COMPOSITIO MATHEMATICA

Integrating morphisms of Lie 2-algebras

Behrang Noohi

Compositio Math. **149** (2013), 264–294.

doi:10.1112/S0010437X1200067X

FOUNDATION COMPOSITIO MATHEMATICA

LONDON MATHEMATICAL SOCIETY

Compositio Math. 149 (2013) 264–294
doi:10.1112/S0010437X1200067X

Integrating morphisms of Lie 2-algebras

Behrang Noohi

Abstract

Given two Lie 2-groups, we study the problem of integrating a weak morphism between the corresponding Lie 2-algebras to a weak morphism between the Lie 2-groups. To do so, we develop a theory of butterflies for 2-term L_∞-algebras. In particular, we obtain a new description of the bicategory of 2-term L_∞-algebras. An interesting observation here is that the role played by 1-connected Lie groups in Lie theory is now played by 2-connected Lie 2-groups. Using butterflies, we also give a functorial construction of 2-connected covers of Lie 2-groups. Based on our results, we expect that a similar pattern generalizes to Lie n-groups and Lie n-algebras.

Contents

1	Introduction	264
2	2-term L_∞-algebras	266
3	Butterflies between 2-term L_∞-algebras	268
4	Homotopy fiber of a morphism of 2-term L_∞-algebras	270
5	The bicategory of Lie 2-algebras and butterflies	272
6	Weak morphisms of Lie crossed-modules and butterflies	274
7	Connected covers of a Lie 2-group	276
8	Functorial properties of connected covers	280
9	The bifunctor from Lie crossed-modules to L_∞-algebras	288
10	Applications	291
	Acknowledgements	293
	Appendix A Functorial n-connected covers for $n \geqslant 3$	293
	References	294

1. Introduction

In this paper, we tackle two main problems in the Lie theory of 2-groups:

(1) integrating weak morphisms of Lie 2-algebras to weak morphisms of Lie 2-groups;

(2) functorial construction of connected covers of Lie 2-groups.

As we will see in our answer to question (1), Theorem 9.4, the role played by simply connected Lie groups in classical Lie theory is played by 2-connected Lie 2-groups in 2-Lie theory. This justifies our interest in question (2).

Received 4 November 2011, accepted in final form 14 August 2012, published online 4 February 2013.
2010 Mathematics Subject Classification 17B99, 18D05 (primary).
Keywords: Lie 2-algebra, Lie 2-group, Lie crossed module, Lie butterfly, L_∞-algebra, connected cover.

Let us explain problems (1) and (2) in detail and outline our solution to them.

Problem 1. A weak morphism $f : \mathbb{H} \to \mathbb{G}$ of Lie 2-groups gives rise to a weak morphism of Lie 2-algebras $\mathrm{Lie}\, f : \mathrm{Lie}\, \mathbb{H} \to \mathrm{Lie}\, \mathbb{G}$. (If we regard $\mathrm{Lie}\, \mathbb{H}$ and $\mathrm{Lie}\, \mathbb{G}$ as 2-term L_∞-algebras, $\mathrm{Lie}\, f$ is then a morphism of 2-term L_∞-algebras in the sense of Definition 2.5.) Problem (1) can be stated as follows: given a morphism $F : \mathrm{Lie}\, \mathbb{H} \to \mathrm{Lie}\, \mathbb{G}$ of Lie 2-algebras, can we integrate it to a weak morphism $\mathrm{Int}\, F : \mathbb{H} \to \mathbb{G}$ of Lie 2-groups?

We answer this question affirmatively by the following theorem (see Theorem 9.4 for a more precise statement).

Theorem 1.1. *Let* $\mathbb{G}$ *and* $\mathbb{H}$ *be (strict) Lie* 2*-groups. Suppose that* $\mathbb{H}$ *is* 2*-connected (Definition 7.3). Then, giving a weak morphism* $f : \mathbb{H} \to \mathbb{G}$ *is equivalent to giving a morphism of Lie* 2*-algebras* $\mathrm{Lie}\, f : \mathrm{Lie}\, \mathbb{H} \to \mathrm{Lie}\, \mathbb{G}$*. The same is true for* 2*-morphisms.*

This theorem is the 2-group version of the well-known fact from Lie theory that a Lie homomorphism $f : H \to G$ is uniquely given by its effect on Lie algebras, $\mathrm{Lie}\, f : \mathrm{Lie}\, H \to \mathrm{Lie}\, G$, whenever H is 1-connected. It implies the following (see Corollary 9.5).

Theorem 1.2. *The bifunctor* $\mathrm{Lie} : \mathbf{LieXM} \to \mathbf{LieAlgXM}$ *has a left adjoint*

$$\mathrm{Int} : \mathbf{LieAlgXM} \to \mathbf{LieXM}.$$

Here, **LieXM** is the bicategory of Lie crossed-modules and weak morphisms, and **LieAlgXM** is the bicategory of Lie algebra crossed-modules and weak morphisms.[1] The bifunctor Int takes a Lie crossed-module to the unique 2-connected (strict) Lie 2-group that integrates it. When restricted to the full subcategory $\mathbf{Lie} \subset \mathbf{LieAlgXM}$ of Lie algebras, it coincides with the standard integration functor which sends a Lie algebra V to the simply connected Lie group $\mathrm{Int}\, V$ with Lie algebra V.

The problem of integrating L_∞-algebras has been studied in [Get09, Hen08], where the authors show how to integrate an L_∞-algebra to a simplicial manifold. The focus of these two papers, however, is different from ours in that we begin with *fixed* Lie 2-groups $\mathbb{H}$ and $\mathbb{G}$ and study the problem of integrating a morphism of Lie 2-algebras $\mathrm{Lie}\, \mathbb{H} \to \mathrm{Lie}\, \mathbb{G}$.

Problem 2. For a Lie group G, its zeroth and first connected covers $G\langle 0\rangle$ and $G\langle 1\rangle$, which are again Lie groups, play an important role in Lie theory. We observe that for (strict) Lie 2-groups it is necessary to go one step further, i.e. one needs to consider the second connected cover as well. We prove the following theorem.

Theorem 1.3. *For* $n = 0, 1, 2$*, there are bifunctors* $(-)\langle n\rangle : \mathbf{LieXM} \to \mathbf{LieXM}$ *sending a Lie crossed-module* $\mathbb{G}$ *to its* n*th connected cover. These bifunctors come with natural transformations* $q_n : (-)\langle n\rangle \Rightarrow \mathrm{id}$ *such that for every* $\mathbb{G}$*,* $q_n : \mathbb{G}\langle n\rangle \to \mathbb{G}$ *induces isomorphisms on* π_i *for* $i \geqslant n+1$*. Furthermore,* $(-)\langle n\rangle$ *is right adjoint to the inclusion of the full sub-bicategory of* n*-connected Lie crossed-modules in* **LieXM**.

The above theorem is essentially the content of §§ 7–8. We will be especially interested in the 2-connected cover $\mathbb{G}\langle 2\rangle$ because, as suggested by Theorem 1.2, it seems to be the correct replacement for the universal cover of a Lie group in the Lie theory of 2-groups.

[1] As we will see in § 6.2, **LieXM** is naturally biequivalent to the 2-category of (strict) Lie 2-groups and weak morphisms. Similarly, the bicategory **LieAlgXM** is naturally biequivalent to the full sub-2-category of the 2-category $\mathbf{2TermL}_\infty$ of 2-term L_∞-algebras consisting of strict 2-term L_∞-algebras, and this in turn is biequivalent to the 2-category of 2-term dglas (see § 9 and Definition 9.2).

Method

To solve problems (1) and (2), we employ the machinery of *butterflies*, which we believe is of independent interest. Roughly speaking, a butterfly (Definition 3.1) between 2-term L_∞-algebras is the Lie algebra-theoretic version of a Morita morphism. We use butterflies to give a new description of the 2-category $\mathbf{2TermL}_\infty$ of 2-term L_∞-algebras introduced in [BC04]. The advantage of using butterflies is twofold. On the one hand, butterflies do away with cumbersome cocycle formulas and are much easier to manipulate. On the other hand, given the diagrammatic nature of butterflies, they are better suited to geometric situations; this is what allows us to prove Theorem 1.2.

Butterflies for 2-term L_∞-algebras parallel the corresponding theory for Lie 2-groups developed in [Noo08, § 9.6] and [AN09]. In fact, taking Lie algebras converts a butterfly in Lie groups to a butterfly in Lie algebras (§ 9). This allows us to study weak morphisms of Lie 2-groups using butterflies between 2-term L_∞-algebras, thereby reducing the problem to one about extensions of Lie algebras. With Theorem 1.2 in hand, we expect this to provide a convenient framework for studying weak morphisms of Lie 2-groups.

Organization of the paper

Sections 2–5 are devoted to setting up the machinery of butterflies and constructing the bicategory $\mathbf{2TermL}^\flat_\infty$ of 2-term L_∞-algebras and butterflies. We show that $\mathbf{2TermL}^\flat_\infty$ is biequivalent to the Baez–Crans 2-category $\mathbf{2TermL}_\infty$ of 2-term L_∞-algebras.

In § 4 we discuss the *homotopy fiber* of a morphism of 2-term L_∞-algebras. The homotopy fiber is the Lie algebra counterpart of what we called the homotopy fiber of a weak morphism of Lie 2-groups in [Noo08, § 9.4]. The homotopy fiber of f measures the deviation of f from being an equivalence, and it sits in a natural exact triangle which gives rise to a 7-term long exact sequence.

The homotopy fiber comes with a rich structure consisting of various brackets and Jacobiators (see § 4.1). Homotopy fibers of morphisms of 2-term L_∞-algebras can also be defined in other ways (e.g. using the corresponding CDGAs), but we are not aware of whether the specific structure discussed here has been studied previously, or whether it is equivalent to a known definition. It is presumably some kind of Lie algebra version of what is called a 'crossed-module in groupoids' in [BG89].

In § 6 we review Lie 2-groups and weak morphisms (butterflies) of Lie 2-groups. Sections 7–8 are devoted to the solution of problem (2). For a Lie crossed-module $\mathbb{G}$ we define its nth connected covers $\mathbb{G}\langle n\rangle$, for $n \leqslant 2$, and show that they are functorial and have the expected adjunction property.

In § 9, we solve problem (1) by proving Theorems 1.2 and 1.1. The proofs rely on the solution of problem (2) given in §§ 7–8 and the theory of butterflies developed in §§ 2–5.

2. 2-term L_∞-algebras

In this section we review some basic facts about 2-term L_∞-algebras. We follow the notation of [BC04] (see also [Roy07]). All modules are over a fixed base commutative unital ring K.

Definition 2.1. A 2-*term* L_∞-*algebra* $\mathbb{V}$ consists of a linear map $\partial : V_1 \to V_0$ of modules together with the following data:

https://doi.org/10.1112/S0010437X1200067X Published online by Cambridge University Press

- three bilinear maps $[\cdot\,,\cdot]: V_i \times V_j \to V_{i+j}$, where $i+j=0,1$;
- an antisymmetric trilinear map (the *Jacobiator*) $\langle\cdot,\cdot,\cdot\rangle: V_0\times V_0\times V_0 \to V_1$.

These maps satisfy the following axioms for all $w, x, y, z \in V_0$ and $h, k \in V_1$:

- $[x, y] = -[y, x]$;
- $[x, h] = -[h, x]$;
- $\partial([x, h]) = [x, \partial h]$;
- $[\partial h, k] = [h, \partial k]$;
- $\partial\langle x, y, z\rangle = [x, [y, z]] + [y, [z, x]] + [z, [x, y]]$;
- $\langle x, y, \partial h\rangle = [x, [y, h]] + [y, [h, x]] + [h, [x, y]]$;
- $[\langle x, y, z\rangle, w] - [\langle w, x, y\rangle, z] + [\langle z, w, x\rangle, y] - [\langle y, z, w\rangle, x] = \langle [x, y], z, w\rangle + \langle [z, w], x, y\rangle$ $+ \langle [x, z], w, y\rangle + \langle [w, y], x, z\rangle + \langle [x, w], y, z\rangle + \langle [y, z], x, w\rangle$.

We sometimes use the notation $[V_1 \to V_0]$ for a 2-term L_∞-algebra.

Definition 2.2. The equality $[\partial h, k] = [h, \partial k]$ allows us to define a bracket on V_1 by setting $[h, k] := [\partial h, k] = [h, \partial k]$.

Lemma 2.3. *For the bracket defined in Definition 2.2, the failure of the Jacobi identity is measured by the equality*

$$\langle \partial h, \partial k, \partial h\rangle = [h, [k, l]] + [k, [l, h]] + [l, [h, k]].$$

Proof. This is straightforward. □

A crossed-module in Lie algebras is the same as a strict 2-term L_∞-algebra, i.e. one for which the Jacobiator $\langle\cdot,\cdot,\cdot\rangle$ is identically zero. More precisely, given a 2-term L_∞-algebra $\mathbb{V}$ with zero Jacobiator, we obtain from Lemma 2.3 a Lie algebra structure on V_1, where the bracket is as in Definition 2.2. This makes ∂ a Lie algebra homomorphism. The action of V_0 on V_1 is the given bracket $[\cdot\,,\cdot]: V_0 \times V_1 \to V_1$. Also, observe that a strict 2-term L_∞-algebra is the same as a 2-term dgla.

Definition 2.4. Let $\mathbb{V} = [\partial: V_1 \to V_0]$ be a 2-term L_∞-algebra. We define

$$H_1(\mathbb{V}) := \ker \partial, \quad H_0(\mathbb{V}) := \operatorname{coker} \partial.$$

Note that $H_0(\mathbb{V})$ and $H_1(\mathbb{V})$ both inherit natural Lie algebra structures, with the latter being necessarily abelian. Furthermore, $H_1(\mathbb{V})$ is naturally an $H_0(\mathbb{V})$-module.

Definition 2.5. A *morphism* $f: \mathbb{W} \to \mathbb{V}$ of 2-term L_∞-algebras consists of the following data:

- linear maps $f_i: W_i \to V_i$, for $i = 0, 1$, which commute with the differentials;
- an antisymmetric bilinear map $\varepsilon: W_0 \times W_0 \to V_1$.

These maps satisfy the following axioms:

- for every $x, y \in W_0$, $[f_0(x), f_0(y)] - f_0[x, y] = \partial\varepsilon(x, y)$;
- for every $x \in W_0$ and $h \in W_1$, $[f_0(x), f_1(k)] - f_1[x, k] = \varepsilon(x, \partial k)$;
- for every $x, y, z \in W_0$,

$$\langle f_0(x), f_0(y), f_0(z)\rangle - f_1(\langle x, y, z\rangle) = \varepsilon(x, [y, z]) + \varepsilon(y, [z, x]) + \varepsilon(z, [x, y]) + [f_0(x), \varepsilon(y, z)] + [f_0(y), \varepsilon(z, x)] + [f_0(z), \varepsilon(x, y)].$$

https://doi.org/10.1112/S0010437X1200067X Published online by Cambridge University Press

A morphism $f : \mathbb{W} \to \mathbb{V}$ of 2-term L_∞-algebras induces a Lie algebra homomorphism $H_0(f) : H_0(\mathbb{W}) \to H_0(\mathbb{V})$ and an $H_0(f)$-equivariant morphism of Lie algebra modules $H_1(f) : H_1(\mathbb{W}) \to H_1(\mathbb{V})$.

Definition 2.6. A morphism $f : \mathbb{W} \to \mathbb{V}$ of 2-term L_∞-algebras is called an *equivalence* (or a *quasi-isomorphism*) if $H_0(f)$ and $H_1(f)$ are isomorphisms.

Definition 2.7. A morphism of 2-term L_∞-algebras is *strict* if ε is identically zero. In the case where $\mathbb{V}$ and $\mathbb{W}$ are crossed-modules in Lie algebras, this means that f is a (strict) morphism of crossed-modules.

Definition 2.8. If $f = (f_0, f_1, \varepsilon) : \mathbb{W} \to \mathbb{V}$ and $g = (g_0, g_1, \delta) : \mathbb{V} \to \mathbb{U}$ are morphisms of 2-term L_∞-algebras, the composition gf is defined to be the triple $(g_0 f_0, g_1 f_1, \gamma)$ where

$$\gamma(x, y) := g_1\varepsilon(x, y) + \delta(f_0(x), f_0(y)) \quad \text{for } x, y \in W_0.$$

Finally, we recall the definition of a transformation between morphisms of 2-term L_∞-algebras. Up to a minor difference in sign conventions, it is the same as [Roy07, Definition 2.20]. It is also equivalent to [BC04, Definition 4.3.7].

Definition 2.9. Given morphisms $f, g : \mathbb{W} \to \mathbb{V}$ of 2-term L_∞-algebras, a *transformation* (or an *L_∞-homotopy*) from g to f is a linear map $\theta : W_0 \to V_1$ such that:

- for every $x \in W_0$, $f_0(x) - g_0(x) = \partial\theta(x)$;
- for every $h \in W_1$, $f_1(h) - g_1(h) = \theta(\partial h)$;
- for every $x, y \in W_0$,

$$[\theta(x), \theta(y)] - \theta([x, y]) = \varepsilon_f(x, y) - \varepsilon_g(x, y) + [g_0(y), \theta(x)] + [\theta(y), g_0(x)].$$

It is easy to see that if f and g are related via a transformation, then $H_i(f) = H_i(g)$ for $i = 0, 1$.

Definition 2.10. If θ is a transformation from f to g and σ is a transformation from g to h, then their composition is the transformation from f to h given by the linear map $\theta + \sigma$.

The following definition is the one in [BC04, Proposition 4.3.8].

Definition 2.11. We define $\mathbf{2TermL_\infty}$ to be the 2-category in which the objects are 2-term L_∞-algebras, the morphisms are as in Definition 2.5, and the 2-morphisms are as in Definition 2.9. (Note that the 2-morphisms are automatically invertible.)

3. Butterflies between 2-term L_∞-algebras

In this section we introduce the notion of a butterfly between 2-term L_∞-algebras and show that butterflies encode morphisms of 2-term L_∞-algebras (Propositions 3.4 and 3.5). A butterfly should be regarded as an analogue of a Morita morphism.

DEFINITION 3.1. Let $\mathbb{V}$ and $\mathbb{W}$ be 2-term L_∞-algebras. A *butterfly* $B : \mathbb{W} \to \mathbb{V}$ is a commutative diagram

$$\begin{array}{ccccc} W_1 & & & & V_1 \\ \Big\downarrow & \overset{\kappa}{\searrow} & & \overset{\iota}{\swarrow} & \Big\downarrow \\ & & E & & \\ & \underset{\sigma}{\swarrow} & & \underset{\rho}{\searrow} & \\ W_0 & & & & V_0 \end{array}$$

of modules in which E is endowed with an antisymmetric bracket $[\cdot\,,\cdot] : E \times E \to E$ satisfying the following axioms:

- both diagonal sequences are complexes and the NE–SW sequence
$$0 \to V_1 \xrightarrow{\iota} E \xrightarrow{\sigma} W_0 \to 0$$
is short exact;
- for every $a, b \in E$,
$$\rho[a, b] = [\rho(a), \rho(b)] \quad \text{and} \quad \sigma[a, b] = [\sigma(a), \sigma(b)];$$
- for every $a \in E$, $h \in V_1$ and $l \in W_1$,
$$[a, \iota(h)] = \iota[\rho(a), h] \quad \text{and} \quad [a, \kappa(l)] = \kappa[\sigma(a), l];$$
- for every $a, b, c \in E$,
$$\iota\langle \rho(a), \rho(b), \rho(c)\rangle + \kappa\langle \sigma(a), \sigma(b), \sigma(c)\rangle = [a, [b, c]] + [b, [c, a]] + [c, [a, b]].$$

In the case where $\mathbb{V}$ and $\mathbb{W}$ are crossed-modules in Lie algebras (i.e. when the Jacobiators are identically zero), the bracket on E makes it a Lie algebra, and all the maps in the butterfly diagram become Lie algebra homomorphisms.

Remark 3.2. The map $\kappa + \iota : W_1 \oplus V_1 \to E$ has a natural 2-term L_∞-algebra structure. Let us denote this 2-term L_∞-algebra by $\mathbb{E}$. The two projections $\mathbb{E} \to \mathbb{W}$ and $\mathbb{E} \to \mathbb{V}$ are strict morphisms of 2-term L_∞-algebras, and the former is a quasi-isomorphism. Thus, we can think of the butterfly B as a zig-zag of strict morphisms from $\mathbb{W}$ to $\mathbb{V}$.

DEFINITION 3.3. Given two butterflies $B, B' : \mathbb{W} \to \mathbb{V}$, a *morphism* of butterflies from B to B' is a linear map $E \to E'$ that commutes with the brackets and all four structure maps of the butterfly. (Note that such a map $E \to E'$ is necessarily an isomorphism.)

In view of Remark 3.2, a morphism of butterflies as in the above definition is the same thing as a morphism of zig-zags $\mathbb{E} \to \mathbb{E}'$.

A butterfly $B : \mathbb{W} \to \mathbb{V}$ induces a Lie algebra homomorphism $H_0(B) : H_0(\mathbb{W}) \to H_0(\mathbb{V})$ and an $H_0(B)$-equivariant morphism $H_1(B) : H_1(\mathbb{W}) \to H_1(\mathbb{V})$. If B and B' are related through a morphism, then $H_i(B) = H_i(B')$ for $i = 0, 1$.

Let $f : \mathbb{W} \to \mathbb{V}$ be a morphism of 2-term L_∞-algebras as in Definition 2.5. Define a bracket on $V_1 \oplus W_0$ by the rule

$$[(k, x), (l, y)] := ([k, l] + [f_0(x), l] + [k, f_0(y)] + \varepsilon(x, y),\ [x, y]).$$

Define the following four maps:

- $\kappa : W_1 \to V_1 \oplus W_0$, $\kappa(l) = (-f_1(l), \partial l)$;
- $\iota : V_1 \to V_1 \oplus W_0$, $\iota(k) = (k, 0)$;

https://doi.org/10.1112/S0010437X1200067X Published online by Cambridge University Press

- $\sigma : V_1 \oplus W_0 \to W_0$, $\sigma(k, x) = x$;
- $\rho : V_1 \oplus W_0 \to V_0$, $\rho(k, x) = \partial k + f_0(x)$.

Proposition 3.4. *With the bracket on $V_1 \oplus W_0$ and the maps κ, ι, ρ and σ defined as above, the diagram*

$$\begin{array}{ccccc} W_1 & & & & V_1 \\ \downarrow & \overset{\kappa}{\searrow} & & \overset{\iota}{\swarrow} & \downarrow \\ & & V_1 \oplus W_0 & & \\ & \underset{\sigma}{\swarrow} & & \underset{\rho}{\searrow} & \\ W_0 & & & & V_0 \end{array}$$

is a butterfly (Definition 3.1). Conversely, given a butterfly as in Definition 3.1 and a linear section $s : W_0 \to E$ to σ, we obtain a morphism of 2-term L_∞-algebras by setting

$$f_0 := \rho s, \quad f_1 := s\partial - \kappa, \quad \varepsilon := [s(\cdot), s(\cdot)] - s[\cdot\,, \cdot].$$

(In the definition of the last two maps we are using the exactness of the NE–SW sequence.) Furthermore, these two constructions are inverse to each other.

Proposition 3.5. *Via the construction introduced in Proposition 3.4, transformations between morphisms of 2-term L_∞-algebras (Definition 2.9) correspond to morphisms of butterflies (Definition 3.3). In other words, we have an equivalence of groupoids between the groupoid of morphisms of 2-term L_∞-algebras from $\mathbb{W}$ to $\mathbb{V}$ and the groupoid of butterflies from $\mathbb{W}$ to $\mathbb{V}$.*

Example 3.6. Let V and W be Lie algebras. Define $\mathbb{D}\mathrm{er}(V)$ to be the crossed-module in Lie algebras $\partial : V \to \mathrm{Der}(V)$, where ∂ sends $v \in V$ to the derivation $[v, \cdot]$. Then the equivalence classes of 2-term L_∞-algebra morphisms $W \to \mathbb{D}\mathrm{er}(V)$ are in bijection with isomorphism classes of extensions of W by V. Here W is regarded as the 2-term L_∞-algebra $[0 \to W]$.

4. Homotopy fiber of a morphism of 2-term L_∞-algebras

We introduce the homotopy fiber (or 'shifted mapping cone') of a butterfly (and also of a morphism of 2-term L_∞-algebras). The homologies of the homotopy fiber sit in a 7-term long exact sequence. We shall see in § 4.1 that the homotopy fiber has a rich structure consisting of various brackets.

Definition 4.1. Let $B : \mathbb{W} \to \mathbb{V}$,

$$\begin{array}{ccccc} W_1 & & & & V_1 \\ \downarrow & \overset{\kappa}{\searrow} & & \overset{\iota}{\swarrow} & \downarrow \\ & & E & & \\ & \underset{\sigma}{\swarrow} & & \underset{\rho}{\searrow} & \\ W_0 & & & & V_0 \end{array}$$

be a butterfly. We define the *homotopy fiber* $\mathrm{hfib}(B)$ of B to be the NW–SE sequence

$$W_1 \xrightarrow{\kappa} E \xrightarrow{\rho} V_0.$$

We will think of W_1, E and V_0 as sitting in degrees 1, 0 and -1.

Using Proposition 3.4, we also get a version of the above definition for morphisms of 2-term L_∞-algebras. More precisely, for a morphism $f = (f_0, f_1, \varepsilon) : \mathbb{W} \to \mathbb{V}$, its homotopy fiber $\mathrm{hfib}(f)$

takes the form

$$W_1 \xrightarrow{(-f_1,\partial)} V_1 \oplus W_0 \xrightarrow{\partial+f_0} V_0.$$

If we forget all the brackets and index the terms of hfib(f) by 2, 1 and 0, we see that hfib(f) coincides with the cone of f in the derived category of chain complexes.

The homotopy fiber measures the deviation of B from being an equivalence. More precisely, we have the following statement.

Proposition 4.2. *There is a long exact sequence*

$$0 \to H_1(\mathrm{hfib}(B)) \to H_1(\mathbb{W}) \xrightarrow{H_1(B)} H_1(\mathbb{V}) \to H_0(\mathrm{hfib}(B)) \to H_0(\mathbb{W}) \xrightarrow{H_0(B)} H_0(\mathbb{V}) \longrightarrow H_{-1}(\mathrm{hfib}(B)) \to 0.$$

Proof. The proof is left as an exercise. □

Except for $H_{-1}(\mathrm{hfib}(B))$, all the terms in the above sequence are Lie algebras and all the maps are Lie algebra homomorphisms; see § 4.1 below.

Corollary 4.3. *A butterfly B is an equivalence (i.e. induces isomorphisms on H_0 and H_1) if and only if its NW–SE sequence is short exact. In this case, the inverse of B is obtained by flipping it along the vertical axis.*

4.1 Structure of the homotopy fiber

The homotopy fiber hfib(B) comes with some additional structure, which we discuss below. This will not be needed in the rest of the paper and can be skipped.

First, let us rename the homotopy fiber in the following way:

$$C_1 \xrightarrow{\partial} C_0 \xrightarrow{\partial} C_{-1}.$$

We have the following data:

- antisymmetric bilinear brackets $[\cdot\,,\cdot]_i : C_i \times C_i \to C_i$ for $i = 1, 0, -1$;
- antisymmetric bilinear brackets $[\cdot\,,\cdot]_{01} : C_0 \times C_1 \to C_1$ and $[\cdot\,,\cdot]_{10} : C_1 \times C_0 \to C_1$;
- antisymmetric trilinear Jacobiators $\langle\cdot,\cdot,\cdot\rangle_i : C_i \times C_i \times C_i \to C_{i+1}$ for $i = -1, 0$.

We write $[\cdot\,,\cdot]_{-1}$ as simply $[\cdot\,,\cdot]$. The following axioms are satisfied:

- $[\cdot\,,\cdot]_{01} = -[\cdot\,,\cdot]_{10}$;
- for every $a \in C_0$ and $h \in C_1$, $\partial([a,h]_{01}) = [a, \partial h]_0$;
- for every $h, k \in C_1$, $[h,k]_1 = [\partial h, k]_{01} = [h, \partial k]_{10}$;
- for every $a, b \in C_0$, $\partial[a,b]_0 = [\partial a, \partial b]$;
- for every $a, b, c \in C_0$,
$$\langle \partial a, \partial b, \partial c\rangle_{-1} + \partial(\langle a,b,c\rangle_0) = [a,[b,c]_0]_0 + [b,[c,a]_0]_0 + [c,[a,b]_0]_0.$$
- for every $a, b \in C_0$ and $h \in C_1$,
$$\langle a, b, \partial h\rangle_0 = [a,[b,h]_{01}]_{01} + [b,[h,a]_{10}]_{01} + [h,[a,b]_0]_{10}.$$

https://doi.org/10.1112/S0010437X1200067X Published online by Cambridge University Press

- for every $a, b, c, d \in C_0$,

$$[\langle a,b,c\rangle_0,d]_{10}-[\langle d,a,b\rangle_0,c]_{10}+[\langle c,d,a\rangle_0,b]_{10}-[\langle b,c,d\rangle_0,a]_{10}$$
$$=\langle [a,b]_0,c,d\rangle_0+\langle [c,d]_0,a,b\rangle_0+\langle [a,c]_0,d,b\rangle_0+\langle [d,b]_0,a,c\rangle_0$$
$$+\langle [a,d]_0,b,c\rangle_0+\langle [b,c]_0,a,d\rangle_0.$$

The butterfly picture of Definition 4.1 gives rise to obvious chain maps

$$\mathbb{V}\to \mathrm{hfib}(B)[-1] \quad \text{and} \quad \mathrm{hfib}(B)\to \mathbb{W}$$

which respect all the brackets on the nose. (Alternatively, one could use this as the definition of all the brackets on hfib(B) introduced above.)

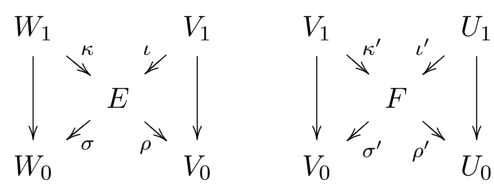
The sequence

$$\mathrm{hfib}(B)\to \mathbb{W}\to \mathbb{V}$$

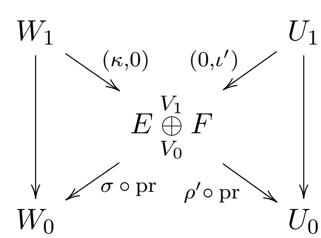
is an exact triangle in the derived category of chain complexes (note the reverse shift due to homological indexing).

5. The bicategory of Lie 2-algebras and butterflies

Given butterflies



we define their composition to be the following butterfly.

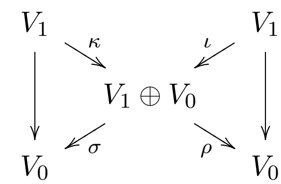


Here $E \overset{V_1}{\underset{V_0}{\oplus}} F$ is, by definition, the fiber product of E and F over V_0 modulo the diagonal image of V_1 via (ι, κ'). The bracket on it is defined componentwise.

In view of Remark 3.2, composition of butterflies corresponds to composition of zig-zags. Under the correspondence between butterflies and morphisms of 2-term L_∞-algebras (Proposition 3.4), composition of butterflies corresponds to composition of morphisms of 2-term L_∞-algebras (see Proposition 5.2).

Proposition 5.1. *With butterflies as morphisms, morphisms of butterflies as 2-morphisms, and composition defined as above, 2-term L_∞-algebras form a bicategory* $\mathbf{2TermL}_\infty^\flat$ *in which all 2-morphisms are invertible.*

For a 2-term L_∞-algebra $\mathbb{V}$, the identity butterfly from $\mathbb{V}$ to itself is defined as follows.



Here, the bracket on $V_1 \oplus V_0$ is defined by

$$[(k,x),(l,y)] := ([k,l] + [x,l] + [k,y], [x,y]).$$

The four structure maps of the butterfly are:

- $\kappa : V_1 \to V_1 \oplus V_0$, $\kappa(l) = (-l, \partial l)$;
- $\iota : V_1 \to V_1 \oplus V_0$, $\iota(k) = (k, 0)$;
- $\sigma : V_1 \oplus V_0 \to V_0$, $\sigma(k,x) = x$;
- $\rho : V_1 \oplus V_0 \to V_0$, $\rho(k,x) = \partial k + x$.

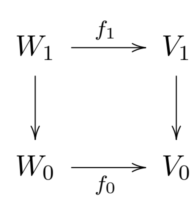
Proposition 5.2. *The construction of Proposition 3.4 induces a biequivalence* $\mathbf{2TermL}_\infty \cong \mathbf{2TermL}_\infty^\flat$ *(see Definition 2.11).*

Proof. This involves straightforward verification. □

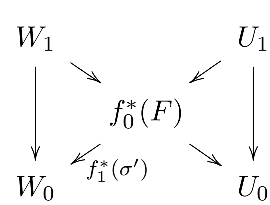
By Lemma 4.3, a butterfly $B : \mathbb{W} \to \mathbb{V}$ is invertible (in the bicategorical sense) if and only if its NW–SE sequence is also short exact. In this case, the inverse of B is obtained by flipping B along the vertical axis.

5.1 Composition of a butterfly with a strict morphism

Composition of butterflies takes a simpler form when one of the butterflies comes from a strict morphism. When the first morphism is strict, say



then the composition is



where $f_0^*(F)$ stands for the pullback of the extension F along $f_0 : W_0 \to V_0$. More precisely, $f_0^*(F) = W_0 \oplus_{V_0} F$ is the fiber product.

https://doi.org/10.1112/S0010437X1200067X Published online by Cambridge University Press

When the second morphisms is strict, say

$$\begin{array}{ccc} V_1 & \xrightarrow{g_1} & U_1 \\ \downarrow & & \downarrow \\ V_0 & \xrightarrow[g_0]{} & U_0 \end{array}$$

then the composition is

$$\begin{array}{ccccc} W_1 & & & & U_1 \\ & \searrow & & \swarrow^{g_{1,*}(\iota)} & \\ \downarrow & & g_{1,*}(E) & & \downarrow \\ & \swarrow & & \searrow & \\ W_0 & & & & U_0 \end{array}$$

where $g_{1,*}(E)$ stands for the pushforward of the extension E along $g_1 : V_1 \to U_1$. More precisely, $g_{1,*}(E) = E \oplus_{V_1} U_1$ is the pushout.

6. Weak morphisms of Lie crossed-modules and butterflies

There are at least three equivalent ways to define weak morphisms of Lie crossed-modules. One way is to localize the 2-category of Lie crossed-modules and strict morphisms with respect to equivalences, and define weak morphisms to be morphisms in this localized category (by definition, an equivalence between Lie crossed-modules is a morphism which induces isomorphisms on π_0 and π_1).

The second definition is that a weak morphism of Lie crossed-modules is a weak morphism (i.e. a monoidal functor) between the associated differentiable group stacks.

The third definition, which is shown in [AN09] to be equivalent to the stack definition, makes use of butterflies and is the subject of this section. It is the butterfly definition that proves to be most suitable for the study of connected covers of Lie crossed-modules as well as for proving our integration result (Theorem 1.2).

6.1 A note on the definition of Lie 2-groups

A Lie 2-group could mean different things to different people, so some clarification of the terminology is in order before we move on.

The definition we use in this paper is the following.

Definition 6.1. A *Lie* 2-*group* is a differentiable group stack which is equivalent to the group stack $\mathcal{G} := [G_0/G_1]$ associated to a Lie crossed-module $\mathbb{G} := [\partial : G_1 \to G_0]$. A *morphism* of Lie 2-groups is a differentiable weak homomorphism of differentiable group stacks.

Although most known examples of Lie 2-groups are of the above form, this is not the most general definition, as it is too strict. Arguably, the correct definition is that a Lie 2-group is simply a differentiable group stack, that is, a (weak) group object $\mathcal{G}$ in the 2-category of differentiable stacks. We have the following result.

Lemma 6.2. *A differentiable group stack $\mathcal{G}$ comes from a Lie crossed-module (i.e. is of the form $[G_0/G_1]$ for a Lie crossed-module $[G_1 \to G_0]$) if and only if it admits an atlas $\varphi : G_0 \to \mathcal{G}$ such that G_0 is a Lie group and φ is a differentiable (weak) homomorphism.*

https://doi.org/10.1112/S0010437X1200067X Published online by Cambridge University Press

Proof. If $\mathcal{G}$ is of the form $[G_0/G_1]$, then the quotient map $\varphi: G_0 \to \mathcal{G}$ has the desired property. Conversely, if $\mathcal{G}$ admits an atlas $\varphi: G_0 \to \mathcal{G}$ where G_0 is a Lie group and φ is a differentiable weak homomorphism, then we set $G_1 := * \times_{1_{\mathcal{G}}, \mathcal{G}, \varphi} G_0$ and let $\partial: G_1 \to G_0$ be the projection map. By general considerations, G_0 has an action on G_1 which makes $[\partial: G_1 \to G_0]$ a crossed-module. □

Therefore, a Lie 2-group in the sense of Definition 6.1 is a differentiable group stack which admits an atlas $\varphi: G_0 \to \mathcal{G}$ as in Lemma 6.2. Although in this paper we have restricted ourselves to such 'strict' Lie 2-groups, we expect that our theory can be extended to arbitrary differentiable group stacks.

Remark 6.3. Another definition of a Lie 2-group (which is presumably equivalent to the stack definition) is discussed in [Hen08, Appendix]. This definition is motivated by the fact that a Lie 2-group gives rise to a simplicial manifold and, conversely, a simplicial manifold with certain fibrancy properties and some conditions on its homotopy groups should come from a Lie 2-group.

Throughout the text, all Lie groups are assumed to be finite-dimensional unless stated otherwise.

6.2 Quick review of Lie butterflies

By Definition 6.1, a Lie 2-group is the differentiable group stack associated to a Lie crossed-module $[G_1 \to G_0]$, and a homomorphism of Lie 2-groups is a weak morphism of differentiable stacks. In this subsection, we give a description of morphisms of Lie 2-groups which avoids the stack language. This is done via butterflies.

For more details on butterflies, see [Noo08] (especially §§ 9.6 and 10.1) and [AN09]. In what follows, by a homomorphism of Lie groups we mean a differentiable homomorphism.

Remark 6.4. In [Noo08, AN09] we used the right-action convention for crossed-modules; but in this article, in order to be compatible with the existing literature on L_∞-algebras, we have used the left-action convention for Lie algebra crossed-modules. Therefore, for the sake of consistency, we will adopt the left-action convention for Lie crossed-modules as well.

Let $\mathbb{G}$ and $\mathbb{H}$ be Lie crossed-modules (i.e. crossed-modules in the category of Lie groups). A *butterfly* $B: \mathbb{H} \to \mathbb{G}$ is a commutative diagram

$$\begin{array}{ccccc} H_1 & & & & G_1 \\ \Big\downarrow & \searrow^{\kappa} & & \swarrow^{\iota} & \Big\downarrow \\ & & E & & \\ \Big\downarrow & \swarrow_{\sigma} & & \searrow_{\rho} & \Big\downarrow \\ H_0 & & & & G_0 \end{array}$$

in which both diagonal sequences are complexes of Lie groups and the NE–SW sequence is short exact. We also require that for every $x \in E$, $\alpha \in G_2$ and $\beta \in H_2$ the following equalities hold:

$$\iota(\rho(x) \cdot \alpha) = x\iota(\alpha)x^{-1}, \quad \kappa(\sigma(x) \cdot \beta) = x\kappa(\beta)x^{-1}.$$

A butterfly between Lie crossed-modules can be regarded as a Morita morphism which respects the group structures. A morphism $B \to B'$ of butterflies is, by definition, a homomorphism $E \to E'$ of Lie groups which commutes with all four structure maps of the butterflies. Note that such a morphism is necessarily an isomorphism.

Remark 6.5. For the reader interested in the topological version of the story, we remark that in the definition of a topological butterfly one needs to assume that the map $\sigma: E \to H_0$, viewed as a continuous map of topological spaces, admits local sections. This is automatic in the Lie case because σ is a submersion.

Thus, with butterflies as morphisms, Lie crossed-modules form a bicategory in which every 2-morphism is an isomorphism. We denote this bicategory by **LieXM**. The following theorem justifies why butterflies provide the right notion of morphism.

Theorem 6.6 [AN09]. *The* 2*-category of Lie* 2*-groups (in the sense of Definition 6.1) and weak morphisms is biequivalent to the bicategory* **LieXM** *of Lie crossed-modules and butterflies.*

We recall (see [Noo08, § 10.1]) how composition of two butterflies $C: \mathbb{K} \to \mathbb{H}$ and $B: \mathbb{H} \to \mathbb{G}$ is defined. Let F and E be the Lie groups appearing in the center of these butterflies, respectively. Then the composition $B \circ C$ is the butterfly

$$\begin{array}{ccccc} K_1 & & & & G_1 \\ \big\downarrow & \searrow & & \swarrow & \big\downarrow \\ & & F \overset{H_1}{\underset{H_0}{\times}} E & & \\ \big\downarrow & \swarrow & & \searrow & \big\downarrow \\ K_0 & & & & G_0 \end{array}$$

where $F \overset{H_1}{\underset{H_0}{\times}} E$ is the fiber product $F \underset{H_0}{\times} E$ modulo the diagonal image of H_1.

In the case where one of the butterflies is strict, the composition takes a simpler form similar to that in the discussion of § 5.1. See [Noo08, § 10.2] for more details.

7. Connected covers of a Lie 2-group

In this section we construct nth connected covers $\mathbb{G}\langle n\rangle$ of a Lie crossed-module $\mathbb{G} = [G_1 \to G_0]$ for $n = 0, 1, 2$. In § 8 we prove that these are functorial with respect to butterflies. Hence, in particular, they are invariant under equivalence of Lie crossed-modules (Corollary 8.7). All Lie groups are assumed to be finite-dimensional unless stated otherwise.

Definition 7.1. By the ith *homotopy group* $\pi_n\mathbb{G}$ of a topological crossed-module $\mathbb{G} = [\partial: G_1 \to G_0]$ we mean the ith homotopy group of the simplicial space associated to it or, equivalently, the ith homotopy group of the quotient stack $[G_0/G_1]$.

Homotopy groups of a topological stack $\mathcal{X}$ can be defined in terms of pointed homotopy classes of maps from spheres or, equivalently, as homotopy groups of a classifying space of $\mathcal{X}$. For details on these two definitions, and why they are equivalent, see [Noo12, Noo05]. Some basic results on homotopy groups of stacks (such as the fiber homotopy exact sequence of a fibration) can be found in [Noo10].

Caveat on notation. When $i = 0, 1$, the homotopy group $\pi_n\mathbb{G}$ should not be confused with the usage of π_0 and π_1 for coker ∂ and ker ∂; the two types of notation agree only when G_0 and G_1 are discrete groups.

Recall that a map $f: X \to Y$ of topological spaces is n-connected if $\pi_i f: \pi_i X \to \pi_i Y$ is an isomorphism for $i \leqslant n$ and a surjection for $i = n + 1$.

https://doi.org/10.1112/S0010437X1200067X Published online by Cambridge University Press

PROPOSITION 7.2. *Let $\mathbb{G} = [G_1 \to G_0]$ be a topological crossed-module and $n \geqslant 0$ an integer. The following are equivalent:*

(i) *the map ∂ is $(n-1)$-connected;*

(ii) *the quotient stack $\mathcal{G} := [G_0/G_1]$ is n-connected (in the sense of [Noo05, § 17]);*

(iii) *the classifying space of $\mathcal{G}$ is n-connected. (We are viewing $\mathcal{G}$ as a stack and ignoring its group structure.)*

Proof. The equivalence of (i) and (ii) follows from the homotopy fiber sequence applied to the fibration sequence of stacks $G_1 \to G_0 \to [G_0/G_1]$. The equivalence of (ii) and (iii) follows from [Noo12, Theorem 10.5]. □

DEFINITION 7.3. We say that a Lie crossed-module $\mathbb{G}$ is *n-connected* if it satisfies the equivalent conditions of Proposition 7.2.

It follows from Proposition 7.2 that the notion of n-connected is invariant under equivalence of Lie crossed-modules.

Remark 7.4. A Lie crossed-module $\mathbb{G}$ is 2-connected if and only if $\pi_i \partial : \pi_i G_1 \to \pi_i G_0$ is an isomorphism for $i = 0, 1$. This is because π_2 of every (finite-dimensional) Lie group vanishes.

7.1 Definition of the connected covers

In this subsection we define the nth connected cover of a Lie crossed-module for $n \leqslant 2$. In the next section we prove that these definitions are functorial with respect to butterflies. In particular, it follows that they are invariant under equivalence of Lie crossed-modules.

The discussions of this and the next section are valid for topological crossed-modules (and for infinite-dimensional Lie crossed-modules) as well.

The zeroth connected cover of $\mathbb{G}$. We have the following result.

LEMMA 7.5. *A Lie 2-group $\mathcal{G}$ is connected if and only if it has a presentation by a Lie crossed-module $[G_1 \to G_0]$ with G_0 connected.*

Proof. Choose an atlas $\varphi : G_0 \to \mathcal{G}$ such that G_0 is a Lie group and φ is a differentiable weak homomorphism (Lemma 6.2).

If G_0 is connected, then $\mathcal{G}$ is clearly connected, being the surjective image of a connected group. Conversely, if $\mathcal{G}$ is connected, we may replace G_0 by its connected component of the identity to obtain an atlas $\varphi : G_0 \to \mathcal{G}$ with G_0 connected. The desired crossed-module is obtained by setting $G_1 := * \times_{1_{\mathcal{G}}, \mathcal{G}, \varphi} G_0$, as in the proof of Lemma 6.2. □

For a given Lie crossed-module $\mathbb{G} = [G_1 \to G_0]$, its zeroth connected cover is defined to be

$$\mathbb{G}\langle 0 \rangle := [\partial^{-1}(G_0^o) \to G_0^o],$$

where G^o stands for the connected component of the identity. The crossed-module $\mathbb{G}\langle 0 \rangle$ should be thought of as the connected component of the identity of $\mathbb{G}$. There is an obvious strict morphism $q_0 : \mathbb{G}\langle 0 \rangle \to \mathbb{G}$ which induces isomorphisms on π_i for $i \geqslant 1$ (see Proposition 7.9).

The first connected cover of $\mathbb{G}$. We have the following result.

LEMMA 7.6. *A Lie 2-group* $\mathcal{G}$ *is* 1*-connected if and only if it has a presentation by a Lie crossed-module* $[G_1 \to G_0]$ *with* G_0 1*-connected and* G_1 *connected.*

Proof. Choose an atlas $\varphi : G_0 \to \mathcal{G}$ such that G_0 is a Lie group and φ is a differentiable weak homomorphism (Lemma 6.2), and let $[G_1 \to G_0]$ be the corresponding Lie crossed-module (as in the proof of Lemma 6.2).

If G_0 is 1-connected and G_1 is connected, a fiber homotopy exact sequence argument applied to the fibration sequence of topological stacks

$$G_1 \to G_0 \to \mathcal{G}$$

implies that $\mathcal{G}$ is 1-connected.

Conversely, suppose that $\mathcal{G}$ is 1-connected. By Lemma 7.5, we may assume that the atlas G_0 is connected. By replacing the atlas G_0 by its universal cover, we may also assume that G_0 is 1-connected. A fiber homotopy exact sequence argument applied to the fibration sequence of topological stacks

$$G_1 \to G_0 \to \mathcal{G}$$

implies that G_1 is connected. □

For a given Lie crossed-module $\mathbb{G} = [G_1 \to G_0]$, its first connected cover is defined to be

$$\mathbb{G}\langle 1\rangle := [L^o \to \widetilde{G_0^o}],$$

where $L := G_1 \times_{G_0} \widetilde{G_0^o}$ and the tilde denotes universal cover. There is an obvious strict morphism $q_1 : \mathbb{G}\langle 1\rangle \to \mathbb{G}$ which factors through q_0 and induces isomorphisms on π_i for $i \geqslant 2$ (see Proposition 7.9).

The second connected cover of $\mathbb{G}$. We have the following result.

LEMMA 7.7. *A Lie 2-group* $\mathcal{G}$ *is* 2*-connected if and only if it has a presentation by a Lie crossed-module* $[G_1 \to G_0]$ *with* G_0 *and* G_1 *both* 1*-connected.*

Proof. Choose an atlas $\varphi : G_0 \to \mathcal{G}$ such that G_0 is a Lie group and φ is a differentiable weak homomorphism (Lemma 6.2), and let $[G_1 \to G_0]$ be the corresponding Lie crossed-module (as in the proof of Lemma 6.2).

If G_0 and G_1 are both 1-connected, a fiber homotopy exact sequence argument applied to the fibration sequence of topological stacks

$$G_1 \to G_0 \to \mathcal{G}$$

implies that $\mathcal{G}$ is 2-connected. (Here we have used the fact that $\pi_2(G_0) = 0$, which is always true for Lie groups.)

Conversely, suppose that $\mathcal{G}$ is 2-connected. By Lemma 7.6, we may assume that the atlas G_0 is 1-connected. A fiber homotopy exact sequence argument applied to the fibration sequence of topological stacks

$$G_1 \to G_0 \to \mathcal{G}$$

implies that G_1 is connected. □

For a given Lie crossed-module $\mathbb{G} = [G_1 \to G_0]$, its second connected cover is defined to be

$$\mathbb{G}\langle 2\rangle := [\widetilde{L^o} \to \widetilde{G_0^o}],$$

where L is as defined above for the first connected cover. There is an obvious strict morphism $q_2 : \mathbb{G}\langle 2\rangle \to \mathbb{G}$ which factors through q_1 and induces isomorphisms on π_i for $i \geqslant 3$ (see Proposition 7.9).

Remark 7.8. Note that a 1-connected Lie group is automatically 2-connected. The same is not true for Lie 2-groups.

7.2 Uniform definition of the n-connected covers

In order to be avoid repetition in the constructions and arguments given in the next section, we phrase the definition of $\mathbb{G}\langle n\rangle$ in a uniform manner for $n = 0, 1, 2$ and highlight the main properties of the connected covers $q_n : G\langle n\rangle \to G$ which will be needed in the next section.[2] Our discussion will be valid for topological crossed-modules (and for infinite-dimensional Lie crossed-modules) as well.

First off, we need functorial n-connected covers $q_n : G\langle n\rangle \to G$ for $n = 0, 1, 2$.[3] We set $G\langle -1\rangle = G$. We take $G\langle 0\rangle := G^o$ and $G\langle 1\rangle = G\langle 2\rangle = \widetilde{G^o}$, where G^o is the connected component of the identity and $\widetilde{G^o}$ is its universal cover. (In the case where G is a topological group, or an infinite-dimensional Lie group, one has to make a different choice for $G\langle 2\rangle$; see Remark 7.10.)

For a crossed-module $\mathbb{G} = [G_1 \to G_0]$ we define $\mathbb{G}\langle n\rangle$ to be

$$\mathbb{G}\langle n\rangle := [\partial : L\langle n-1\rangle \to G_0\langle n\rangle],$$

where $L := G_1 \times_{G_0, q_n} G_0\langle n\rangle$ and $\partial = \mathrm{pr}_2 \circ q_{n-1}$. The action of $G_0\langle n\rangle$ on $L\langle n-1\rangle$ is defined as follows. There is an action of $G_0\langle n\rangle$ on L defined componentwise (on the first component it is obtained, via q_n, from the action of G_0 on G_1, and on the second component it is given by right conjugation). By functoriality of the nth connected cover construction (applied to L), this action lifts to $L\langle n-1\rangle$. For $\mathbb{G}\langle n\rangle$ to be a crossed-module, we use the following property.

(★0) For every $x \in G\langle n-1\rangle$, the action of $q_{n-1}(x) \in G$ on $G\langle n-1\rangle$ obtained (by functoriality) from the conjugation action of $q_{n-1}(x)$ on G is equal to conjugation by x.

There is a strict morphism of crossed-modules $q_n : \mathbb{G}\langle n\rangle \to \mathbb{G}$ defined as follows.

$$\begin{array}{ccc} L\langle n-1\rangle & \xrightarrow{\mathrm{pr}_1 \circ q_{n-1}} & G_1 \\ \partial \downarrow & & \downarrow \partial \\ G_0\langle n\rangle & \xrightarrow[q_n]{} & G_0 \end{array}$$

We will also need the following property.

(★1) The map $q_{n-1} : G\langle n-1\rangle \to G$ admits local sections near every point in its image (and hence is a fibration with open-closed image).

[2] Apart from improving the clarity of proofs in the next section, there is another purpose for emphasizing properties of connected covers in the form of axioms ★: in contexts other than Lie crossed-modules, it may be possible to formulate the axioms ★ for, say, other values of n, or by using different constructions for $G\langle n\rangle$. In such cases, our proofs apply verbatim.

[3] This, in fact, can be arranged for any n in the category of topological groups.

Proposition 7.9. *For $i \leqslant n$, we have $\pi_i(\mathbb{G}\langle n\rangle) = \{0\}$. For $i \geqslant n+1$, the morphism $q_n : \mathbb{G}\langle n\rangle \to \mathbb{G}$ induces isomorphisms $\pi_i(q_n) : \pi_i(\mathbb{G}\langle n\rangle) \to \pi_i(\mathbb{G})$.*

Proof. Consider the following commutative diagram.

$$\begin{array}{ccccc} L\langle n-1\rangle & \xrightarrow{\partial} & G_0\langle n\rangle & \longrightarrow & \mathbb{G}\langle n\rangle \\ {\scriptstyle \mathrm{pr}_1 \circ q_{n-1}}\downarrow & & \downarrow{\scriptstyle q_n} & & \downarrow{\scriptstyle q_n} \\ G_1 & \xrightarrow[\partial]{} & G_0 & \longrightarrow & \mathbb{G} \end{array}$$

Both rows are fibration sequences of crossed-modules (and so induce fibration sequences on the classifying spaces). The first claim follows by applying the fiber homotopy exact sequence to the first row. For the second claim, use the fact that $L \to G_1$ is a fibration (because of (★1)) with the same fiber as $q_n : G_0\langle n\rangle \to G_0$, and apply the fiber homotopy exact sequence to the two rows of the above diagram (together with the five lemma). □

Remark 7.10. In the definition of $\mathbb{G}\langle n\rangle = [L\langle n-1\rangle \to G_0\langle n\rangle]$, the fact that $G_0\langle n\rangle$ is an n-connected cover of G_0 is not really needed. All we need (e.g. for the discussion in the next section and the proof of Proposition 8.8) is to have a functorial replacement $q : G' \to G$ such that q is a fibration and $\pi_i G'$ is trivial for $i \leqslant n$. (For $L\langle n-1\rangle$, however, we do still need to take the $(n-1)$st connected cover of L.)

For instance, we could take G' to be the group $\mathrm{Path}_1(G)$ of paths originating at the identity element. To illustrate this by means of an example, let $\mathbb{G} = [1 \to G]$ be an arbitrary group. In this case, for the zeroth, first and second connected covers we find, respectively,

$$[\Omega_1(G) \to \mathrm{Path}_1(G)], \quad [\Omega_1(G)^o \to \mathrm{Path}_1(G)], \quad [\widetilde{\Omega_1(G)^o} \to \mathrm{Path}_1(G)],$$

where $\Omega_1(G) = L$ is the based loop group. Note that in the finite-dimensional context this construction would not be suitable, as $\mathrm{Path}_1(G)$ is infinite-dimensional. That is why we chose $G\langle 2\rangle := \widetilde{G^o}$ instead.

In the above discussion, the fact that we still need to use the $(n-1)$st connected cover of L in our construction of $\mathbb{G}\langle n\rangle$ is somewhat unsatisfactory, and one would hope that the same trick that was applied to G can be applied to L as well. This is indeed possible, at the cost of using a higher group model for $L\langle n-1\rangle$ (and thus for $\mathbb{G}\langle n\rangle$). For instance, instead of using

$$[\Omega_1(G)^o \to \mathrm{Path}_1(G)]$$

as a model for $\mathbb{G}\langle 1\rangle$, we could use the 3-group

$$[\Omega_1\Omega_1(G) \to \mathrm{Path}_1\,\Omega_1(G) \to \mathrm{Path}_1(G)].$$

In general, this suggests that there is natural model of $\mathbb{G}\langle n\rangle$ as a Lie $(n+2)$-group which is constructed solely using the Path_1 and Ω_1 functors.

8. Functorial properties of connected covers

For $n \leqslant 2$ we prove that our definition of the nth connected cover $\mathbb{G}\langle n\rangle$ of a Lie crossed-module is functorial in Lie butterflies and satisfies the expected adjunction property (Proposition 8.8). We will need the following property of the connected covers.

https://doi.org/10.1112/S0010437X1200067X Published online by Cambridge University Press

(★2) For any homomorphism $f : H \to G$ such that $\pi_i f : \pi_i H \to \pi_i G$ is an isomorphism for $0 \leqslant i \leqslant n-1$, the diagram

$$\begin{array}{ccc} H\langle n-1\rangle & \xrightarrow{f\langle n-1\rangle} & G\langle n-1\rangle \\ {\scriptstyle q_{n-1}}\downarrow & & \downarrow{\scriptstyle q_{n-1}} \\ H & \xrightarrow[f]{} & G \end{array}$$

is cartesian.

8.1 Construction of the nth connected cover functor

Consider the following Lie butterfly $B : \mathbb{H} \to \mathbb{G}$.

$$\begin{array}{ccccc} H_1 & & & & G_1 \\ & \overset{\kappa}{\searrow} & & \overset{\iota}{\swarrow} & \\ \downarrow & & E & & \downarrow \\ & \underset{\sigma}{\swarrow} & & \underset{\rho}{\searrow} & \\ H_0 & & & & G_0 \end{array}$$

The butterfly $B\langle n\rangle : \mathbb{H}\langle n\rangle \to \mathbb{G}\langle n\rangle$ is defined as follows.

$$\begin{array}{ccccc} L_H\langle n-1\rangle & & & & L_G\langle n-1\rangle \\ & \overset{\kappa_n}{\searrow} & & \overset{\iota_n}{\swarrow} & \\ \downarrow & & F\langle n-1\rangle & & \downarrow \\ & \underset{\sigma_n}{\swarrow} & & \underset{\rho_n}{\searrow} & \\ H_0\langle n\rangle & & & & G_0\langle n\rangle \end{array}$$

Let us explain what the terms appearing in this diagram mean. The groups L_G and L_H are what we called L in the definition of the n-connected cover (see § 7.2). For example, $L_H = H_1 \times_{H_0} H_0\langle n\rangle$. The Lie group F appearing in the center of the butterfly is defined to be

$$F := H_0\langle n\rangle \times_{H_0} E \times_{G_0} G_0\langle n\rangle.$$

The maps ρ_n and σ_n are obtained by composing $q_{n-1} : F\langle n-1\rangle \to F$ with the corresponding projections. The map κ_n is obtained by applying the functoriality of $(-)\langle n-1\rangle$ to $(\mathrm{pr}_2, \kappa \circ \mathrm{pr}_1, 1) : L_H \to F$. The definition of ι_n is less trivial and is given in the next paragraphs. We need to show that the kernel of $\sigma_n : F\langle n-1\rangle \to H_0\langle n\rangle$ is naturally isomorphic to $L_G\langle n-1\rangle$.

There is an equivalent way of defining F which is somewhat more illuminating. Set

$$K := H_0\langle n\rangle \times_{H_0} E.$$

Let $\sigma' : K \to H_0\langle n\rangle$ be the first projection map and $\rho' : K \to G_0$ the second projection map composed with ρ. Then

$$F = K \times_{\rho', G_0} G_0\langle n\rangle.$$

Now, observe that we have a short exact sequence

$$1 \to G_1 \xrightarrow{\alpha} K \xrightarrow{\sigma'} H_0\langle n\rangle \to 1.$$

https://doi.org/10.1112/S0010437X1200067X Published online by Cambridge University Press

Therefore, we have a cartesian diagram

$$\begin{array}{ccc} L_G & \overset{\beta}{\hookrightarrow} & F \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_1} \\ G_1 & \underset{\alpha}{\hookrightarrow} & K \end{array}$$

and the sequence

$$1 \to L_G \xrightarrow{\beta} F \xrightarrow{\sigma' \circ \mathrm{pr}_1} H_0\langle n\rangle \to 1$$

is short exact. (Exactness at the right end follows from (★1) and the fact that $H_0\langle n\rangle$ is connected.) A homotopy fiber sequence argument applied to this short exact sequence shows that α induces isomorphisms $\pi_i G_1 \to \pi_i K$ for $0 \leqslant i < n$. By (★2), we have a cartesian diagram as follows.

$$\begin{array}{ccc} L_G\langle n-1\rangle & \overset{\beta\langle n-1\rangle}{\hookrightarrow} & F\langle n-1\rangle \\ {\scriptstyle q_{n-1}}\downarrow & & \downarrow{\scriptstyle q_{n-1}} \\ L_G & \underset{\beta}{\hookrightarrow} & F \end{array}$$

Therefore

$$1 \to L_G\langle n-1\rangle \xrightarrow{\beta\langle n-1\rangle} F\langle n-1\rangle \xrightarrow{\sigma_n} H_0\langle n\rangle \to 1$$

is short exact, where $\sigma_n := \sigma' \circ \mathrm{pr}_1 \circ q_{n-1}$. (Exactness at the right end follows from (★1) and the fact that $H_0\langle n\rangle$ is connected.) Setting $\iota_n := \beta\langle n-1\rangle$ completes the construction of our butterfly diagram. The equivariance axioms for this butterfly follow from the functoriality of the $(n-1)$st connected cover.

Remark 8.1. In the case where we have a strict morphism $f : \mathbb{H} \to \mathbb{G}$, we can define a natural strict morphism $f\langle n\rangle : \mathbb{H}\langle n\rangle \to \mathbb{G}\langle n\rangle$ componentwise. It is natural to ask whether this morphism coincides with the one we constructed above using butterflies. The answer is yes. The proof uses the following property of connected covers.

(★3) If G is an n-connected group acting on H, then $(\mathrm{id}, q_{n-1}) : G \ltimes H\langle n-1\rangle \to G \ltimes H$ is the $(n-1)$-connected cover of $G \ltimes H$. That is, the map (id, q_{n-1}) is isomorphic to the q_{n-1} map of $G \ltimes H$.

8.2 Effect on the composition of butterflies

The proof that the construction of the previous subsection respects composition of butterflies is somewhat intricate. We will only consider Lie butterflies and assume that $0 \leqslant n \leqslant 2$, but the exact same proofs apply verbatim to topological butterflies (and also to infinite-dimensional Lie butterflies). We begin with a few lemmas.

Lemma 8.2. *Let $m \geqslant 0$ be an integer. Consider a homotopy cartesian diagram of topological spaces as follows.*

$$\begin{array}{ccc} X & \xrightarrow{h} & Y \\ \downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow[g]{} & W \end{array}$$

https://doi.org/10.1112/S0010437X1200067X Published online by Cambridge University Press

Suppose that W is $(m+1)$-connected and Z is m-connected. Then h induces isomorphisms $\pi_i h : \pi_i X \to \pi_i Y$ for $i \leqslant m$.

Proof. The connectivity assumptions on Z and T imply that the homotopy fiber of g is m-connected. Since the diagram is homotopy cartesian, the same is true of the homotopy fiber of h. A homotopy fiber exact sequence implies the claim. □

COROLLARY 8.3. *Let $f : Y \to W$ and $g : Z \to W$ be homomorphisms of Lie groups and suppose that W is $(m+1)$-connected. Suppose that either f or g is a fibration (e.g. surjective). Then we have natural isomorphisms*

$$Z\langle m\rangle \times_W Y\langle m\rangle \cong (Z\langle m\rangle \times_W Y)\langle m\rangle \cong (Z \times_W Y\langle m\rangle)\langle m\rangle.$$

In particular, all three groups are m-connected.

Proof. We prove the first equality. Apply Lemma 8.2 to the diagram

$$\begin{array}{ccc} X & \xrightarrow{h} & Y \\ \downarrow & & \downarrow{\scriptstyle f} \\ Z\langle m\rangle & \xrightarrow[g \circ q_m]{} & W \end{array}$$

where $X := Z\langle m\rangle \times_W Y$. The diagram is homotopy cartesian because either f or $g \circ q_m$ is a fibration. Now apply ($\bigstar$2) to $h = \mathrm{pr}_2 : Z\langle m\rangle \times_W Y \to Y$. □

The next lemma is the technical core of this subsection.

LEMMA 8.4. *Consider the commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{h} & Y \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow[g]{} & W \end{array}$$

of Lie groups. Suppose that W acts on X so that $[f \circ h : X \to W]$ is a Lie crossed-module. Also, suppose that the induced action of Y on X via f makes the map h Y-equivariant (the action of Y on itself being the right conjugation). Assume the same for the induced action of Z on X via g. Suppose that W is $(m+1)$-connected, f is surjective, and k is closed injective normal with $(m+1)$-connected cokernel. Then the sequence

$$1 \to X\langle m\rangle \xrightarrow{(k\langle m\rangle, h\langle m\rangle)} Z\langle m\rangle \times_W Y\langle m\rangle \xrightarrow{u} (Z \overset{X}{\underset{W}{\times}} Y)\langle m\rangle \to 1$$

is short exact. Here, u is the composition $(q_m, \mathrm{id})\langle m\rangle \circ \phi$ where $\phi : Z\langle m\rangle \times_W Y\langle m\rangle \xrightarrow{\sim} (Z\langle m\rangle \times_W Y)\langle m\rangle$ is the isomorphism of Corollary 8.3. (For the definition of $Z \overset{X}{\underset{W}{\times}} Y$ see the end of § 6.). In other words, we have a natural isomorphism

$$Z\langle m\rangle \overset{X\langle m\rangle}{\underset{W}{\times}} Y\langle m\rangle \cong (Z \overset{X}{\underset{W}{\times}} Y)\langle m\rangle.$$

Proof. We start with the short exact sequence

$$1 \to X \to Z \times_W Y \to Z \overset{X}{\underset{W}{\times}} Y \to 1,$$

which is essentially the definition of $Z \overset{X}{\underset{W}{\times}} Y$. From it we construct the exact sequence

$$1 \to X\langle m\rangle \xrightarrow{\alpha} Z\langle m\rangle \times_W Y \xrightarrow{(q_m,\mathrm{id})} Z \overset{X}{\underset{W}{\times}} Y$$

where $\alpha : X\langle m\rangle \to Z\langle m\rangle \times_W Y$ is $(k\langle m\rangle, h \circ q_m)$. To see why this sequence is exact, we calculate the kernel of the homomorphism $(q_m, \mathrm{id}) : Z\langle m\rangle \times_W Y \to Z \times_W Y$:

$$X \times_{Z \times_W Y} (Z\langle m\rangle \times_W Y) \cong X \times_{Z \times_W Y} ((Z \times_W Y) \times_Z Z\langle m\rangle) \cong X \times_Z Z\langle m\rangle \cong X\langle m\rangle.$$

For the first equality we used

$$Z\langle m\rangle \times_W Y \cong (Z \times_W Y) \times_Z Z\langle m\rangle,$$

and for the last equality we used (★2) for $k : X \to Z$. (Note that since $\operatorname{coker} k$ is $(m+1)$-connected, $k : X \to Z$ induces isomorphisms on π_i for all $i \leqslant m$.)

Observe that the last map in the above sequence is a fibration with (open-closed) image $I \subseteq Z \overset{X}{\underset{W}{\times}} Y$. This fibration has an m-connected kernel $X\langle m\rangle$, so, using the homotopy fiber exact sequence, we see that it induces isomorphisms on π_i for $i \leqslant m$. By (★2) we get the following cartesian square.

$$\begin{array}{ccc}
(Z\langle m\rangle \times_W Y)\langle m\rangle & \xrightarrow{(q_m,\mathrm{id})\langle m\rangle} & (Z \overset{X}{\underset{W}{\times}} Y)\langle m\rangle \\
{\scriptstyle q_m}\downarrow & & \downarrow{\scriptstyle q_m} \\
Z\langle m\rangle \times_W Y & \xrightarrow[(q_m,\mathrm{id})]{} & I
\end{array}$$

(Note that $I\langle m\rangle = (Z \overset{X}{\underset{W}{\times}} Y)\langle m\rangle$, because the mth connected cover depends only on the connected component of the identity, which is contained in I.) Precomposing the top row with the isomorphism $\phi : Z\langle m\rangle \times_W Y\langle m\rangle \xrightarrow{\sim} (Z\langle m\rangle \times_W Y)\langle m\rangle$ of Corollary 8.3, and calling the composition u as in the statement of the lemma, we find the following commutative diagram in which the square on the right is cartesian.

$$\begin{array}{ccccccccc}
1 & \longrightarrow & X\langle m\rangle & \xrightarrow{(k\langle m\rangle, h\langle m\rangle)} & Z\langle m\rangle \times_W Y\langle m\rangle & \xrightarrow{u} & (Z \overset{X}{\underset{W}{\times}} Y)\langle m\rangle & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle (\mathrm{id}, q_m)} & & \downarrow{\scriptstyle q_m} & & \\
1 & \longrightarrow & X\langle m\rangle & \xrightarrow[(k\langle m\rangle, h\circ q_m)]{} & Z\langle m\rangle \times_W Y & \xrightarrow[(q_m,\mathrm{id})]{} & I & \longrightarrow & 1
\end{array}$$

Since the bottom row is short exact, so is the top row. The proof of the lemma is therefore complete. □

https://doi.org/10.1112/S0010437X1200067X Published online by Cambridge University Press

We need one more technical lemma.

LEMMA 8.5. *Consider a commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{h} & Y \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow[g]{} & W \end{array}$$

of topological groups. Suppose that W acts on X so that $[f \circ h : X \to W]$ is a topological crossed-module. Also, suppose that the induced action of Y on X via f makes the map h Y-equivariant (the action of Y on itself being the right conjugation). Assume the same for the induced action of Z on X via g. Suppose that f is surjective. Let $\alpha : W' \to W$ be a homomorphism with normal image, and denote its cokernel by W_0. Denote the pullback of the above diagram along α by adding prime superscripts. Denote the images of X and Z in W_0 by X_0 and Z_0, respectively. (Note that X_0 is normal in W_0.) Then the sequence

$$1 \to Z' \overset{X'}{\underset{W'}{\times}} Y' \to Z \overset{X}{\underset{W}{\times}} Y \to Z_0/X_0 \to 1$$

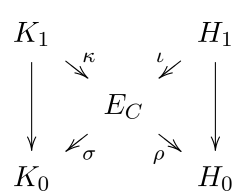
is exact. In particular, if the image of W' is open in W, then $Z' \overset{X'}{\underset{W'}{\times}} Y'$ is a union of connected components of $Z \overset{X}{\underset{W}{\times}} Y$.

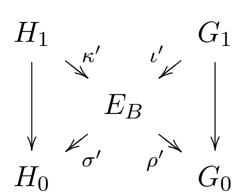
Proof. The proof is elementary group theory. □

We are now ready to prove that our construction of n-connected covers is functorial, that is, it respects composition of butterflies.

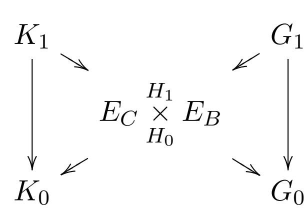
PROPOSITION 8.6. *Let $C : \mathbb{K} \to \mathbb{H}$ and $B : \mathbb{H} \to \mathbb{G}$ be Lie butterflies, and let $B \circ C : \mathbb{K} \to \mathbb{G}$ be their composition. Then there is a natural isomorphism of butterflies $B\langle n\rangle \circ C\langle n\rangle \Rightarrow (B \circ C)\langle n\rangle$ which makes the assignment $\mathbb{G} \mapsto \mathbb{G}\langle n\rangle$ a bifunctor from the bicategory* **LieXM** *of Lie crossed-modules and butterflies to itself.*

Proof. Let C and B be given by





respectively. Then the composition $B \circ C$ is the following butterfly.



https://doi.org/10.1112/S0010437X1200067X Published online by Cambridge University Press

Recall the following notation from § 8.1:

$$F_C = K_0\langle n\rangle \times_{K_0} E_C \times_{H_0} H_0\langle n\rangle \quad \text{and} \quad F_B = H_0\langle n\rangle \times_{H_0} E_B \times_{G_0} G_0\langle n\rangle.$$

The group appearing in the center of the butterfly $B\langle n\rangle \circ C\langle n\rangle$ is

$$F_C\langle n-1\rangle \overset{L_H\langle n-1\rangle}{\underset{H_0\langle n\rangle}{\times}} F_B\langle n-1\rangle.$$

The group appearing in the center of the butterfly $(B\circ C)\langle n\rangle$ is

$$F_{B\circ C}\langle n-1\rangle = (K_0\langle n\rangle \times_{K_0} (E_C \overset{H_1}{\underset{H_0}{\times}} E_B) \times_{G_0} G_0\langle n\rangle)\langle n-1\rangle.$$

We show that there is a natural isomorphism from the former to the latter. For this, we first apply Lemma 8.4 with $m = n-1$ and

$$X = L_H, \quad Z = F_C, \quad Y = F_B, \quad W = H_0\langle n\rangle$$

to get

$$F_C\langle n-1\rangle \overset{L_H\langle n-1\rangle}{\underset{H_0\langle n\rangle}{\times}} F_B\langle n-1\rangle \cong (F_C \overset{L_H}{\underset{H_0\langle n\rangle}{\times}} F_B)\langle n-1\rangle.$$

It is now enough to construct a natural isomorphism

$$F_C \overset{L_H}{\underset{H_0\langle n\rangle}{\times}} F_B \to F_{B\circ C},$$

that is,

$$(K_0\langle n\rangle \times_{K_0} E_C \times_{H_0} H_0\langle n\rangle) \overset{L_H}{\underset{H_0\langle n\rangle}{\times}} (H_0\langle n\rangle \times_{H_0} E_B \times_{G_0} G_0\langle n\rangle)$$
$$\longrightarrow (K_0\langle n\rangle \times_{K_0} (E_C \overset{H_1}{\underset{H_0}{\times}} E_B) \times_{G_0} G_0\langle n\rangle).$$

This, however, may not be the case. More precisely, there is such a natural homomorphism, but it is not necessarily an isomorphism. It is, however, an isomorphism between the connected components of the identity elements (and that is enough for our purposes). To see this, use Lemma 8.5 with

$$X = H_1, \quad Z = K_0\langle n\rangle \times_{K_0} E_C, \quad Y = E_B \times_{G_0} G_0\langle n\rangle,$$
$$W = H_0, \quad W' = H_0\langle n\rangle \quad \text{and} \quad \alpha = q_n.$$

(Recall that $L_H = H_1 \times_{H_0} H_0\langle n\rangle$.) Here we are using the fact that $\alpha = q_n : H_0\langle n\rangle \to H_0$ surjects onto the connected component of the identity element in H_0.

We omit the verification that the isomorphism $B\langle n\rangle \circ C\langle n\rangle \Rightarrow (B\circ C)\langle n\rangle$ respects isomorphisms of butterflies and that it commutes with the associator isomorphisms in **LieXM**. □

The proposition is valid in the topological setting as well, and the proof is identical.

Corollary 8.7. *Let $f : \mathbb{H} \to \mathbb{G}$ be an equivalence of Lie crossed-modules. Then the induced morphism $f\langle n\rangle : \mathbb{H}\langle n\rangle \to \mathbb{G}\langle n\rangle$, for $n = 0, 1, 2$, is also an equivalence of Lie crossed-modules.*

8.3 Adjunction property of connected covers

We show that n-connected covers of Lie crossed-modules satisfy the expected adjunction property, namely that a weak morphism $f : \mathbb{H} \to \mathbb{G}$ from an n-connected Lie crossed-module $\mathbb{H}$ uniquely factors through $q_n : \mathbb{G}\langle n\rangle \to \mathbb{G}$ (Proposition 8.8).

https://doi.org/10.1112/S0010437X1200067X Published online by Cambridge University Press

As in the previous section, we will assume that $n \leqslant 2$. What we say remains valid for topological crossed-modules (and also for infinite-dimensional Lie crossed-modules). We will use the following adjunction property for groups.

($\bigstar$4) For any homomorphism $f: H \to G$ with H being $(n-1)$-connected, f factors uniquely through $q_{n-1}: H\langle n-1\rangle \to H$.

Proposition 8.8. *Let $\mathbb{G}$ and $\mathbb{H}$ be Lie crossed-modules, and suppose that $\mathbb{H}$ is n-connected (Definition 7.3). Then the morphism $q = q_n : \mathbb{G}\langle n\rangle \to \mathbb{G}$ induces an equivalence of hom-groupoids*

$$q_* : \mathbf{LieXM}(\mathbb{H}, \mathbb{G}\langle n\rangle) \xrightarrow{\sim} \mathbf{LieXM}(\mathbb{H}, \mathbb{G}).$$

Proof. We construct an inverse functor (quasi-inverse, to be precise) to q_*. The construction is very similar to the construction of the n-connected cover of a butterfly given in the previous subsection.

Since $\mathbb{H} = [H_1 \to H_0]$ is n-connected, we may assume that H_0 is n-connected and H_1 is $(n-1)$-connected (this was discussed in § 7.1). Consider a butterfly B,

$$\begin{array}{ccccc} H_1 & & & & G_1 \\ \downarrow & \overset{\kappa}{\searrow} & & \overset{\iota}{\swarrow} & \downarrow \\ & & E & & \\ \downarrow & \underset{\sigma}{\swarrow} & & \underset{\rho}{\searrow} & \downarrow \\ H_0 & & & & G_0 \end{array}$$

in $\mathbf{LieXM}(\mathbb{H}, \mathbb{G})$. Define $F := E \times_{G_0} G_0\langle n\rangle$. Let $\tau : F \to H_0$ be $\sigma \circ \mathrm{pr}_1$. Since τ is a (locally trivial) fibration and H_0 is connected, τ is surjective. On the other hand, $\ker \tau$ is the inverse image of $\iota(G_1)$ under the projection $\mathrm{pr}_1 : F \to E$; this is exactly $G_1 \times_{G_0} G_0\langle n\rangle = L_G$. In other words, we have a short exact sequence

$$1 \to L_G \xrightarrow{\beta} F \xrightarrow{\tau} H_0 \to 1.$$

It follows from ($\bigstar$2) applied to β that the sequence

$$1 \to L_G\langle n-1\rangle \xrightarrow{\beta\langle n-1\rangle} F\langle n-1\rangle \xrightarrow{\tau\circ q_{n-1}} H_0 \to 1$$

is also short exact.

Define the butterfly B' to be

$$\begin{array}{ccccc} H_1 & & & & L_G\langle n-1\rangle \\ \downarrow & \overset{\kappa'}{\searrow} & & \overset{\beta\langle n-1\rangle}{\swarrow} & \downarrow \\ & & F\langle n-1\rangle & & \\ \downarrow & \underset{\tau\circ q_{n-1}}{\swarrow} & & \underset{\varrho}{\searrow} & \downarrow \\ H_0 & & & & G_0\langle n\rangle \end{array}$$

where $\varrho = \mathrm{pr}_2 \circ q_{n-1}$ and κ' is obtained by the adjunction property ($\bigstar$4) applied to $q_{n-1}: F\langle n-1\rangle \to F$.

It is easy to verify that $B \mapsto B'$ is an inverse to q_*. (For this, use the fact that E is the pushout of $F\langle n-1\rangle$ along $\mathrm{pr}_1 \circ q_{n-1} : L_G\langle n-1\rangle \to G_1$ and apply [Noo08, § 10.2].) □

Corollary 8.9. *For $n = 0, 1, 2$, the inclusion of the full sub-bicategory of* **LieXM** *consisting of n-connected Lie crossed-modules is left adjoint to the n-connected cover bifunctor $(-)\langle n\rangle$:* **LieXM** $\to$ **LieXM**.

https://doi.org/10.1112/S0010437X1200067X Published online by Cambridge University Press

9. The bifunctor from Lie crossed-modules to L_∞-algebras

In this section we prove our main integration results for weak morphisms of 2-term L_∞-algebras (Theorem 9.4 and Corollary 9.5). Throughout the section, we fix the base ring to be $\mathbb{R}$ or $\mathbb{C}$. All Lie groups and Lie algebras are finite-dimensional (real or complex). We also have a slight change of notation: from now on **LieAlgXM** only contains *finite-dimensional* Lie algebra crossed-modules.

Definition 9.1. To a Lie crossed-module $\mathbb{G} = [G_1 \to G_0]$ we associate a crossed-module in Lie algebras

$$\mathrm{Lie}\,\mathbb{G} := [\mathrm{Lie}\,G_1 \to \mathrm{Lie}\,G_0],$$

where $\mathrm{Lie}\,G$ stands for the Lie algebra associated to the Lie group G. To a crossed-module in Lie algebras $\mathbb{V} = [V_1 \to V_0]$ we associate a Lie crossed-module

$$\mathrm{Int}\,\mathbb{V} := [\mathrm{Int}\,V_1 \to \mathrm{Int}\,V_0],$$

where $\mathrm{Int}\,V$ is the connected and simply connected Lie group associated to the Lie algebra V.

Definition 9.2. We define the bicategory **LieAlgXM** to be the full sub-bicategory of $\mathbf{2TermL}_\infty^\flat$ consisting of strict 2-term L_∞-algebras (i.e. Lie algebra crossed-modules).

Note that, by Proposition 5.2, **LieAlgXM** is biequivalent to a full sub-2-category of the 2-category $\mathbf{2TermL}_\infty$.

Before proving our main result (Theorem 9.4), we need a lemma.

Lemma 9.3. *Let H, K and K' be connected Lie groups. Suppose that H acts on K and K' by automorphisms, and let $f : K \to K'$ be a Lie homomorphism. If the induced map* $\mathrm{Lie}\,f : \mathrm{Lie}\,K \to \mathrm{Lie}\,K'$ *is H-equivariant, then so is f itself.*

Proof. This follows from the fact that if two group homomorphisms induce the same map on Lie algebras, then they are equal. □

Theorem 9.4. *Taking* Lie *(as in Definition 9.1) induces a bifunctor*

$$\mathrm{Lie} : \mathbf{LieXM} \to \mathbf{2TermL}_\infty^\flat.$$

The bifunctor Lie *factors through and essentially surjects onto* **LieAlgXM**. *Furthermore, for* $\mathbb{H}, \mathbb{G} \in \mathbf{LieXM}$, *the induced functor*

$$\mathrm{Lie} : \mathbf{LieXM}(\mathbb{H}, \mathbb{G}) \to \mathbf{2TermL}_\infty^\flat(\mathrm{Lie}\,\mathbb{H}, \mathrm{Lie}\,\mathbb{G})$$

on hom-groupoids is:

(i) *faithful if $\mathbb{H}$ is connected;*

(ii) *fully faithful if $\mathbb{H}$ is 1-connected;*

(iii) *an equivalence if $\mathbb{H}$ is 2-connected.*

Proof. That $\mathrm{Lie} : \mathbf{LieXM} \to \mathbf{2TermL}_\infty^\flat$ is a bifunctor follows from the fact that taking Lie algebras is exact and commutes with fiber products of Lie groups.

Proof of (i). Let $\mathbb{G} = [G_1 \to G_0]$ and $\mathbb{H} = [H_1 \to H_0]$. Let $B, B' : \mathbb{H} \to \mathbb{G}$ be two butterflies. Since $\mathbb{H}$ is connected, we may assume that H_0 is connected (see § 7.1). Write the NE–SW short exact

https://doi.org/10.1112/S0010437X1200067X Published online by Cambridge University Press

sequences for B and B' as

$$0 \to G_1 \to E \to H_0 \to 0,$$
$$0 \to G_1 \to E' \to H_0 \to 0.$$

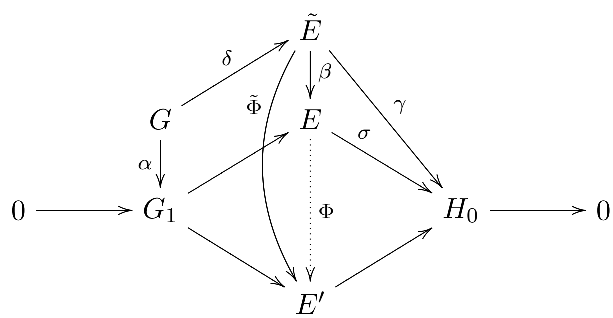
Consider two isomorphisms $B \Rightarrow B'$, given by $\Phi, \Psi : E \to E'$, such that

$$\mathrm{Lie}\,\Phi = \mathrm{Lie}\,\Psi : \mathrm{Lie}\,E \to \mathrm{Lie}\,E'.$$

Then Φ and Ψ are equal on the connected component E^o and also on G_1. Since H_0 is connected, E^o and G_1 generate E, so Φ and Ψ are equal on the whole of E.

Proof of (*ii*). With notation as in the previous part, we may assume that H_0 is connected and simply connected and that H_1 is connected (see § 7.1). Consider an isomorphism $\mathrm{Lie}\,B \Rightarrow \mathrm{Lie}\,B'$ given by $f : \mathrm{Lie}\,E \to \mathrm{Lie}\,E'$. We show that f integrates to $\Phi : E \to E'$.

Let $\tilde{E} \to E$ be the universal cover of E. Integrate f to a homomorphism $\tilde{\Phi} : \tilde{E} \to E'$. Consider the following diagram.

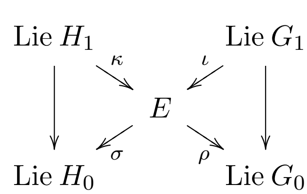


Here G is the kernel of $\gamma := \sigma\beta : \tilde{E} \to H_0$. Note that $G \cong G_1 \times_E \tilde{E}$; that is, G is the pullback of $\tilde{E}$ along the map $G_1 \to E$. Since $\pi_i H_0 = 0$ for $i = 1, 2$, a fiber homotopy exact sequence argument shows that $\pi_1 G_1 \to \pi_1 E$ is an isomorphism. Hence G is the universal cover of G_1 and, in particular, is connected.

If we apply Lie to the above diagram, we obtain a commutative diagram of Lie algebras. Therefore, since all the groups involved are connected, the original diagram of Lie groups is also commutative. Since the top left square is cartesian, δ induces an isomorphism $\delta : \ker\alpha \to \ker\beta$. Commutativity of the diagram then implies that $\tilde{\Phi}$ vanishes on $\ker\beta$. Therefore, $\tilde{\Phi}$ induces a homomorphism $\Phi : E \to E'$ which makes the diagram commute.

By looking at the corresponding Lie algebra maps, we see that if f commutes with the other two maps of the butterflies, then so does Φ; that is, Φ is indeed a morphism of butterflies from B to B'.

Proof of (*iii*). We may assume that H_0 and H_1 are connected and simply connected (see § 7.1). In view of the previous part, we have to show that every butterfly $B : \mathrm{Lie}\,\mathbb{H} \to \mathrm{Lie}\,\mathbb{G}$,

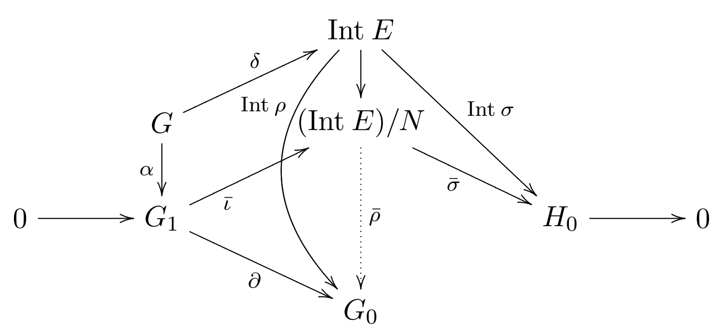


https://doi.org/10.1112/S0010437X1200067X Published online by Cambridge University Press

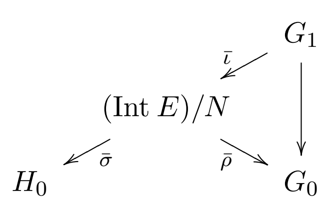
integrates to a butterfly $\mathrm{Int}\, B : \mathbb{H} \to \mathbb{G}$. Let $\mathrm{Int}\, E$ be the simply connected Lie group whose Lie algebra is E. Let G be the kernel of $\mathrm{Int}\,\sigma : \mathrm{Int}\, E \to H_0$. Since $\pi_i H_0 = 0$ for $i = 0, 1, 2$, an easy homotopy fiber exact sequence argument implies that G is connected and simply connected.

We identify the Lie algebras of G and G_1 via $\iota : \mathrm{Lie}\, G_1 \to E$ and regard them as equal. Since G is simply connected and G_1 is connected, we have a natural isomorphism $\bar{\iota} : G_1 \to G/N$ for some discrete central subgroup $N \subseteq G$. We claim that N is a normal subgroup of $\mathrm{Int}\, E$. To prove this, we compare the conjugation action of $\mathrm{Int}\, E$ on G with the action of $\mathrm{Int}\, E$ on G_1 obtained via $\mathrm{Int}\,\rho : \mathrm{Int}\, E \to G_0$. (The latter is the integration of the Lie algebra homomorphism $\rho : E \to \mathrm{Lie}\, G_0$.) The equivariance axiom of the butterfly for the map ρ, plus the fact that $\bar{\iota}^{-1} \circ \mathrm{pr} : G \to G_1$ induces the identity map on the Lie algebras, implies (by Lemma 9.3) that $\bar{\iota}^{-1} \circ \mathrm{pr} : G \to G_1$ is $\mathrm{Int}\, E$-equivariant. Therefore, its kernel N is invariant under the conjugation action of $\mathrm{Int}\, E$; that is, $N \subseteq \mathrm{Int}\, E$ is normal.

An argument similar to the one used in the previous part shows that the map $\mathrm{Int}\,\rho : \mathrm{Int}\, E \to G_0$ vanishes on N. More precisely, repeat the same argument with the following diagram.



Thus, we obtain an induced homomorphism $\bar{\rho} : (\mathrm{Int}\, E)/N \to G_0$. Denote the map $(\mathrm{Int}\, E)/N \to H_0$ induced from $\mathrm{Int}\,\sigma$ by $\bar{\sigma}$. Collecting what we have so far, we obtain a partial butterfly diagram as follows.



(Observe that applying Lie to this partial butterfly gives us back the corresponding portion of the original butterfly B.) Finally, using the fact that H_1 is connected and simply connected, we can complete the butterfly by integrating κ to $\bar{\kappa} : H_1 \to (\mathrm{Int}\, E)/N$. It is easily verified that the resulting diagram satisfies the butterfly axioms; this is the sought after butterfly $\mathrm{Int}\, B : \mathbb{H} \to \mathbb{G}$. The proof is complete. □

Corollary 9.5. *The bifunctor* $\mathrm{Int} : \mathbf{LieAlgXM} \to \mathbf{LieXM}$ *is left adjoint to the bifunctor* $\mathrm{Lie} : \mathbf{LieXM} \to \mathbf{LieAlgXM}$ *(see Definition 9.1).*

Proof. By Proposition 7.2, for any crossed-module in Lie algebras $\mathbb{V}$, the associated Lie crossed-module $\mathrm{Int}\, \mathbb{V}$ is 2-connected. The corollary now follows from Theorem 9.4. □

https://doi.org/10.1112/S0010437X1200067X Published online by Cambridge University Press

Remark 9.6. Presumably, the adjunction of Corollary 9.5 can be extended to the following.

$$\begin{array}{ccc} \mathrm{Int} : \mathbf{LieAlgXM} & \leftrightharpoons & \mathbf{LieXM} : \mathrm{Lie} \\ \downarrow & & \downarrow \\ \mathrm{Int} : \mathbf{2TermL_\infty} & \leftrightharpoons & \mathbf{DiffGpSt} : \mathrm{Lie} \end{array}$$

Here, by **DiffGpSt** we mean the 2-category of differentiable group stacks. The inclusion on the right is given by the fully faithful bifunctor

$$\mathbf{LieXM} \to \mathbf{DiffGpSt},$$
$$[G_1 \to G_0] \mapsto [G_0/G_1].$$

10. Applications

In Lie theory, integration results are tools to linearize problems. For instance, to study a one-parameter group of automorphisms of a manifold M, one looks at the corresponding vector field. Integrating vector fields reduces the problem of studying symmetries of a manifold to the study of the Lie algebra of vector fields. More precisely, integration results allow us to study actions of a Lie group G on a manifold M by considering infinitesimal actions of the corresponding Lie algebra $\mathrm{Lie}\, G$ on M.

Now replace M by a 'higher' object, say a differentiable stack $\mathcal{M}$. In this case, the symmetries of $\mathcal{M}$ form a Lie 2-group. (In general, symmetries of an object in an n-category form an n-group.) To study symmetries of $\mathcal{M}$, one looks at actions of Lie 2-groups $\mathcal{G}$ on $\mathcal{M}$. Integration results, such as the ones proved in this paper, allow us to reduce the study of such actions to the linear problem of studying infinitesimal actions of the Lie 2-algebra $\mathrm{Lie}\, \mathcal{G}$ on $\mathcal{M}$.

In this section we illustrate these ideas by two simple examples.

10.1 Actions on weighted projective stacks

Let $n_1, n_2, \ldots, n_r$ be a sequence of positive integers, and consider the weight-$(n_1, n_2, \ldots, n_r)$ action of $\mathbb{C}^*$ on $\mathbb{C}^r - \{0\}$ (that is, $t \in \mathbb{C}^*$ acts by multiplication by $(t^{n_1}, t^{n_2}, \ldots, t^{n_r})$). The stack quotient of this action is the weighted projective stack $\mathbb{P}(n_1, n_2, \ldots, n_r)$.

The weighted projective general linear 2-group

$$\mathrm{PGL}(n_1, n_2, \ldots, n_r)$$

(see [BN06, § 8]) is defined to be the complex (algebraic) Lie 2-group associated to the crossed-module

$$[\partial : \mathbb{C}^* \to G_{n_1, n_2, \ldots, n_r}],$$

where $G_{n_1, n_2, \ldots, n_r}$ is the group of all $\mathbb{C}^*$-equivariant (for the above weighted action) complex automorphisms $f : \mathbb{C}^r - \{0\} \to \mathbb{C}^r - \{0\}$. The homomorphism $\partial : \mathbb{C}^* \to G_{n_1, n_2, \ldots, n_r}$ is the one induced from the $\mathbb{C}^*$-action. We take the action of $G_{n_1, n_2, \ldots, n_r}$ on $\mathbb{C}^*$ to be trivial.

By [BN06, Theorem 8.1], $\mathrm{PGL}(n_1, n_2, \ldots, n_r)$ is equivalent to the 2-group of complex automorphisms of $\mathbb{P}(n_1, n_2, \ldots, n_r)$.

https://doi.org/10.1112/S0010437X1200067X Published online by Cambridge University Press

Now, let $\mathcal{G}$ be a 2-connected complex Lie 2-group. By the cited theorem, giving an action of $\mathcal{G}$ on $\mathbb{P}(n_1, n_2, \ldots, n_r)$ is the same as giving a weak homomorphism of Lie 2-groups

$$\mathcal{G} \to \mathrm{PGL}(n_1, n_2, \ldots, n_r).$$

By Theorem 9.4, this is equivalent to giving a weak morphism of Lie algebra crossed-modules

$$[\mathrm{Lie}\, G_1 \to \mathrm{Lie}\, G_0] \;\to\; [\mathrm{Lie}\, \mathbb{C}^* \to \mathrm{Lie}\, G_{n_1,n_2,\ldots,n_r}],$$

where $[G_1 \to G_0]$ is a Lie crossed-module presenting $\mathcal{G}$. Observe that the map $\mathrm{Lie}\, \mathbb{C}^* \to \mathrm{Lie}\, G_{n_1,n_2,\ldots,n_r}$ is injective, so the crossed-module on the right-hand side is equivalent to the honest Lie algebra

$$\mathrm{Lie}(G_{n_1,n_2,\ldots,n_r})/\,\mathrm{Lie}(\mathbb{C}^*) \cong \mathrm{Lie}(G_{n_1,n_2,\ldots,n_r}/\mathbb{C}^*) \cong \mathfrak{pgl}\left(\frac{n_1}{d}, \frac{n_2}{d}, \ldots, \frac{n_r}{d}\right),$$

where $d = \gcd(n_1, n_2, \ldots, n_r)$ and $\mathfrak{pgl}(n_1/d, n_2/d, \ldots, n_r/d) := \mathrm{Lie}(\mathrm{PGL}(n_1/d, n_2/d, \ldots, n_r/d))$ is now an honest Lie algebra. If we let G be the cokernel of $G_1 \to G_0$, we conclude from the above discussion that there is a bijection

$$\{\text{actions of } \mathcal{G} \text{ on } \mathbb{P}(n_1, n_2, \ldots, n_r)\} \longleftrightarrow \left\{\text{Lie algebra maps } \mathrm{Lie}\, G \to \mathfrak{pgl}\left(\frac{n_1}{d}, \frac{n_2}{d}, \ldots, \frac{n_r}{d}\right)\right\}.$$

The structure of the algebraic group $G_{n_1,n_2,\ldots,n_r}$ is studied in detail in [Noo07]. This gives us a good handle on the Lie algebra $\mathfrak{pgl}(n_1, n_2, \ldots, n_d)$ and hence on 2-group actions on $\mathbb{P}(n_1, n_2, \ldots, n_r)$.

Remark 10.1. Observe that $\mathrm{PGL}(n_1/d, n_2/d, \ldots, n_r/d)$ is the automorphism 2-group of the reduced orbifold $\mathbb{P}(n_1/d, n_2/d, \ldots, n_r/d)$, and that $\mathbb{P}(n_1, n_2, \ldots, n_r)$ is a μ_d-gerbe over $\mathbb{P}(n_1/d, n_2/d, \ldots, n_r/d)$. The above discussion implies that for every 2-connected complex Lie 2-group $\mathcal{G}$, any action of $\mathcal{G}$ on $\mathbb{P}(n_1/d, n_2/d, \ldots, n_r/d)$ lifts uniquely (up to 2-isomorphism) to an action on $\mathbb{P}(n_1, n_2, \ldots, n_r)$. For instance, when $n_1 = n_2 = \cdots = n_r = d$, giving an action of $\mathcal{G}$ on $\mathbb{P}(d, d, \ldots, d)$ would be equivalent to giving an action of $\mathcal{G}$ on its coarse moduli space $\mathbb{CP}^{r-1}$.

10.2 2-representation theory

In the classical theory, the functors Lie and Int (see Definition 9.1) relate representation theory of a Lie group G to the representation theory of its Lie algebra $\mathrm{Lie}\, G$. For Lie 2-groups, the same correspondence relates 2-representations of a Lie 2-group $\mathcal{G}$ to representations of the Lie 2-algebra $\mathrm{Lie}\, \mathcal{G}$.

We outline a possible application of our methods to the representation theory of 2-groups on an abelian category. Such actions arise, for instance, in the work of Frenkel and Gaitsgory [FG06] and Frenkel and Zhu [FZ12] in the context of the geometric Langlands program, where they are used to study representations of double loop groups.

It is argued in [FZ12] that the correct double loop group analogues of the projective representations of loop groups are 'gerbal' representations of the double loop group on certain abelian categories (e.g. on the category of Fock representations of a certain Clifford algebra; this is the higher analogue of the fermionic Fock representation of $\mathfrak{gl}_\infty$). This led the authors to study weak homomorphisms

$$G \to \mathbb{GL}(\mathcal{C}), \qquad \text{representation of } G \text{ on the abelian category } \mathcal{C},$$
$$G \to \pi_0\, \mathbb{GL}(\mathcal{C}), \quad \text{'gerbal' reprsentation of } G \text{ on } \mathcal{C},$$

https://doi.org/10.1112/S0010437X1200067X Published online by Cambridge University Press

and the corresponding Lie algebra representations. In [Zhu09] the Lie 2-algebra Lie $\mathbb{GL}(\mathcal{C})$ is described. This, in conjunction with the butterfly method of § 3, provides a new way of studying representations of a Lie 2-algebra on Lie $\mathbb{GL}(\mathcal{C})$. Our integration results (§ 9) would then enable one to promote these to Lie 2-group representations.

It should be stressed that many interesting examples of representations of Lie groups or algebras on abelian categories involve actions of infinite-dimensional Lie groups or algebras (e.g. $\mathrm{GL}_{\infty,\infty}$ and $\mathfrak{gl}_{\infty,\infty}$). Studying these requires generalizing the results of this paper to infinite dimensions, which is a subject for further investigation.

ACKNOWLEDGEMENTS

I thank Gustavo Granja and Christoph Wockel for helpful conversations on the subject of this paper. Some of the results of this paper are answers to questions asked by Chenchang Zhu, to whom I am grateful. I also thank Fernando Muro, Jim Stasheff, Tim Porter, Dmitry Roytenberg and Marco Zambon for making useful comments on an earlier version of the paper.

Appendix A. Functorial n-connected covers for $n \geqslant 3$

Axioms (★1–4) discussed in §§ 7–8 have a certain iterative property which we would like to point out in this appendix. To simplify the notation, we will replace $n-1$ by m.

We saw in §§ 7–8 that, for $m \leqslant 1$, the standard choices for the m-connected cover functors $(-)\langle m\rangle$ on the category of topological groups automatically satisfy (★1–4). Using this, we constructed our m-connected cover bifunctor $(-)\langle m\rangle$ on the bicategory of topological (or Lie) crossed-modules for $m \leqslant 2$. It can be shown that these bifunctors again satisfy (a categorified version of) axioms (★1–4).

A magic seems to have occurred here: we managed to raise m from 1 to 2! This may sound contradictory, as we do not expect to have a functorial 2-connected cover functor $(-)\langle 2\rangle$ on the category of topological groups which satisfies either the pullback property (★2) or the adjunction property (★4).

This apparent contradiction is explained by noticing that our definition of $(-)\langle 2\rangle$ indeed yields a crossed-module, even if the input is a topological group. More precisely, for a topological group G we get

$$G\langle 2\rangle = [\widetilde{L^o} \to G'],$$

where $q : G' \to G$ is a choice of a 2-connected replacement for G and $L = \ker q$. (For example, take $G' = \mathrm{Path}_1(G)$, the space of paths starting at 1; see Remark 7.10.) It is also interesting to note that for different choices of the 2-connected replacement $q : G' \to G$, the resulting crossed-modules $G\langle 2\rangle$ are canonically (up to a unique isomorphism of butterflies) equivalent.

The upshot of this discussion is that 2-connected covers of topological groups seem to exist more naturally as topological crossed-modules. Another implication is that we can now iterate the process. For example, we get a functorial construction of a 3-connected cover $G\langle 3\rangle$ of a topological group G as a 2-crossed-module, and this (essentially unique) construction enjoys a categorified version of (★1–4).

This seems to hint at the following general philosophy: for any $m \leqslant k+1$, there should be a (essentially unique) construction of m-connected covers $\mathbb{G}\langle m\rangle$ for topological k-crossed-modules $\mathbb{G}$ which enjoys a categorified version of (★1–4).

https://doi.org/10.1112/S0010437X1200067X Published online by Cambridge University Press

We point out that the notion of k-crossed-module exists for $k \leqslant 3$ (see [Con84, AKU09]). The butterfly construction of the tricategory of 2-crossed-modules (and weak morphisms) is being developed in [AN]. For higher values of k, the simplicial approach is perhaps a better alternative, as k-crossed-modules tend to become immensely complicated as k increases.

Remark. The above discussion applies to the case where topological groups are replaced by infinite-dimensional Lie groups.

References

AN E. Aldrovandi and B. Noohi, *Butterflies III: 2-butterflies and 2-group stacks*, in preparation.

AN09 E. Aldrovandi and B. Noohi, *Butterflies I: morphisms of 2-group stacks*, Adv. Math. **221** (2009), 687–773.

AKU09 Z. Arvasi, T. S. Kuzpinari and E. Ö. Uslu, *Three crossed modules*, Homology, Homotopy Appl. **11** (2009), 161–187.

BC04 J. Baez and A. Crans, *Higher-dimensional algebra. VI. Lie 2-algebras*, Theory Appl. Categ. **12** (2004), 492–538.

BN06 K. Behrend and B. Noohi, *Uniformization of Deligne–Mumford analytic curves*, J. Reine Angew. Math. **599** (2006), 111–153.

BG89 R. Brown and N. D. Gilbert, *Algebraic models of 3-types and automorphism structures for crossed modules*, Proc. Lond. Math. Soc. (3) **59** (1989), 51–73.

Con84 D. Conduché, *Modules croisés généralisés de longueur* 2, J. Pure Appl. Algebra **34** (1984), 155–178.

FG06 E. Frenkel and D. Gaitsgory, *Local geometric Langlands correspondence and affine Kac–Moody algebras*, in *Algebraic geometry and number theory*, Progress in Mathematics, vol. 253 (Birkhäuser, Boston, 2006), 69–260.

FZ12 E. Frenkel and X. Zhu, *Gerbal representations of double loop groups*, Int. Math. Res. Not. IMRN **2012** (2012), 3929–4013, doi:10.1093/imrn/rnr159.

Get09 E. Getzler, *Lie theory for nilpotent L_∞-algebras*, Ann. of Math. (2) **170** (2009), 271–301.

Hen08 A. Henriques, *Integrating L_∞-algebras*, Compositio Math. **144** (2008), 1017–1045.

Noo05 B. Noohi, *Foundations of topological stacks I*, Preprint (2005), arXiv:math/0503247.

Noo07 B. Noohi, *Automorphism 2-group of a weighted projective stack*, Preprint (2007), arXiv:0704.1010.

Noo08 B. Noohi, *On weak maps between* 2*-group*, Preprint (2008), arXiv:math/0506313v3.

Noo10 B. Noohi, *Fibrations of topological stacks*, Preprint (2010), arXiv:1010.1748.

Noo12 B. Noohi, *Homotopy types of topological stacks*, Adv. Math. **230** (2012), 2014–2047.

Roy07 D. Roytenberg, *On weak Lie 2-algebras*, in *XXVI Workshop on geometrical methods in physics*, AIP Conference Proceedings, vol. 956 (American Institute of Physics, Melville, NY, 2007).

Zhu09 X. Zhu, *The 2-group of linear auto-equivalences of an abelian category and its Lie 2-algebra*, Preprint (2009), arXiv:0910.5699.

Behrang Noohi noohi@maths.qmul.ac.uk
School of Mathematical Sciences, Queen Mary, University of London,
Mile End Road, London E1 4NS, UK