k^*} &= -2^{k-2}c_2^k, \quad 2 \nmid b_1c_1,\\
-2^{k-2} \cdot 3c_1^k + b_1^{k^*} &= -c_2^k, \quad 2 \nmid b_1c_2,\\
-3c_1^k + b_1^{k^*} &= -2^{k-2}c_2^k, \quad 2 \nmid b_1c_1
\end{aligned}$$

by the first and second equations from (3.23)–(3.26), since k and k^* are even, which are impossible by taking these equations mod 4.

Similarly, we can prove that the only rational points (x, y) on the curve (1.5) are the trivial ones with $y = 0$.

This completes the proof of Theorem 1.2. □

We are left with the case $k = 3$. For (1.4), by (3.7), (3.11), (3.12), (3.15), (3.16) and (3.19)–(3.22), we obtain, respectively,

$$\begin{aligned}
a &= c_1^3, & b = d &= c_3^3 - c_2^3 > 0, & c &= c_1c_2c_3, & c_1^3 + 2c_3^3 &= 3c_2^3,\\
a &= 2c_1^3, & b = d &= c_3^3 - 4c_2^3 > 0, & c &= 2c_1c_2c_3, & c_1^3 + c_3^3 &= 6c_2^3,\\
a &= 4c_1^3, & b = d &= c_3^3 - 2c_2^3 > 0, & c &= 2c_1c_2c_3, & 2c_1^3 + c_3^3 &= 3c_2^3,\\
a &= 3c_1^3, & b = d &= 9c_3^3 - c_2^3 > 0, & c &= 3c_1c_2c_3, & c_1^3 + 6c_3^3 &= c_2^3,\\
a &= 9c_1^3, & b = d &= 3c_3^3 - c_2^3 > 0, & c &= 3c_1c_2c_3, & 3c_1^3 + 2c_3^3 &= c_2^3,\\
a &= 36c_1^3, & b = d &= 3c_3^3 - 2c_2^3 > 0, & c &= 6c_1c_2c_3, & 3c_1^3 + 2c_3^3 &= c_2^3,\\
a &= 18c_1^3, & b = d &= 3c_3^3 - 4c_2^3 > 0, & c &= 6c_1c_2c_3, & 3c_1^3 + c_3^3 &= 2c_2^3,\\
a &= 12c_1^3, & b = d &= 9c_3^3 - 2c_2^3 > 0, & c &= 6c_1c_2c_3, & 2c_1^3 + 3c_3^3 &= c_2^3,\\
a &= 6c_1^3, & b = d &= 9c_3^3 - 4c_2^3 > 0, & c &= 6c_1c_2c_3, & c_1^3 + 3c_3^3 &= 2c_2^3,
\end{aligned}$$

for certain pairwise coprime integers c_1, c_2, c_3.

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Similarly, for (1.5), we obtain

$$\begin{aligned}
&a = c_1^3, && b = d = c_2^3 - c_1^3 > 0, && c = c_1c_2c_3, && c_3^3 + 2c_1^3 = 3c_2^3,\\
&a = c_1^3, && b = d = 4c_2^3 - c_1^3 > 0, && c = 2c_1c_2c_3, && c_1^3 + c_3^3 = 6c_2^3,\\
&a = c_1^3, && b = d = 2c_2^3 - c_1^3 > 0, && c = 2c_1c_2c_3, && c_1^3 + 2c_3^3 = 3c_2^3,\\
&a = 9c_1^3, && b = d = c_2^3 - 9c_1^3 > 0, && c = 3c_1c_2c_3, && 6c_1^3 + c_3^3 = c_2^3,\\
&a = 3c_1^3, && b = d = c_2^3 - 3c_1^3 > 0, && c = 3c_1c_2c_3, && 3c_1^3 + 2c_3^3 = c_2^3,\\
&a = 3c_1^3, && b = d = 2c_2^3 - 3c_1^3 > 0, && c = 6c_1c_2c_3, && 6c_3^3 + c_1^3 = c_2^3,\\
&a = 3c_1^3, && b = d = 4c_2^3 - 3c_1^3 > 0, && c = 6c_1c_2c_3, && 2c_1^3 + 3c_3^3 = c_2^3,\\
&a = 9c_1^3, && b = d = 2c_2^3 - 9c_1^3 > 0, && c = 6c_1c_2c_3, && 3c_1^3 + 2c_3^3 = c_2^3,\\
&a = 9c_1^3, && b = d = 4c_2^3 - 9c_1^3 > 0, && c = 6c_1c_2c_3, && 3c_1^3 + c_3^3 = 2c_2^3,
\end{aligned}$$

for certain pairwise coprime integers c_1, c_2, c_3.

Therefore Corollaries 1.3 and 1.4 follow from the above 18 equations and Lemma 2.9.

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