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On classification of singular matrix difference equations of mixed order

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This paper is concerned with singular matrix difference equations of mixed order. The existence and uniqueness of initial value problems for these equations are derived, and then the classification of them is obtained with a similar classical Weyl's method by selecting a suitable quasi-difference. An equivalent characterization of this classification is given in terms of the number of linearly independent square summable solutions of the equation. The influence of off-diagonal coefficients on the classification is illustrated by two examples. In particular, two limit point criteria are established in terms of coefficients of the equation.

Keywords: block operator matrix; matrix differential equation; matrix difference equation; limit point case; limit circle case

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1. Introduction

Consider the matrix difference expressions of mixed order:

$$\mathcal{L}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}(t) := \begin{cases} \begin{pmatrix} -\nabla p\Delta + q & -\nabla c + h \\ c\Delta + h & d \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}(t), \ t \in \mathcal{I}, \\ \begin{pmatrix} 0 & 0 \\ c\Delta + h & d \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}(t), \ t = a-1, \end{cases}$$

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where $\mathcal{I} := \{t\}_{t=a}^{+\infty}$ is an integer set with a being a finite integer; ∇ and Δ are the backward and forward difference operators, respectively, i.e., $\nabla y(t) = y(t) - y(t-1)$ and $\Delta y(t) = y(t+1) - y(t)$; p, q, c, h, and d are real-valued functions on $\mathcal{I}' := \mathcal{I} \cup \{a-1\}$ with $p(t) \neq 0$ for all $t \in \mathcal{I}'$. Let λ be a spectral parameter. Then, equation $\mathcal{L}(y) = \lambda y$ on $\mathcal{I}'$ can be expressed as follows:

$$\begin{cases} -\nabla\left(p(t)\Delta y_1(t)\right) + q(t)y_1(t) - \nabla\left(c(t)y_2(t)\right) + h(t)y_2(t) = \lambda y_1(t), \ t \in \mathcal{I}, \\ c(t)\Delta y_1(t) + h(t)y_1(t) + d(t)y_2(t) = \lambda y_2(t), \ t \in \mathcal{I}'. \end{cases} \tag{1.1$_\lambda$}$$

The difference expressions $\mathcal{L}$ or equations (1.1$_\lambda$) is called singular since one of the endpoints of $\mathcal{I}$ is infinity. If $\mathcal{I} := \{t\}_{t=a}^{b}$ with a and b being finite integers, then $\mathcal{L}$ is called regular. In the case of $h = c \equiv 0$ on $\mathcal{I}'$ and $d(t) \neq \lambda$ for $t \in \mathcal{I}'$, the equation (1.1$_\lambda$) becomes the classical Sturm–Liouville difference equation

$$\tau(y_1)(t) := -\nabla\left(p(t)\Delta y_1(t)\right) + q(t)y_1(t) = \lambda y_1(t), \quad t \in \mathcal{I}. \tag{1.2}$$

Therefore, equations (1.1$_\lambda$) contain classical Sturm–Liouville difference equations as their special ones. Moreover, if $y = (y_1, y_2)^{\mathrm{T}}$ (the superscript T denotes the transpose of a vector) satisfies (1.1$_\lambda$), then the first component y_1 is a solution of the following Sturm–Liouville difference equation with coefficients depending rationally on the spectral parameter:

$$-\nabla\left(\widetilde{p}(t,\lambda)\Delta y_1(t)\right) + \tilde{q}(t,\lambda)y_1(t) = \lambda y_1(t), \quad t \in \mathcal{I}, \tag{1.3}$$

where $\widetilde{p}(t, \lambda)$ and $\widetilde{q}(t, \lambda)$ are given by

$$\begin{aligned} \widetilde{p}(t,\lambda) &:= p(t) + \frac{c^2(t)}{\lambda - d(t)} - \frac{h(t)c(t)}{\lambda - d(t)}, \quad t \in \mathcal{I}' \\ \widetilde{q}(t,\lambda) &:= q(t) + \frac{h^2(t)}{\lambda - d(t)} - \nabla\left(\frac{h(t)c(t)}{\lambda - d(t)}\right), \ t \in \mathcal{I}. \end{aligned} \tag{1.4}$$

In addition, y_2 can be expressed in terms of y_1 as follows:

$$y_2(t) = \frac{c(t)}{\lambda - d(t)}\Delta y_1(t) + \frac{h(t)}{\lambda - d(t)}y_1(t), \quad t \in \mathcal{I}'. \tag{1.5}$$

Conversely, if y_1 and y_2 satisfy (1.3) and (1.5), then y_1 with y_2 is a solution of (1.1$_\lambda$). Hence, equation (1.1$_\lambda$) is equivalent to (1.3) and (1.5) when λ is given such that (1.4) and (1.5) are well-defined.

Matrix differential expressions of mixed order arise in fluid mechanics, magnetohydrodynamics, and quantum mechanics, etc. Essential spectra of operators generated by a class of 3×3 matrix differential expressions of mixed order for ideal magnetohydrodynamics models were studied by Kako in [**29**]. This work was generalized and developed by many authors (cf., e.g., [**16**, **17**, **19**, **40**, **44**]), and then spectral properties of this class of differential expressions were gotten more clear understanding. Up to now, the spectral theory for this class of differential expressions has been studied intensively (cf., [**9**, **24**, **25**, **31**–**33**, **45**, **46**] and the references

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cited therein). It is noted that most existing relevant results are concerned with the following 2×2 matrix differential equations of mixed order:

$$\mathbb{L}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}(t) := \begin{pmatrix} -DpD + q & -Dc + h \\ \bar{c}D + \bar{h} & d \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}(t) = \lambda \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}(t), \quad t \in (a, b), \quad (1.6)$$

where $-\infty < a < b \leqslant \infty$; p^{-1}, q, c, h, and d are local integrable functions on (a, b) with $p(t) \neq 0$ for all $t \in (a, b)$; $D = \mathrm{d}/\mathrm{dt}$, λ is a spectral parameter. Although equations (1.6) are more simple forms of matrix differential expressions of mixed order, they may contain more complicated examples including 3×3 ones which were considered, e.g., in [**16**, **17**, **19**, **40**, **44**], when $c(t), h(t) \in \mathbb{C}^n$, and $d(t) \in \mathbb{C}^{n\times n}$, $t \in (a, b)$, and $\bar{c}(t)$ and $\bar{h}(t)$ are replaced by $c^*(t)$ and $h^*(t)$, where $c^*(t)$ denotes the complex conjugate transpose of $c(t)$, $t \in (a, b)$. Essential spectra of equations (1.6) with the above vector and matrix coefficients have been investigated by Ibrogimov, Siegl, and Tretter in great detail under considerably weaker assumptions [**24**]. For the study of non-self-adjoint matrix differential expressions, the reader can be referred to [**25**]. At the same time, the spectral theory for abstract block operator matrices has been developed and some elegant results have been established for the various essential spectra, spectral decomposition, spectral enclosure, spectral inclusion, quadratic numerical range, and Friedrichs extension (cf., [**2**, **4**, **18**, **26**, **27**, **30**, **34**, **39**, **50**, **51**]). As everyone knows, there are a large number of discrete mathematical models in applications. The spectral theory of discrete systems has attracted a great deal of interest (cf., [**3**, **12**, **22**, **28**, **41**, **42**, **47**, **54**] and the references cited therein). Equations (1.1_λ) can be regarded as a discrete analogue of the singular equations (1.6). However, as far as we know, there are little attention on equations (1.1_λ) including the regular case and the singular case.

For classical differential operators, the Weyl–Titchmarsh theory is extremely useful in the spectral analysis, which goes back to H. Weyl's work [**53**]. He initially classified singular second-order symmetric differential equations into two cases: the limit point case and the limit circle case, based on geometrical properties of a certain limiting set. In the limit circle case, the essential spectrum of the associated operator is empty. In addition, this classification is closely related to characterizations of self-adjoint extensions of the minimal operators generated by symmetric differential expressions. This work was followed and developed extensively and intensively and many good results have been obtained for differential and difference expressions including symmetric and non-symmetric cases (cf., e.g., [**6**–**8**, **10**, **13**, **23**, **28**, **36**–**38**, **42**, **43**, **47**–**49**, **52**, **54**]). Some limit point and limit circle criteria have been established for singular differential and difference expressions [**11**, **13**–**15**, **28**, **35**, **41**, **48**, **52**]. Sturm–Liouville differential equations with coefficients depending rationally on the spectral parameter attracted people's interest in the past because of their floating singularities (cf., e.g., [**1**, **5**, **20**]). Also, there is an analogue of the limit point and limit circle classification for this class of singular differential equations [**20**]. Furthermore, equations (1.6) satisfying certain conditions can also be classified into the limit point case and the limit circle case by transforming them into symmetric Hamiltonian systems [**45**, **46**]. Especially, a similar classification has been made for more general equations (1.6) with real coefficients by using classical Weyl's method [**21**].

It is known that (1.1_λ) can be transformed into (1.3) and (1.5) with floating singularities which depend on the value of λ. Equation (1.6) can also be transformed into two equations similar to (1.3) and (1.5) with floating singularities. Here, we point out that (1.6) is said to be singular at $t = a$ or $t = b$ if some of the coefficients of these two equations are singular at $t = a$ or $t = b$ rather than the floating singularities. Unlike classical differential operators, matrix differential operators with singular endpoints have their interesting and unexpected spectral properties. Their essential spectrum consists of two parts: a regular part and a singular part (cf. e.g., [**17**, **19**, **24**, **31**, **45**, **46**]). Since the second part appears due to singularities of coefficients at the endpoint, it is empty when the matrix differential expression is in the limit circle case (cf. [**45**, **46**]). Therefore, this classification is crucial for the study of spectral properties of the class of matrix differential equations. Inspired by the work of [**21**], we shall consider the classification of equations (1.1_λ) by using Weyl's method, and investigate spectral properties of them in the subsequent study. In this paper, the existence and uniqueness of initial value problem of equation (1.1_λ) are derived, and then the classification is obtained by selecting a suitable quasi-difference. Similarly to classical differential systems, it is proved that an equivalent characterization of this classification can also be given in terms of the number of linearly independent square summable solutions of equation (1.1_λ). The influence of off-diagonal coefficients $c(t)$ and $h(t)$ on this classification is illustrated by two examples. In particular, two limit point criteria are established in terms of coefficients of equation (1.1_λ).

The paper is organized as follows. In section 2, the Green's formula and the existence and uniqueness of initial value problem for equation (1.1_λ) are derived. In section 3, the classification is shown, and the equivalent characterization is given. Section 4 is devoted to the influence of off-diagonal coefficients on this classification. Section 5 gives two limit-point criteria.

2. Preliminaries

In this section, Green's formula for $\mathcal{L}$ or (1.1_λ) is obtained, and the existence and uniqueness of initial value problem for (1.1_λ) are derived.

First, let $l(\mathcal{I}) = \left\{y = \{y(t)\}_{t=a-1}^{+\infty} \subset \mathbb{C}^2\right\}$ and then we introduce the following space:

$$l^2(\mathcal{I}) := \left\{y \in l(\mathcal{I}) : \sum_{t\in\mathcal{I}} y^*(t)y(t) < +\infty\right\}$$

with the inner product $\langle y, z\rangle := \sum_{t\in\mathcal{I}} z^*(t)y(t)$, where z^* denotes the complex conjugate transpose of z. The induced norm is $\|y\| := \langle y, y\rangle^{1/2}$ for $y \in l^2(\mathcal{I})$. For $N \in \mathcal{I} \setminus \{a\}$, set $\mathcal{I}_N := \{t\}_{t=a}^{N}$,

$$l(\mathcal{I}_N) = \left\{y = \{y(t)\}_{t=a-1}^{N+1} \subset \mathbb{C}^2\right\},$$

and let the definition of $l^2(\mathcal{I}_N)$ be similar to that of $l^2(\mathcal{I})$ with $\mathcal{I}$ replaced by $\mathcal{I}_N$. By$\langle\cdot, \cdot\rangle_N$ and $\|\cdot\|_N$, we denote the inner product and its induced norm of $l^2(\mathcal{I}_N)$.

Next, for $y = (y_1,\, y_2)^{\mathrm{T}} \in l(\mathcal{I})$, the quasi-difference operator $y^{[1]}(t)$ is defined by

$$y^{[1]}(t) := p(t)\Delta y_1(t) + c(t)y_2(t),\ \ t \in \mathcal{I}'. \tag{2.1}$$

Further, the Lagrange bracket of $y = (y_1,\, y_2)^{\mathrm{T}}$ and $z = (z_1,\, z_2)^{\mathrm{T}}$ is defined by

$$[y(t), z(t)] := y_1(t+1)\overline{z}^{[1]}(t) - y^{[1]}(t)\overline{z}_1(t+1), \quad t \in \mathcal{I}'. \tag{2.2}$$

Then, Green's formula for $\mathcal{L}$ or (1.1_λ) can be given as follows.

Lemma 2.1. *For $y,\, z \in l(\mathcal{I}_N)$, it holds that*

$$\langle \mathcal{L}(y), z\rangle_N - \langle y, \mathcal{L}(z)\rangle_N = [y(t), z(t)]\Big|_{t=a-1}^{N}. \tag{2.3}$$

Proof. Using (2.1), we have

$$\begin{aligned}
&\langle \mathcal{L}(y), z\rangle_N - \langle y, \mathcal{L}(z)\rangle_N \\
&\quad = \sum_{t\in\mathcal{I}_N} \Big\{ (\overline{z}_1, \overline{z}_2) \begin{pmatrix} -\nabla\,(p\Delta y_1 + c y_2) + q y_1 + h y_2 \\ c\Delta y_1 + h y_1 + d y_2 \end{pmatrix}(t) \\
&\qquad - \Big(-\nabla\,(p\Delta \overline{z}_1 + c\overline{z}_2) + q\overline{z}_1 + h\overline{z}_2, c\Delta\overline{z}_1 + h\overline{z}_1 + d\overline{z}_2\Big) \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}(t)\Big\} \\
&\quad = \Big[p(t)\big(y_1(t+1)\Delta\overline{z}_1(t) - \overline{z}_1(t+1)\Delta y_1(t)\big) + c(t)\big(y_1(t+1)\overline{z}_2(t) \\
&\quad - y_2(t)\overline{z}_1(t+1)\big)\Big]\Big|_{t=a-1}^{N} \\
&\quad = \Big[y_1(t+1)\overline{z}^{[1]}(t) - y^{[1]}(t)\overline{z}_1(t+1)\Big]\Big|_{t=a-1}^{N} \\
&\quad = [y(t), z(t)]\Big|_{t=a-1}^{N}.
\end{aligned}$$

This completes the proof. □

The following lemma is a consequence of the Green's formula given by (2.3).

Lemma 2.2. *Let $\varphi(t,\, \lambda)$ be a solution of (1.1_λ) and $\psi(t,\, \mu)$ a solution of (1.1_μ). Then*

$$(\lambda - \overline{\mu}) \sum_{t\in\mathcal{I}_N} \psi^*(t,\mu)\varphi(t,\lambda) = [\varphi(t,\lambda), \psi(t,\mu)]\big|_{t=a-1}^{N} \tag{2.4}$$

holds for all $N \in \mathcal{I}$.

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Proof. Since $\mathcal{L}(\varphi) = \lambda\varphi$ and $\mathcal{L}(\psi) = \mu\psi$, we have

$$\langle \mathcal{L}(\varphi), \psi\rangle_N - \langle \varphi, \mathcal{L}(\psi)\rangle_N = (\lambda - \overline{\mu}) \sum_{t\in\mathcal{I}_N} \psi^*(t,\mu)\varphi(t,\lambda).$$

Then, (2.4) holds by Lemma 2.1. This completes the proof. □

Now, set $\sigma(f) := \{\lambda \in \mathbb{R} : \inf_{t\in\mathcal{I}'} |\lambda - f(t)| = 0\}$ and $\Omega(f) := \mathbb{C} \setminus \sigma(f)$ for a function $f(t)$, $t \in \mathcal{I}'$. If $\lambda \in \Omega(d)$, then $\widetilde{p}(t, \lambda)$ is well-defined on $\mathcal{I}'$. Further, if $y = (y_1, y_2)^{\mathrm{T}}$ is a solution of (1.1_λ) with $\lambda \in \Omega(d)$, then by (1.5) and (2.1), we get

$$y^{[1]}(t) = \widetilde{p}(t,\lambda)\Delta y_1(t) + \frac{h(t)c(t)}{\lambda - d(t)} y_1(t+1),\ t \in \mathcal{I}'. \tag{2.5}$$

LEMMA 2.3. *Let $\varphi = (\varphi_1, \varphi_2)^{\mathrm{T}}$ and $\psi = (\psi_1, \psi_2)^{\mathrm{T}}$ be solutions of (1.1_λ). Then, for $\lambda \in \Omega(d)$ and $t \in \mathcal{I}'$,*

$$[\psi(t,\lambda), \overline{\varphi}(t,\lambda)] = \widetilde{p}(t,\lambda)\Big(\psi_1(t+1,\lambda)\Delta\varphi_1(t,\lambda) - \varphi_1(t+1,\lambda)\Delta\psi_1(t,\lambda)\Big), \tag{2.6}$$

and further, $[\psi(t, \lambda), \overline{\varphi}(t, \lambda)]$ is a constant on $\mathcal{I}'$.

Proof. Let $\lambda \in \Omega(d)$ and $\varphi = (\varphi_1, \varphi_2)^{\mathrm{T}}$ and $\psi = (\psi_1, \psi_2)^{\mathrm{T}}$ be solutions of (1.1_λ). Then (1.3) and (2.5) hold for φ and ψ, respectively. Then, (2.6) can be easily obtained by (2.5). Moreover, using (1.3) and (2.6), one has

$$\begin{aligned}\nabla[\psi(t,\lambda), \overline{\varphi}(t,\lambda)] &= \psi_1(t,\lambda)\nabla\big(\widetilde{p}(t,\lambda)\Delta\varphi_1(t,\lambda)\big) - \varphi_1(t,\lambda)\nabla\big(\widetilde{p}(t,\lambda)\Delta\psi_1(t,\lambda)\big)\\ &= \psi_1(t,\lambda)(\widetilde{q}(t,\lambda) - \lambda)\varphi_1(t,\lambda)\\ &\quad - \varphi_1(t,\lambda)(\widetilde{q}(t,\lambda) - \lambda)\psi_1(t,\lambda) = 0, \quad t \in \mathcal{I},\end{aligned}$$

which implies that $[\psi(t, \lambda), \overline{\varphi}(t, \lambda)]$ is a constant on $\mathcal{I}'$. This completes the proof. □

For convenience, let

$$\mathcal{M}(t) := d(t) - \frac{c^2(t) - h(t)c(t)}{p(t)}, \quad t \in \mathcal{I}', \quad \Omega'(\mathcal{M}, d) := \mathbb{C}\backslash(\sigma(\mathcal{M}) \cup \sigma(d)).$$

The existence and uniqueness of initial value problems for (1.1_λ) are given below.

THEOREM 2.4. *Let $\lambda \in \Omega'(\mathcal{M}, d)$ and $c_1, c_2 \in \mathbb{C}$. Then the initial value problem*

$$(\mathcal{L} - \lambda)y = 0, \quad y_1(a) = c_1, \quad y^{[1]}(a-1) = c_2, \tag{2.7}$$

has a unique solution $y = y(t, \lambda)$ on $\mathcal{I}'$.

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Proof. First, equations (1.1_λ) can be transformed into the following system of three term recurrence relations:

$$\begin{cases} -p(t)y_1(t+1) + (h(t) - c(t))\, y_2(t) = \kappa_1(t, \lambda)\ t \in \mathcal{I}, \\ c(t)y_1(t+1) + (d(t) - \lambda)\, y_2(t) = \kappa_2(t),\ \ t \in \mathcal{I}', \end{cases} \tag{2.8}$$

where

$$\kappa_1(t, \lambda) = \Big(\lambda - p(t) - p(t-1) - q(t)\Big) y_1(t) + p(t-1)y_1(t-1) - c(t-1)y_2(t-1),$$

$$\kappa_2(t) = \big(c(t) - h(t)\big) y_1(t).$$

Now, let $y = (y_1,\, y_2)^{\mathrm{T}} \in l(\mathcal{I})$ satisfy (2.8) with $y_1(a) = c_1$ and $y^{[1]}(a-1) = c_2$. Then, from the second relation of (2.8) and (2.1), we get a system of linear equations about $y_1(a-1)$ and $y_2(a-1)$ as follows:

$$\begin{cases} \big(h(a-1) - c(a-1)\big) y_1(a-1) + \big(d(a-1) - \lambda\big) y_2(a-1) = -c(a-1)c_1, \\ p(a-1)y_1(a-1) - c(a-1)y_2(a-1) = p(a-1)c_1 - c_2. \end{cases} \tag{2.9}$$

The determinant of coefficients of (2.9) is equal to $p(a-1)(\lambda - \mathcal{M}(a-1))$ which is nonzero since $\lambda \in \Omega'(\mathcal{M},\, d)$. Therefore, (2.9) has a unique solution $(y_1(a-1), y_2(a-1))^{\mathrm{T}}$. Inserting this solution and $y_1(a) = c_1$ into (2.8) with $t = a$, we get a system of linear equations about $y_1(a+1)$ and $y_2(a)$, i.e.,

$$\begin{cases} -p(a)y_1(a+1) + (h(a) - c(a))\, y_2(a) = \kappa_1(a, \lambda), \\ c(a)y_1(a+1) + (d(a) - \lambda)\, y_2(a) = \kappa_2(a). \end{cases} \tag{2.10}$$

The determinant of coefficients of (2.10) is equal to $p(a)(\lambda - \mathcal{M}(a))$ which is nonzero since $\lambda \in \Omega'(\mathcal{M},\, d)$ again. Then (2.10) has a unique solution $(y_1(a+1),\, y_2(a))^{\mathrm{T}}$. By repeating the above process and by noting that $\lambda \in \Omega'(\mathcal{M},\, d)$, a unique solution $y(t)$ of (1.1_λ) can be obtained satisfying the initial value problem (2.7). This completes the proof. □

For $\lambda \in \Omega'(\mathcal{M},\, d)$, it follows from Theorem 2.4 that mapping $y \mapsto (y_1(a),\, y^{[1]}(a-1))^{\mathrm{T}} \in \mathbb{C}^2$ is bijective for solutions y of equation (1.1_λ). Hence, we have

Corollary 2.5. *For $\lambda \in \Omega'(\mathcal{M},\, d)$, the set of solutions of equation (1.1_λ) is a vector space of dimension* 2.

3. Classification of singular matrix difference equations of mixed order

It has been known that singular Sturm–Liouville differential and difference equations can be classified into the limit point case and the limit circle case, respectively, by the Weyl's method, i.e., in terms of geometrical properties of the limiting set of a sequence of nested Weyl's circles [**53**]. This work has been developed intensively, and especially, Hassi *et al* [**21**] founded that matrix differential equations (1.6) can also be classified with a similar method when the coefficients are real-valued. Motivated by the work given by [**21**], we shall give the classification for difference expressions $\mathcal{L}$ by constructing nested Weyl's circles. This section consists of two subsections.

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3.1. Weyl's circles and limit point and limit circle classification

In this subsection, we shall construct Weyl's circles for difference expressions $\mathcal{L}$ over finite intervals, which are nested and converge to a limiting set. Difference expressions $\mathcal{L}$ will be classified in terms of properties of the limiting set. We present it in detail for the convenience of the reader.

First, we consider $\mathcal{L}$ on the finite interval $\mathcal{I}'_N := \{t\}_{t=a-1}^{N+1}$, $N \in \mathcal{I} \setminus \{a\}$, satisfying the following boundary conditions:

$$\begin{cases} U_1(y) := y_1(a)\sin\alpha - y^{[1]}(a-1)\cos\alpha = 0, \\ U_2(y) := y_1(N+1)\cos\beta + y^{[1]}(N)\sin\beta = 0, \end{cases} \tag{3.1}$$

where $0 \leqslant \alpha, \beta < \pi$. By Theorem 2.4, for $\lambda \in \Omega'(\mathcal{M}, d)$, let $\varphi(t, \lambda) = (\varphi_1(t, \lambda), \varphi_2(t, \lambda))^{\mathrm{T}}$ and $\psi(t, \lambda) = (\psi_1(t, \lambda), \psi_2(t, \lambda))^{\mathrm{T}}$ be solutions of (1.1_λ) with the initial conditions:

$$\begin{aligned} &\varphi_1(a,\lambda) = \sin\alpha,\ \varphi^{[1]}(a-1,\lambda) = -\cos\alpha; \\ &\psi_1(a,\lambda) = \cos\alpha,\ \psi^{[1]}(a-1,\lambda) = \sin\alpha. \end{aligned} \tag{3.2}$$

From Lemma 2.3 and (3.2), we can get that $\widetilde{\varphi}(t, \lambda) := (\varphi_1(t+1, \lambda), \varphi^{[1]}(t, \lambda))^{\mathrm{T}}$ and $\widetilde{\psi}(t, \lambda) := (\psi_1(t+1, \lambda), \psi^{[1]}(t, \lambda))^{\mathrm{T}}$ are linearly independent on $\mathcal{I}'_N$. Then, we claim that $\varphi(t, \lambda)$ and $\psi(t, \lambda)$ are linearly independent on $\mathcal{I}'_N$. In fact, suppose on the contrary that $\varphi(t, \lambda)$ and $\psi(t, \lambda)$ are linearly dependent, i.e., $\varphi(t, \lambda) = k\psi(t, \lambda)$, $t \in \mathcal{I}'_N$, for some k. Then $\varphi^{[1]}(t, \lambda) = k\psi^{[1]}(t, \lambda)$ by (2.1), which yields that $\widetilde{\varphi}(t, \lambda)$ and $\widetilde{\psi}(t, \lambda)$ are linearly dependent on $\mathcal{I}'_N$, which is a contradiction. Hence, $\varphi(t, \lambda)$ and $\psi(t, \lambda)$ are linearly independent on $\mathcal{I}'_N$. Furthermore, the following result holds:

Lemma 3.1. *A number $\lambda \in \Omega'(\mathcal{M}, d)$ is an eigenvalue of boundary value problem (1.1_λ) with (3.1) if and only if $U_2(\psi(\cdot, \lambda)) = 0$.*

Proof. It is evident that $U_1(\psi(\cdot, \lambda)) = 0$ for $\lambda \in \Omega'(\mathcal{M}, d)$ by (3.2). Therefore, in the case that $U_2(\psi(\cdot, \lambda)) = 0$, this λ is an eigenvalue of (1.1_λ) with (3.1) and ψ is the associated eigenvector. Hence, the sufficiency is proved.

Now, we show the necessity. Suppose that $\lambda \in \Omega'(\mathcal{M}, d)$ is an eigenvalue of (1.1_λ) with (3.1) and $y = (y_1, y_2)^{\mathrm{T}}$ is the associated eigenvector. Then y can be expressed as $y(t, \lambda) = c_1\varphi(t, \lambda) + c_2\psi(t, \lambda)$, $t \in \mathcal{I}'$, $c_1, c_2 \in \mathbb{C}$, since it is a solution of (1.1_λ). Inserting this expression of y into $U_1(y) = 0$ and using (3.2), we get that $c_1 = 0$. Furthermore, inserting this expression of y with $c_1 = 0$ into $U_2(y) = 0$, we have $c_2 U_2(\psi(\cdot, \lambda)) = 0$. Since y is a nontrivial solution of (1.1_λ), we have $c_2 \neq 0$. Hence, it follows from the above relation that $U_2(\psi(\cdot, \lambda)) = 0$. This completes the proof. □

From the proof of Theorem 2.4, we get that $\psi_1(N+1, \lambda)$ and $\psi_2(N+1, \lambda)$ are rational fractions of λ for $\lambda \in \Omega'(\mathcal{M}, d)$, respectively. So is $\psi^{[1]}(N+1, \lambda)$ by (2.1). Therefore, from Lemma 3.1 and definition of $U_2(\psi(\cdot, \lambda))$, the boundary value problem (1.1_λ) with (3.1) has eigenvalues and the number of all the eigenvalues is finite. In addition, for $y = (y_1, y_2)^{\mathrm{T}}$ and $z = (z_1, z_2)^{\mathrm{T}} \in l(\mathcal{I}_N)$ satisfying (3.1), we

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get from (3.1) that

$$[y(a-1), z(a-1)] = [y(N), z(N)] = 0.$$

Then, it follows that $\langle \mathcal{L}(y), z\rangle_N = \langle y, \mathcal{L}(z)\rangle_N$ by Lemma 2.1, which implies that boundary value problem (1.1_λ) with (3.1) is symmetric in the space $l^2(\mathcal{I}_N)$. Hence, all eigenvalues of boundary value problem (1.1_λ) with (3.1) are real numbers.

For $\lambda \in \Omega'(\mathcal{M}, d)$, let

$$A(\lambda, N) = \varphi_1(N+1, \lambda), \quad B(\lambda, N) = \varphi^{[1]}(N, \lambda),$$
$$C(\lambda, N) = \psi_1(N+1, \lambda), \quad D(\lambda, N) = \psi^{[1]}(N, \lambda).$$

For simplicity, $A(\lambda, N)$, $B(\lambda, N)$, $C(\lambda, N)$, $D(\lambda, N)$ are written as A, B, C, D. Then, it is evident that

$$A\bar{D} - B\bar{C} = [\varphi(N,\lambda), \psi(N,\lambda)], \quad C\bar{D} - \bar{C}D = [\psi(N,\lambda), \psi(N,\lambda)]. \tag{3.3}$$

By (3.3) and Lemma 2.2, we have

$$C\bar{D} - \bar{C}D = 2i\mathrm{Im}\,\lambda \sum_{t\in\mathcal{I}_N} \psi^*(t,\lambda)\psi(t,\lambda), \quad i = \sqrt{-1}. \tag{3.4}$$

In addition, by Lemma 2.3 and (3.2), we have

$$AD - BC = [\varphi(N,\lambda), \overline{\psi}(N,\lambda)] = 1. \tag{3.5}$$

Now, let $\lambda \in \mathbb{C}\setminus\mathbb{R}$. Then $\lambda \in \Omega'(\mathcal{M}, d)$ since $\mathcal{M}$ and d are real-valued. Hence, λ is not an eigenvalue of (1.1_λ) with (3.1) since $\mathrm{Im}\,\lambda \neq 0$. Therefore, $U_2(\psi(\cdot, \lambda)) \neq 0$ by Lemma 3.1 since $U_1(\psi(\cdot, \lambda)) = 0$. Next, let $\chi(t, \lambda, m) = (\chi_1(t, \lambda, m), \chi_2(t, \lambda, m))^{\mathrm{T}}$ be given by

$$\chi(t,\lambda,m) := \varphi(t,\lambda) + m\psi(t,\lambda), \quad t \in \mathcal{I}'. \tag{3.6}$$

Then $\chi(\cdot, \lambda, m)$ is a solution of equation (1.1_λ). Let $\chi(\cdot, \lambda, m)$ satisfy the boundary condition $U_2(\chi(\cdot, \lambda, m)) = 0$. Then this gives rise to the following formula with m depending on z, λ, and N,

$$m(z,\lambda,N) = -\frac{U_2(\varphi)}{U_2(\psi)} = -\frac{A(\lambda,N)z + B(\lambda,N)}{C(\lambda,N)z + D(\lambda,N)}, \tag{3.7}$$

where $z = \cot\beta$, $0 \leqslant \beta < \pi$. For simplicity, $m(z, \lambda, N)$ is written as m in what follows. It is noted that (3.7) describes a circle, denoted by $C_N(\lambda)$, in the complex plane as z varies. We shall give the characteristics of the circle $C_N(\lambda)$ below.

THEOREM 3.2. *The center $O_N(\lambda)$ and radius $r_N(\lambda)$ of circle $C_N(\lambda)$ are respectively given by*

$$O_N(\lambda) = -\frac{[\varphi(N,\lambda), \psi(N,\lambda)]}{[\psi(N,\lambda), \psi(N,\lambda)]}, \quad r_N(\lambda) = \frac{1}{\left|[\psi(N,\lambda), \psi(N,\lambda)]\right|}, \tag{3.8}$$

and further, the equation and interior of circle $C_N(\lambda)$ are respectively given by

$$\sum_{t\in\mathcal{I}_N} \chi^*(t,\lambda,m)\chi(t,\lambda,m) = \frac{\mathrm{Im}\,m}{\mathrm{Im}\,\lambda}, \quad \sum_{t\in\mathcal{I}_N} \chi^*(t,\lambda,m)\chi(t,\lambda,m) < \frac{\mathrm{Im}\,m}{\mathrm{Im}\,\lambda}. \tag{3.9}$$

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Proof. Let $\chi(\cdot,\, \lambda,\, m)$ be defined by (3.6) satisfying $U_2(\chi(\cdot,\, \lambda,\, m)) = 0$. Then, by the fact that $\cos\beta$ and $\sin\beta$ are real numbers, it can be concluded that $\chi_1(N+1,\, \lambda,\, m)\overline{\chi}^{[1]}(N,\, \lambda,\, m)$ is a real number. Since it holds that

$$[\chi(N,\lambda,m),\chi(N,\lambda,m)] = 2i\,\mathrm{Im}\left\{\chi_1(N+1,\lambda,m)\overline{\chi}^{[1]}(N,\lambda,m)\right\},$$

we have

$$[\chi(N,\lambda,m),\chi(N,\lambda,m)] = 0. \tag{3.10}$$

Conversely, if (3.10) holds, then $\chi_1(N+1,\, \lambda,\, m)\overline{\chi}^{[1]}(N,\, \lambda,\, m)$ is a real number. Hence, from (3.10) and the fact that $\chi_1(N+1,\, \lambda,\, m)\overline{\chi}^{[1]}(N,\, \lambda,\, m)$ is a real number, it can be verified that there exists $\beta \in [0,\, \pi)$ such that $U_2(\chi(\cdot,\, \lambda,\, m)) = 0$.

Note that $C\bar{D} - \bar{C}D \neq 0$ by (3.4). Then, by using (3.3) and the definition of $\chi(t,\, \lambda,\, m)$, we have

$$\frac{[\chi(N,\lambda,m),\chi(N,\lambda,m)]}{[\psi(N,\lambda),\psi(N,\lambda)]} = m\bar{m} - \frac{\bar{A}D - \bar{B}C}{C\bar{D} - \bar{C}D}m + \frac{A\bar{D} - B\bar{C}}{C\bar{D} - \bar{C}D}\bar{m} - \frac{\bar{A}B - \bar{B}A}{C\bar{D} - \bar{C}D}. \tag{3.11}$$

As a result, we get from (3.10) that

$$m\bar{m} - \frac{\bar{A}D - \bar{B}C}{C\bar{D} - \bar{C}D}m + \frac{A\bar{D} - B\bar{C}}{C\bar{D} - \bar{C}D}\bar{m} - \frac{\bar{A}B - \bar{B}A}{C\bar{D} - \bar{C}D} = 0. \tag{3.12}$$

Equation (3.12) gives a clear expression of circle $C_N(\lambda)$. From (3.12) and (3.3), the center $O_N(\lambda)$ of circle $C_N(\lambda)$ is given by

$$O_N(\lambda) = \frac{B\bar{C} - A\bar{D}}{C\bar{D} - \bar{C}D} = -\frac{[\varphi(N,\lambda),\psi(N,\lambda)]}{[\psi(N,\lambda),\psi(N,\lambda)]}.$$

Further, it follows from (3.5) and (3.12) that

$$r_N(\lambda)^2 = |O_N(\lambda)|^2 + \frac{\bar{A}B - \bar{B}A}{C\bar{D} - \bar{C}D} = \left|\frac{AD - BC}{C\bar{D} - \bar{C}D}\right|^2 = \frac{1}{|[\psi(N,\lambda),\psi(N,\lambda)]|^2}.$$

This completes the proof of (3.8).

In addition, it is easy to verify that

$$[\chi(a-1,\lambda,m),\chi(a-1,\lambda,m)] = -2i\,\mathrm{Im}\, m.$$

Thus, it follows from Lemma 2.2 that

$$\begin{aligned}[\chi(N,\lambda,m),\chi(N,\lambda,m)] &= 2i\,\mathrm{Im}\,\lambda \sum_{t\in\mathcal{I}_N}\chi^*(t,\lambda,m)\chi(t,\lambda,m) \\ &\quad + [\chi(a-1,\lambda,m),\chi(a-1,\lambda,m)] \\ &= 2i\Big(\mathrm{Im}\,\lambda \sum_{t\in\mathcal{I}_N}\chi^*(t,\lambda,m)\chi(t,\lambda,m) - \mathrm{Im}\, m\Big).\end{aligned} \tag{3.13}$$

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Then, from (3.4) and (3.13), we get

$$\frac{[\chi(N,\lambda,m),\chi(N,\lambda,m)]}{[\psi(N,\lambda),\psi(N,\lambda)]} = \frac{\sum_{t\in\mathcal{I}_N}\chi^*(t,\lambda,m)\chi(t,\lambda,m) - \dfrac{\operatorname{Im} m}{\operatorname{Im}\lambda}}{\sum_{t\in\mathcal{I}_N}\psi^*(t,\lambda)\psi(t,\lambda)}. \tag{3.14}$$

It is noted that (3.12) is the equation of the circle $C_N(\lambda)$. By (3.11) and (3.14), we get that m is on the circle $C_N(\lambda)$ if and only if the first formula of (3.9) holds. Moreover, if m is inside $C_N(\lambda)$, then it holds that

$$\frac{[\chi(N,\lambda,m),\chi(N,\lambda,m)]}{[\psi(N,\lambda),\psi(N,\lambda)]} < 0.$$

Hence, by (3.14), m is inside the circle $C_N(\lambda)$ if and only if the second formula of (3.9) holds. This completes the proof. □

Corollary 3.3. *If $N < N'$, then $C_{N'}(\lambda)$ is inside the circle $C_N(\lambda)$.*

Proof. Let $m \in C_{N'}(\lambda)$. Then, it is evident that

$$\sum_{t\in\mathcal{I}_N}\chi^*(t,\lambda,m)\chi(t,\lambda,m) < \sum_{t\in\mathcal{I}_{N'}}\chi^*(t,\lambda,m)\chi(t,\lambda,m) = \frac{\operatorname{Im} m}{\operatorname{Im}\lambda}. \tag{3.15}$$

Therefore, m is inside $C_N(\lambda)$ by Theorem 3.2 which implies that $C_{N'}(\lambda)$ is inside $C_N(\lambda)$. This completes the proof. □

By Corollary 3.3, the sequence of circles $\{C_N(\lambda)\}$ converges as $N \to +\infty$ for $\lambda \in \mathbb{C}\setminus\mathbb{R}$. The limiting set is either a circle or a point. Correspondingly, the classification of $\mathcal{L}$ can be given as follows.

Definition 3.4. *If $\{C_N(\lambda)\}$ converges to a circle, then $\mathcal{L}$ is called to be in the limit circle case (LCC) at $t = \infty$ for $\lambda \in \mathbb{C}\setminus\mathbb{R}$; and if $\{C_N(\lambda)\}$ converges to a point, then $\mathcal{L}$ is called to be in the limit point case (LPC) at $t = \infty$ for $\lambda \in \mathbb{C}\setminus\mathbb{R}$.*

In fact, $\mathcal{L}$ is in the LCC or LPC at $t = \infty$ for some $\lambda \in \mathbb{C}\setminus\mathbb{R}$, hence for all $\lambda \in \mathbb{C}\setminus\mathbb{R}$, see Theorem 3.5 below.

3.2. Relationships between square summable solutions and the classification

In this subsection, we shall derive an equivalent characterization of the classification in terms of the number of linearly independent solutions of (1.1_λ) in $l^2(\mathcal{I})$. Here, we remark that solutions of (1.1_λ) in $l^2(\mathcal{I})$ are also called square summable solutions of (1.1_λ). The following is the main result of this section:

Theorem 3.5. *If there exists $\lambda_0 \in \Omega'(\mathcal{M}, d)$ such that (1.1_{λ_0}) has two linearly independent solutions in $l^2(\mathcal{I})$, then $\mathcal{L}$ is in the LCC at $t = \infty$ for all $\lambda \in \mathbb{C}\setminus\mathbb{R}$. Otherwise, $\mathcal{L}$ is in the LPC at $t = \infty$ for all $\lambda \in \mathbb{C}\setminus\mathbb{R}$.*

Before proving Theorem 3.5, we need derive three results in what follows. First, we can get the following result:

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Lemma 3.6. *If $\mathcal{L}$ is in the LCC at $t=\infty$ for $\lambda\in\mathbb{C}\backslash\mathbb{R}$, then (1.1_λ) has exactly two linearly independent solutions in $l^2(\mathcal{I})$, and if $\mathcal{L}$ is in the LPC at $t=\infty$ for $\lambda\in\mathbb{C}\backslash\mathbb{R}$, then (1.1_λ) has exactly one linearly independent solutions in $l^2(\mathcal{I})$.*

Proof. If $\mathcal{L}$ is in the LCC at $t=\infty$ for $\lambda\in\mathbb{C}\backslash\mathbb{R}$, then $\{C_N(\lambda)\}$ converges to a circle. We take a point of this circle as m. If $\mathcal{L}$ is in the LPC at $t=\infty$ for $\lambda\in\mathbb{C}\backslash\mathbb{R}$, then $\{C_N(\lambda)\}$ converges to a point. In this case, we take this point as m. Then m is inside $C_N(\lambda)$ for all $N>a$ by Corollary 3.3. Let $\chi(t,\,\lambda,\,m)$ be given by (3.6) with this m. Then by the second formula of (3.9), we have

$$\sum_{t\in\mathcal{I}_N}\chi^*(t,\lambda,m)\chi(t,\lambda,m)<\frac{\operatorname{Im}m}{\operatorname{Im}\lambda},\quad N>a,$$

which implies that $\chi(\cdot,\,\lambda,\,m)\in l^2(\mathcal{I})$.

Furthermore, if $\{C_N(\lambda)\}$ converges to a circle, then $\{r_N(\lambda)\}$ converges to a positive number. Then, from (3.3), (3.4), and (3.8), we get $\psi(\cdot,\,\lambda)\in l^2(\mathcal{I})$. Therefore (1.1_λ) has two linearly independent solutions in $l^2(\mathcal{I})$ since $\psi(\cdot,\,\lambda)$ and $\chi(\cdot,\,\lambda,\,m)$ are linearly independent on $\mathcal{I}$. If $\{C_N(\lambda)\}$ converges to a point, then $\{r_N(\lambda)\}$ converges to 0. From (3.4) and (3.8), we get that $\psi(\cdot,\,\lambda)\notin l^2(\mathcal{I})$. Hence, $\chi(\cdot,\,\lambda,\,m)$ is the only linearly independent solution of (1.1_λ) in $l^2(\mathcal{I})$. This completes the proof. □

Lemma 3.7. *Let $\lambda_0,\,\lambda\in\Omega'(\mathcal{M},\,d)$ and $\varphi(t,\,\lambda_0)=(\varphi_1(t,\,\lambda_0),\,\varphi_2(t,\,\lambda_0))^{\mathrm{T}}$ and $\psi(t,\,\lambda_0)=(\psi_1(t,\,\lambda_0),\,\psi_2(t,\,\lambda_0))^{\mathrm{T}}$ be linearly independent solutions of (1.1_{λ_0}) satisfying initial conditions (3.2). Then for a solution $z=(z_1,\,z_2)^{\mathrm{T}}$ of (1.1_λ), there exist two constants k_1 and k_2 independently of t, i.e., only depending on $N\in\mathcal{I}$, λ, and λ_0, such that for $t>N+2$,*

$$z_1(t)=k_1\psi_1(t)+k_2\varphi_1(t)+(\lambda_0-\lambda)\sum_{s=N+1}^{t-1}\Big(\psi_1(t)\varphi^{\mathrm{T}}(s)-\varphi_1(t)\psi^{\mathrm{T}}(s)\Big)z(s),\tag{3.16}$$

and

$$\begin{aligned}z_2(t)=\frac{\lambda_0-\mathcal{M}(t)}{\lambda-\mathcal{M}(t)}\Big\{&k_1\psi_2(t)+k_2\varphi_2(t)\\&+(\lambda_0-\lambda)\sum_{s=N+1}^{t}\Big(\psi_2(t)\varphi^{\mathrm{T}}(s)-\varphi_2(t)\psi^{\mathrm{T}}(s)\Big)z(s)\Big\}.\end{aligned}\tag{3.17}$$

Proof. Let $\lambda_0,\,\lambda\in\Omega'(\mathcal{M},\,d)$ and $z=(z_1,\,z_2)^{\mathrm{T}}$ be a solution of (1.1_λ), and for $t\in\mathcal{I}'$ set

$$\begin{aligned}A(t)&:=\widetilde{p}(t,\lambda_0)\left[z_1(t+1,\lambda)\varphi_1(t,\lambda_0)-z_1(t,\lambda)\varphi_1(t+1,\lambda_0)\right],\\B(t)&:=\widetilde{p}(t,\lambda_0)\left[z_1(t+1,\lambda)\psi_1(t,\lambda_0)-z_1(t,\lambda)\psi_1(t+1,\lambda_0)\right].\end{aligned}$$

Then, we claim that for $t\in\mathcal{I}$,

$$z_1(t,\lambda)=A(t-1)\psi_1(t,\lambda_0)-B(t-1)\varphi_1(t,\lambda_0),\tag{3.18}$$

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$$z_2(t,\lambda)=\frac{\lambda_0-d(t)}{\lambda-d(t)}\Big(A(t)\psi_2(t,\lambda_0)-B(t)\varphi_2(t,\lambda_0)\Big). \tag{3.19}$$

In fact, by Lemma 2.3 and the initial conditions (3.2), one has

$$\widetilde{p}(t,\lambda_0)\left[\psi_1(t+1,\lambda_0)\varphi_1(t,\lambda_0)-\psi_1(t,\lambda_0)\varphi_1(t+1,\lambda_0)\right]=1,\ t\in\mathcal{I}', \tag{3.20}$$

and thus, by the definitions of $A(t)$ and $B(t)$, it is easy to verify that

$$z_1(t,\lambda)=A(t)\psi_1(t,\lambda_0)-B(t)\varphi_1(t,\lambda_0),\quad t\in\mathcal{I}'. \tag{3.21}$$

Since z is a solution of (1.1_λ), we have (1.3) and (1.5) hold for z. From (1.3) and (1.5) for z and by (1.4), it can be derived that

$$\begin{aligned}
&-\nabla\left(\widetilde{p}(t,\lambda_0)\Delta z_1(t,\lambda)\right)+\left(\tilde{q}\,(t,\lambda_0)-\lambda_0\right)z_1(t,\lambda)\\
&\quad=\nabla\left[\left(\widetilde{p}(t,\lambda)-\widetilde{p}(t,\lambda_0)\right)\Delta z_1(t,\lambda)\right]+\left(\tilde{q}\,(t,\lambda_0)-\tilde{q}\,(t,\lambda)\right)z_1(t,\lambda)+(\lambda-\lambda_0)z_1(t,\lambda)\\
&\quad=(\lambda_0-\lambda)\left\{\nabla\left[\frac{c(t)}{\lambda_0-d(t)}z_2(t,\lambda)-\frac{h(t)c(t)z_1(t+1,\lambda)}{(\lambda_0-d(t))(\lambda-d(t))}\right]\right.\\
&\qquad\left.-\frac{h^2(t)z_1(t,\lambda)}{(\lambda_0-d(t))(\lambda-d(t))}+\nabla\left(\frac{h(t)c(t)}{(\lambda_0-d(t))(\lambda-d(t))}\right)z_1(t,\lambda)-z_1(t,\lambda)\right\}\\
&\quad=(\lambda_0-\lambda)\left[\nabla\left(\frac{c(t)}{\lambda_0-d(t)}z_2(t,\lambda)\right)-\frac{h(t)}{\lambda_0-d(t)}z_2(t,\lambda)-z_1(t,\lambda)\right],\quad t\in\mathcal{I}.
\end{aligned} \tag{3.22}$$

In addition, (1.3) holds for φ_1, i.e.,

$$-\nabla\left(\widetilde{p}(t,\lambda_0)\Delta\varphi_1(t,\lambda_0)\right)+\left(\tilde{q}\,(t,\lambda_0)-\lambda_0\right)\varphi_1(t,\lambda_0)=0,\quad t\in\mathcal{I}. \tag{3.23}$$

Here, we remark that similarly to those in (3.16) and (3.17), we omit λ and λ_0 for simplicity, e.g., write $\varphi_1(t,\lambda_0)$ as $\varphi_1(t)$ and $z_1(t,\lambda)$ as $z_1(t)$, in what follows. Multiplying both sides of (3.22) by $-\varphi_1(t)$ and (3.23) by $z_1(t)$, and adding them give that

$$\nabla A(t)=-\left(\lambda_0-\lambda\right)\left[\nabla\left(\frac{c(t)}{\lambda_0-d(t)}z_2(t)\right)-\frac{h(t)}{\lambda_0-d(t)}z_2(t)-z_1(t)\right]\varphi_1(t),\quad t\in\mathcal{I}. \tag{3.24}$$

With a similar argument to that of (3.24), we have

$$\nabla B(t)=-\left(\lambda_0-\lambda\right)\left[\nabla\left(\frac{c(t)}{\lambda_0-d(t)}z_2(t)\right)-\frac{h(t)}{\lambda_0-d(t)}z_2(t)-z_1(t)\right]\psi_1(t),\quad t\in\mathcal{I}. \tag{3.25}$$

Multiplying both sides of (3.24) by $-\psi_1(t)$ and (3.25) by $\varphi_1(t)$, and adding them give that

$$A(t)\psi_1(t)-B(t)\varphi_1(t)=A(t-1)\psi_1(t)-B(t-1)\varphi_1(t),\quad t\in\mathcal{I}, \tag{3.26}$$

which yields that (3.18) holds by (3.21). From (3.18) and (3.26), it can be obtained that

$$\Delta z_1(t)=A(t)\Delta\psi_1(t)-B(t)\Delta\varphi_1(t),\quad t\in\mathcal{I}. \tag{3.27}$$

Then, from (1.5) for z, φ, and ψ, respectively, (3.21), and (3.27), we get that (3.19) holds.

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Next, for $t > N+2$, summing up (3.24) from $N+1$ to t gives

$$\begin{aligned} A(t) &= A(N) - (\lambda_0 - \lambda) \\ &\quad \times \sum_{s=N+1}^{t} \left[\nabla \left(\frac{c(s)}{\lambda_0 - d(s)} z_2(s) \right) - \frac{h(s)}{\lambda_0 - d(s)} z_2(s) - z_1(s) \right] \varphi_1(s) \\ &= A(N) - (\lambda_0 - \lambda) \left[L(t) - L(N) - \sum_{s=N+1}^{t} \big(z_2(s)\varphi_2(s) + z_1(s)\varphi_1(s) \big) \right], \\ &= A(N) - (\lambda_0 - \lambda) \Big(L(t) - L(N) - \sum_{s=N+1}^{t} \varphi^{\mathrm{T}}(s) z(s) \Big), \end{aligned} \tag{3.28}$$

where $L(s) := \dfrac{c(s)}{\lambda_0 - d(s)} z_2(s)\varphi_1(s+1)$ for all $s \in \mathcal{I}$. Similarly, for $t > N+2$, we get

$$B(t) = B(N) - (\lambda_0 - \lambda) \Big(K(t) - K(N) - \sum_{s=N+1}^{t} \psi^{\mathrm{T}}(s) z(s) \Big), \tag{3.29}$$

where $K(s) := \dfrac{c(s)}{\lambda_0 - d(s)} z_2(s)\psi_1(s+1)$ for all $s \in \mathcal{I}$. On the other hand, we get that

$$L(t-1)\psi_1(t) - K(t-1)\varphi_1(t) = 0, \ \ t > N+2, \tag{3.30}$$

and from (1.5) and (3.20) that

$$L(t)\psi_2(t) - K(t)\varphi_2(t) = \frac{c^2(t) - h(t)c(t)}{(\lambda_0 - d(t))^2 \widetilde{p}(t, \lambda_0)} z_2(t), \quad t > N+2. \tag{3.31}$$

Then, inserting (3.28)-(3.30) into (3.18), we get that there exist two constants k_1 and k_2 depending on N, λ, and λ_0, such that (3.16) holds for $t > N+2$. Similarly, inserting (3.28), (3.29), and (3.31) into (3.19), we get

$$\begin{aligned} z_2(t) = \frac{\lambda_0 - d(t)}{\lambda - d(t)} \Big\{ & k_1\psi_2(t) + k_2\varphi_2(t) + (\lambda_0 - \lambda) \Big[\frac{h(t)c(t) - c^2(t)}{(\lambda_0 - d(t))^2 \widetilde{p}(t, \lambda_0)} z_2(t) \\ & + \sum_{s=N+1}^{t} \Big(\psi_2(t)\varphi^{\mathrm{T}}(s) - \varphi_2(t)\psi^{\mathrm{T}}(s) \Big) z(s) \Big] \Big\}, \quad t > N+2, \end{aligned} \tag{3.32}$$

where k_1 and k_2 are the same as those in (3.16). It can be easily verified that

$$1 - \frac{(\lambda_0 - \lambda)(h(t)c(t) - c^2(t))}{(\lambda - d(t))(\lambda_0 - d(t))\widetilde{p}(t, \lambda_0)} = \frac{\lambda_0 - d(t)}{\lambda - d(t)} \frac{\lambda - \mathcal{M}(t)}{\lambda_0 - \mathcal{M}(t)},$$

which, together with (3.32), yields that (3.17) holds for $t > N+2$. This completes the proof. □

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LEMMA 3.8. *If there exists* $\lambda_0 \in \Omega'(\mathcal{M}, d)$ *such that* (1.1_{λ_0}) *has two linearly independent solutions in* $l^2(\mathcal{I})$, *then it is true for all* $\lambda \in \Omega'(\mathcal{M}, d)$.

Proof. Suppose that $\varphi = (\varphi_1, \varphi_2)^{\mathrm{T}}$ and $\psi = (\psi_1, \psi_2)^{\mathrm{T}}$ are linearly independent solutions of (1.1_{λ_0}) in $l^2(\mathcal{I})$ for some $\lambda_0 \in \Omega'(\mathcal{M}, d)$ by the assumption. If $z = (z_1, z_2)^{\mathrm{T}}$ is a solution of (1.1_λ) with $\lambda \in \Omega'(\mathcal{M}, d)$, then (3.16) and (3.17) hold. Applying the Cauchy–Schwarz inequality, we get from (3.16) that

$$|z_1(t)| \leqslant (|\varphi_1(t)| + |\psi_1(t)|)\Big[(|k_1| + |k_2|) + |\lambda - \lambda_0|\,(\|\varphi\| + \|\psi\|)\|z\|_{N+1}^{t-1}\Big],$$

where $\|z\|_{N+1}^{t-1} = \big(\sum_{s=N+1}^{t-1} z^*(s)z(s)\big)^{1/2}$. Then, from the above relation we get that there exists $K_1 > 0$ such that for $\tau > N_0 > N+2$,

$$\|z_1\|_{N_0}^{\tau} \leqslant K_1\gamma_{N_0}\Big[1 + |\lambda - \lambda_0|\left(\|z\|_N^{N_0} + \|z\|_{N_0}^{\tau}\right)\Big], \tag{3.33}$$

where $\|z_1\|_{N_0}^{\tau} = \big(\sum_{s=N_0}^{\tau} |z_1(s)|^2\big)^{1/2}$ and

$$\gamma_{N_0} = \left(\sum_{t=N_0}^{\infty} \varphi^*(t)\varphi(t)\right)^{1/2} + \left(\sum_{t=N_0}^{\infty} \psi^*(t)\psi(t)\right)^{1/2}.$$

Furthermore, since $\lambda \in \Omega'(\mathcal{M}, d)$, we get $\inf_{t\in\mathcal{I}'}\Big|\lambda - \mathcal{M}(t)\Big| > 0$, and thus

$$\left|\frac{\lambda_0 - \mathcal{M}(t)}{\lambda - \mathcal{M}(t)}\right| = 1 + \frac{|\lambda_0 - \lambda|}{\inf_{t\in\mathcal{I}'}\Big|\lambda - \mathcal{M}(t)\Big|} < \infty. \tag{3.34}$$

Similarly, we can get from (3.17) and (3.34) that there exists $K_2 > 0$ such that for $\tau > N_0 > N+2$,

$$\|z_2\|_{N_0}^{\tau} \leqslant K_2\gamma_{N_0}\Big[1 + |\lambda - \lambda_0|\left(\|z\|_N^{N_0} + \|z\|_{N_0}^{\tau}\right)\Big]. \tag{3.35}$$

Since $\varphi, \psi \in l^2(\mathcal{I})$, we have $\gamma_{N_0} \to 0$ as $N_0 \to \infty$. Thus, letting $K_0 := \max\{K_1, K_2\}$, we can choose sufficiently large N_0 satisfying $K_0\,|\lambda - \lambda_0|\,\gamma_{N_0} \leqslant 1/4$. Then, from (3.33) and (3.35), we have

$$\|z_j\|_{N_0}^{\tau} \leqslant K_0\gamma_{N_0}\left(1 + |\lambda - \lambda_0|\|z\|_N^{N_0}\right) + \frac{1}{4}\|z\|_{N_0}^{\tau},\ j = 1, 2, \tag{3.36}$$

which implies that $z \in l^2(\mathcal{I})$. This completes the proof. □

Finally, we prove Theorem 3.5.

Proof of Theorem 3.5. It is noted that $\mathbb{C}\setminus\mathbb{R} \subset \Omega'(\mathcal{M}, d)$ since the coefficients of (1.1_λ) are real-valued. Suppose that there exists $\lambda_0 \in \Omega'(\mathcal{M}, d)$ such that (1.1_{λ_0}) has two linearly independent solutions in $l^2(\mathcal{I})$. Then it is true for all $\lambda \in \mathbb{C}\setminus\mathbb{R}$

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by Lemma 3.8, which implies that $\mathcal{L}$ is in the LCC at $t=\infty$ for all $\lambda\in\mathbb{C}\setminus\mathbb{R}$ by Lemma 3.6. Otherwise, there exists at most one linearly independent solutions of (1.1_λ) in $l^2(\mathcal{I})$ for each $\lambda\in\Omega'(\mathcal{M}, d)$. Then $\mathcal{L}$ is in the LPC at $t=\infty$ for all $\lambda\in\mathbb{C}\setminus\mathbb{R}$ by Lemma 3.6. This completes the proof. □

Remark 3.9. (1) From Theorem 3.5, the difference expression $\mathcal{L}$ is in the LCC or LPC at $t=\infty$ for some $\lambda\in\mathbb{C}\setminus\mathbb{R}$, hence for all $\lambda\in\mathbb{C}\setminus\mathbb{R}$. The classification of $\mathcal{L}$ is independent of $\lambda\in\mathbb{C}\setminus\mathbb{R}$. Then, a simpler expression of the classification of $\mathcal{L}$ can be given as:
If $\{C_N(\lambda)\}$ converges to a circle for $\lambda\in\mathbb{C}\setminus\mathbb{R}$, then $\mathcal{L}$ is in LCC at $t=\infty$; and if $\{C_N(\lambda)\}$ converges to a point for $\lambda\in\mathbb{C}\setminus\mathbb{R}$, then $\mathcal{L}$ is in LPC at $t=\infty$.

(2) It is noted that equations (1.1_λ) contain (1.2) as their special case. Therefore, Definition 3.4 and Theorem 3.5 are also applied to (1.2) which is useful in the next section. In fact, Jirari [**28**] has considered singular Sturm–Liouville difference equations $\tau y_1=\lambda w y_1$ on $\mathcal{I}$, where τ is given by (1.2) and $w>0$ is a weight function. Similar classification and result to those given by Definition 3.4 and Theorem 3.5 were obtained for $\tau y_1=\lambda w y_1$ on $\mathcal{I}$ in [**28**]. Definition 3.4 and Theorem 3.5 for (1.2) are their special case of $w\equiv 1$.

4. On perturbations of matrix difference equations

The difference expression $\mathcal{L}$ can be interpreted as

$$\mathcal{L}=\mathcal{L}^{(0)}+\mathcal{L}^{(1)}, \tag{4.1}$$

where

$$\mathcal{L}^{(0)}\begin{pmatrix} y_1\\ y_2\end{pmatrix}(t):=\begin{cases}\operatorname{diag}\{-\nabla p\Delta+q, d\}\begin{pmatrix} y_1\\ y_2\end{pmatrix}(t),\ t\in\mathcal{I},\\ \operatorname{diag}\{0, d\}\begin{pmatrix} y_1\\ y_2\end{pmatrix}(t),\ t=a-1,\end{cases}$$

$$\mathcal{L}^{(1)}\begin{pmatrix} y_1\\ y_2\end{pmatrix}(t):=\begin{cases}\begin{pmatrix} 0 & -\nabla c+h\\ c\Delta+h & 0\end{pmatrix}\begin{pmatrix} y_1\\ y_2\end{pmatrix}(t),\ t\in\mathcal{I},\\ \begin{pmatrix} 0 & 0\\ c\Delta+h & 0\end{pmatrix}\begin{pmatrix} y_1\\ y_2\end{pmatrix}(t),\ t=a-1.\end{cases}$$

If c and h are bounded on $\mathcal{I}'$, then it can be verified that the limit point or limit circle type of $\mathcal{L}^{(0)}$ is equal to that of $\mathcal{L}$. A natural question is whether the limit type is invariant if c or h is unbounded on $\mathcal{I}'$. Hassi, Möller, and Snoo considered equation (1.6) on the interval $[0,\infty)$ with real-valued coefficients p, q, c, h, and d. It was shown that the limit type of $\mathbb{L}^{(0)}$ is different from that of $\mathbb{L}$ in general when c or h is unbounded on $[0,\infty)$ by [**21**, Examples 6.3 and 6.4], where $\mathbb{L}^{(0)}$ and $\mathbb{L}^{(1)}$ are given by

$$\mathbb{L}^{(0)}=\begin{pmatrix} -DpD+q & 0\\ 0 & d\end{pmatrix},\ \mathbb{L}^{(1)}=\begin{pmatrix} 0 & -Dc+h\\ cD+h & 0\end{pmatrix}.$$

Here, we shall show that this is also true for $\mathcal{L}^{(0)}$ and $\mathcal{L}$ when $\mathcal{L}^{(1)}$ is given here with c or h being unbounded on $\mathcal{I}$ by two examples. The first example shows that

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$\mathcal{L}^{(0)}$ is in the LCC at $t=\infty$ while $\mathcal{L}$ is in the LPC at $t=\infty$, and the second one shows that $\mathcal{L}^{(0)}$ is in the LPC at $t=\infty$ while $\mathcal{L}$ is in the LCC at $t=\infty$.

EXAMPLE 4.1. Consider $\mathcal{L}^{(0)}$ with $p(t)=-4^t$, $q(t)=4^t$, and $d=1$ for $t\in\mathcal{I}'=\{t\}_{t=-1}^{+\infty}$. It is evident that $\widetilde{p}(t,\lambda)$ and $\tilde{q}(t,\lambda)$ associated with $\mathcal{L}^{(0)}(y)=\lambda y$ are given by

$$\widetilde{p}(t,\lambda)=-4^t, t\in\mathcal{I}';\ \tilde{q}(t,\lambda)=4^t,\ t\in\mathcal{I}=\{t\}_{t=0}^{+\infty}.$$

Therefore, the corresponding equation (1.3) becomes as

$$\tilde{\tau}(y_1)(t):=\nabla\big(4^t\Delta y_1(t)\big)+4^t y_1(t)=\lambda y_1(t),\ t\in\mathcal{I}. \tag{4.2}$$

By [**11**, Example 3.2], $\tilde{\tau}$ is in the LCC at $t=\infty$. Then all its solutions y_1 satisfy $\sum\limits_{t=0}^{\infty}|y_1(t)|^2<\infty$. In addition, we get from (1.5) that $y_2=0$ with $\lambda\neq 1$ since $c=h=0$ on $\mathcal{I}'$ and $d(t)\neq 0$ for $t\in\mathcal{I}'$, which implies that all solutions of $\mathcal{L}^{(0)}(y)=\lambda y$ with $\lambda\neq 1$ are in $l^2(\mathcal{I})$. Therefore, $\mathcal{L}^{(0)}$ is in the LCC at $t=\infty$.

Now, take $\mathcal{L}^{(1)}$ with $h(t)=2^t+2^{-t}$ and $c(t)=0,\ t\in\mathcal{I}'$. Then $\mathcal{M}=d=1$ on $\mathcal{I}'$, and hence $\Omega'(\mathcal{M},d)=\mathbb{C}\setminus\{1\}$. For $\mathcal{L}(y)=\lambda y$ with $\lambda\in\mathbb{C}\setminus\{1\}$ and $\mathcal{L}$ given by (4.1), $\widetilde{p}(t,\lambda)$ and $\tilde{q}(t,\lambda)$ are given by

$$\widetilde{p}(t,\lambda)=-4^t, t\in\mathcal{I}';\quad \tilde{q}(t,\lambda)=4^t+\frac{4^t+4^{-t}+2}{\lambda-1},\quad t\in\mathcal{I}.$$

Therefore, the corresponding equation (1.3) with $\lambda=0$ becomes as

$$\nabla\big(4^t\Delta y_1(t)\big)-(4^{-t}+2)y_1(t)=0,\quad t\in\mathcal{I}. \tag{4.3}$$

By [**28**, Theorem 3.11.6], (4.3) has a solution y_1 satisfying $\sum\limits_{t=0}^{\infty}|y_1(t)|^2=\infty$. Let $y=(y_1,\,y_2)^{\mathrm{T}}$ with y_2 given by (1.5) with $\lambda=0$. Then y is a solution of $\mathcal{L}(y)=0$. Clearly $y\notin l^2(\mathcal{I})$. Hence, $\mathcal{L}$ is in the LPC at $t=\infty$.

EXAMPLE 4.2. Consider $\mathcal{L}^{(0)}$ with $p=1$, $q(t)=4^t$, and $d(t)=4^t$ for $t\in\mathcal{I}'=\{t\}_{t=-1}^{+\infty}$. It is evident that $\widetilde{p}(t,\lambda)$ and $\tilde{q}(t,\lambda)$ associated with $\mathcal{L}^{(0)}(y)=\lambda y$ are given by

$$\widetilde{p}(t,\lambda)=1, t\in\mathcal{I}';\ \tilde{q}(t,\lambda)=4^t,\ t\in\mathcal{I}=\{t\}_{t=0}^{+\infty},$$

Therefore, the corresponding equation (1.3) becomes as

$$-\nabla(\Delta y_1(t))+4^t y_1(t)=\lambda y_1(t),\ t\in\mathcal{I}. \tag{4.4}$$

By [**11**, Corollary 3.1], (4.4) has a solution y_1 satisfying $\sum\limits_{t=0}^{\infty}|y_1(t)|^2=\infty$. Then, $\mathcal{L}^{(0)}(y)=\lambda y$ for λ with $\operatorname{Im}\lambda\neq 0$ has a solution $y\notin l^2(\mathcal{I})$. Therefore, $\mathcal{L}^{(0)}$ is in the LPC at $t=\infty$.

Now, take $\mathcal{L}^{(1)}$ with $c(t)=\sqrt{4^{2t}+4^t}$ and $h(t)=0,\quad t\in\mathcal{I}'$. Then $\mathcal{M}(t)=-4^{2t},\,t\in\mathcal{I}'$, and then $\sigma(\mathcal{M})\cup\sigma(d)=\{-4^{2t},\,4^t:t\in\mathcal{I}'\}$. Thus $\Omega'(\mathcal{M},d)=\mathbb{C}\setminus$

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$\{-4^{2t},\, 4^t : t \in \mathcal{I}'\}$. For $\mathcal{L}(y) = \lambda y$ with $\lambda \in \Omega'(\mathcal{M},\, d)$ and $\mathcal{L}$ given by (4.1), $\widetilde{p}(t,\, \lambda)$ and $\tilde{q}(t,\, \lambda)$ are given by

$$\widetilde{p}(t, \lambda) = 1 + \frac{4^{2t} + 4^t}{\lambda - 4^t}, \quad t \in \mathcal{I}';\ \tilde{q}(t, \lambda) = 4^t, \quad t \in \mathcal{I}.$$

Note that $\lambda = 0 \in \Omega'(\mathcal{M},\, d)$. Then take $\lambda = 0$ and the corresponding equation (1.3) becomes as $\tilde{\tau}(y_1)(t) = 0,\ t \in \mathcal{I}$, where $\tilde{\tau}$ is given by (4.2). Then $\tilde{\tau}$ is in the LCC at $t = \infty$, which implies that all solutions $y_1(t)$ of $\tilde{\tau}(y_1)(t) = 0$ satisfy $\sum_{t=0}^{\infty} |y_1(t)|^2 < \infty$, In addition, (1.5) becomes as

$$y_2(t) = \frac{\sqrt{4^{2t} + 4^t}}{-4^t} \Delta y_1(t), \quad t \in \mathcal{I}', \tag{4.5}$$

which, together with $\sum_{t=0}^{\infty} |y_1(t)|^2 < \infty$, yields that $\sum_{t=0}^{\infty} |y_2(t)|^2 < \infty$. Let $y = (y_1,\, y_2)^{\mathrm{T}}$ with y_2 given by (4.5). Then y is a solution of $\mathcal{L}(y) = 0$. Clearly $y = (y_1,\, y_2)^{\mathrm{T}} \in l^2(\mathcal{I})$. Hence, $\mathcal{L}$ is in the LCC at $t = \infty$.

5. Limit point criteria

In this section, we shall establish two criteria of the limit point case for $\mathcal{L}$ in terms of its coefficients which extend the existing results for Sturm–Liouville differential and difference expressions to matrix difference expressions $\mathcal{L}$.

Theorem 5.1. *If there exist $N \in \mathcal{I}$ and K such that $\left|\frac{c(t)}{p(t)}\right| \leqslant K$ for $t > N$, and $\sum_{t=N}^{\infty} \frac{1}{|p(t)|} = \infty$ then $\mathcal{L}$ is in the LPC at $t = \infty$.*

Proof. Suppose on the contrary that $\mathcal{L}$ is in the LCC at $t = \infty$. Then, $\varphi = (\varphi_1,\, \varphi_2)^{\mathrm{T}}$ and $\psi = (\psi_1,\, \psi_2)^{\mathrm{T}}$ given in Section 3 satisfying (3.2) are linearly independent solutions of (1.1_λ) in $l^2(\mathcal{I})$. Further, by Lemma 2.3 and (3.2), one has $[\varphi(t),\, \overline{\psi}(t)] = 1$ on $\mathcal{I}'$, which, together with (2.1), yields that

$$p(t)\left[\varphi_1(t+1)\psi_1(t) - \varphi_1(t)\psi_1(t+1)\right] + c(t)\left[\psi_1(t+1)\varphi_2(t) - \psi_2(t)\varphi_1(t+1)\right] = -1.$$

Since $\left|\frac{c(t)}{p(t)}\right| \leqslant K$ for $t > N$, it follows that for $t > N$,

$$|\psi_1(t+1)|\left(|\varphi_1(t)| + K\,|\varphi_2(t)|\right) + |\varphi_1(t+1)|\left(|\psi_1(t)| + K\,|\psi_2(t)|\right) \geqslant \frac{1}{|p(t)|}. \tag{5.1}$$

By the Cauchy's inequality, the left-hand side of (5.1) is summable, which contradicts to $\sum_{t=N}^{\infty} \frac{1}{|p(t)|} = \infty$. Therefore, $\mathcal{L}$ is in the LPC at $t = \infty$. This completes the proof. □

Remark 5.2. (1) It is noted that the criterion given by Theorem 5.1 only depends on the coefficients $p(t)$ and $c(t)$ for $t > N$.

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(2) By Theorem 5.1, $\mathcal{L}^{(0)}$ is in the LPC at $t=\infty$ if $\sum_{t=N}^{\infty} \frac{1}{|p(t)|}=\infty$, and this limit point case is invariant under the perturbation $\mathcal{L}^{(1)}$ under condition $\left|\frac{c(t)}{p(t)}\right| \leqslant K$, $t>N$.

(3) Hinton and Lewis [22] considered equation $\tau y_1=\lambda w y_1$, i.e.,

$$\tau y_1 := -\nabla(p(t)\Delta y_1(t))+q(t)y_1(t)=\lambda w(t)y_1(t), \quad t\in\mathcal{I}, \tag{5.2}$$

where $w(t)>0$, p and q are real-valued on $\mathcal{I}$. By [22, Theorem 10], if

$$\sum_{t\in\mathcal{I}} \frac{(w(t)w(t+1))^{1/2}}{|p(t)|}=\infty,$$

then τ is in the LPC at $t=\infty$. It is noted that (1.1_λ) contains (5.2) as its special case of $h(t)=c(t)\equiv 0$ and $w(t)\equiv 1$ on $\mathcal{I}'$ and $d(t)\neq\lambda$ for $t\in\mathcal{I}'$. Then, Theorem 5.1 is a generalization of [22, Theorem 10] with $w(t)\equiv 1$.

THEOREM 5.3. *If $p(t)>0$, $t\in\mathcal{I}$ and there exist $N\in\mathcal{I}$, a sequence of positive numbers $\{M(t)\}_{t=N}^{\infty}$, and positive constants k_j, $1\leqslant j\leqslant 4$, such that for all $t>N$,*

1) $|c(t)|+|c(t-1)|\leqslant k_1 M(t), \quad |h(t)|\leqslant k_2 M(t)$,
2) $q(t)\geqslant -k_3 M(t)$,
3) $\dfrac{p^{1/2}(t-1)|\nabla M(t)|}{M^{1/2}(t)M(t-1)}\leqslant k_4$,
4) $\displaystyle\sum_{t=N}^{\infty}\frac{1}{(p^2(t-1)+c^2(t-1))^{1/4}M^{1/2}(t)}=\infty$,

then $\mathcal{L}$ is in the LPC at $t=\infty$.

Proof. Suppose that $y=(y_1,\, y_2)^{\mathrm{T}}$ is a solution of (1.1_λ) with $\lambda=i$. Then, we have

$$-\nabla(p(t)\Delta y_1(t))-\nabla(c(t)y_2(t))+h(t)y_2(t)+(q(t)-i)y_1(t)=0, \quad t\in\mathcal{I}. \tag{5.3}$$

Multiplying both side of (5.3) by $\frac{\overline{y}_1(t)}{M(t)}$ and with a simple calculation, we get that

$$\nabla\left(\frac{p(t)(\Delta y_1(t))\overline{y}_1(t)}{M(t)}\right)=\frac{p(t-1)|\nabla y_1(t)|^2}{M(t)}-\frac{p(t-1)(\nabla M(t))(\nabla y_1(t))\overline{y}_1(t-1)}{M(t)M(t-1)} \\ +\frac{(q(t)-i)|y_1(t)|^2}{M(t)}+\frac{h(t)y_2(t)\overline{y}_1(t)}{M(t)}-\frac{\nabla(c(t)y_2(t))\overline{y}_1(t)}{M(t)}. \tag{5.4}$$

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Summing up (5.4) from N to t yields

$$\frac{p(t)(\Delta y_1(t))\overline{y}_1(t)}{M(t)} = G(t) - \sum_{s=N}^{t} \frac{p(s-1)(\nabla M(s))(\nabla y_1(s))\overline{y}_1(s-1)}{M(s)M(s-1)} + \sum_{s=N}^{t} \frac{(q(s)-i)|y_1(s)|^2}{M(s)} + \sum_{s=N}^{t} \frac{h(s)y_2(s)\overline{y}_1(s)}{M(s)} - \sum_{s=N}^{t} \frac{\nabla(c(s)y_2(s))\overline{y}_1(s)}{M(s)} + c_0 \tag{5.5}$$

where

$$G(t) = \sum_{s=N}^{t} \frac{p(s-1)|\nabla y_1(s)|^2}{M(s)} \quad \text{and} \quad c_0 = \frac{p(N-1)(\Delta y_1(N-1))\overline{y}_1(N-1)}{M(N-1)}.$$

Since $p(t) > 0$ and $M(t) > 0$, $\lim_{t\to\infty} G(t)$ exists which may be infinity. Now suppose that $y = (y_1,\, y_2)^{\mathrm{T}} \in l^2(\mathcal{I})$. We shall show that $\lim_{t\to\infty} G(t) < \infty$ in this case. By the assumptions 1)-3), the Cauchy's inequality, $y \in l^2(\mathcal{I})$, it follows from (5.5) that there exist $\tilde{k}_1,\, \tilde{k}_2 \in \mathbb{R}$ such that for $t > N$,

$$\mathrm{Re}\left\{\frac{p(t)(\Delta y_1(t))\overline{y}_1(t)}{M(t)}\right\} \geqslant G(t) - \tilde{k}_1 G^{1/2}(t) + \tilde{k}_2. \tag{5.6}$$

Assume on the contrary that $\lim_{t\to\infty} G(t) = \infty$. Then, (5.6) yields that there exists a positive integer $N_1 \geqslant N$ such that

$$\mathrm{Re}\{(\Delta y_1(t))\overline{y}_1(t)\} > 0, \quad t > N_1, \tag{5.7}$$

i.e.,

$$\frac{1}{2}(y_1(t+1)\overline{y}_1(t) + \overline{y}_1(t+1)y_1(t)) - \overline{y}_1(t)y_1(t) > 0, \quad t > N_1. \tag{5.8}$$

It is obtained from (5.7) that $y_1(t) \neq 0$ for $t > N_1$. Therefore, (5.8) implies that

$$\mathrm{Re}\left\{\frac{y_1(t+1)}{y_1(t)}\right\} > 1, \quad t > N_1.$$

Hence, $\sum_{t=N}^{\infty} |y_1(t)|^2 = \infty$, which is contrary to assumption $y \in l^2(\mathcal{I})$. Therefore, we have $\lim_{t\to\infty} G(t) < \infty$.

Now, let $\varphi = (\varphi_1,\, \varphi_2)^{\mathrm{T}}$ and $\psi = (\psi_1,\, \psi_2)^{\mathrm{T}}$ be solutions of (1.1_λ) with $\lambda = i$ satisfying (3.2). Then φ and ψ are linearly independent. Further, by Lemma 2.3, (2.1), and (3.2), we get

$$\begin{aligned} &p(t-1)[\psi_1(t)\nabla\varphi_1(t)-\varphi_1(t)\nabla\psi_1(t)] \\ &\quad +c(t-1)[\psi_1(t)\varphi_2(t-1)-\varphi_1(t)\psi_2(t-1)]=-1, \quad t\in\mathcal{I}, \end{aligned} \tag{5.9}$$

which implies that

$$\begin{aligned} &\frac{p^{1/2}(t-1)}{M^{1/2}(t)}\Big(|\psi_1(t)\nabla\varphi_1(t)|+|(\nabla\psi_1(t))\varphi_1(t)|\Big) \\ &\qquad +\frac{c^{1/2}(t-1)}{M^{1/2}(t)}\Big(|\psi_1(t)\varphi_2(t-1)|+|\psi_2(t-1)\varphi_1(t)|\Big) \\ &\quad \geqslant \frac{1}{(p^2(t-1)+c^2(t-1))^{\frac{1}{4}}\,M^{1/2}(t)}. \end{aligned} \tag{5.10}$$

If φ, $\psi\in l^2(\mathcal{I})$, then

$$\sum_{t=N}^{\infty}\frac{p(t-1)|\nabla\varphi_1(t)|^2}{M(t)}<\infty \text{ and } \sum_{t=N}^{\infty}\frac{p(t-1)|\nabla\psi_1(t)|^2}{M(t)}<\infty$$

by the above discussions. Then, by the Cauchy's inequality and the first assumption in 1), we get from (5.10) that

$$\sum_{t=N}^{\infty}\frac{1}{(p^2(t-1)+c^2(t-1))^{\frac{1}{4}}\,M^{1/2}(t)}<\infty,$$

which contradicts to assumption 4). Then $\mathcal{L}$ is in the LPC at $t=\infty$. This completes the proof. □

Remark 5.4. (1) By Theorem 5.3, $\mathcal{L}^{(0)}$ is in the LPC at $t=\infty$ under the following conditions: $p(t)>0$, 2) and 3) of Theorem 5.3, and

$$\sum_{t=N}^{\infty}\frac{1}{(p(t-1)M(t))^{\frac{1}{2}}}=\infty. \tag{5.11}$$

This limit point type of $\mathcal{L}^{(0)}$ is invariant under the perturbation $\mathcal{L}^{(1)}$ in the case that conditions 1) and 4) of Theorem 5.3 hold.

(2) There were some limit point criteria in terms of coefficients for Sturm–Liouville differential equations, e.g., [**14**, **15**, **35**]. Among them, there is a well-known limit point criterion for Sturm–Liouville differential equations given by Levinson [**35**, Theorem IV]. Mingarelli [**41**] extended it to equation (5.2) with $w(t)\equiv 1$ on $\mathcal{I}'$, i.e., equation (1.2). By [**41**, Theorem 1], if there

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exist k_1, $k_2 > 0$, and $N \in \mathcal{I}$ such that

$$\text{i) } q(t) \geqslant -k_1 M(t), \quad t > N,$$

$$\text{ii) } \frac{p^{1/2}(t-1)|\nabla M(t)|}{M^{1/2}(t)M(t-1)} \leqslant k_2, \quad t > N,$$

and (5.11) holds, then τ in (5.2) is in the LPC at $t = \infty$. Clearly, Theorem 5.3 is a generalization of [**41**, Theorem 1] with $w(t) \equiv 1$ for Sturm–Liouville difference equation to matrix difference equation (1.1_λ).

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