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A PROOF OF ISBELL'S ZIGZAG THEOREM

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Abstract

We provide a short, intuitive proof of Isbell's zigzag theorem.

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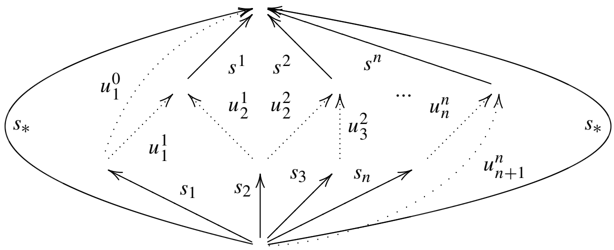
The elegant, elementary proof of the zigzag theorem published by Higgins [1, 2] is unfortunately incorrect (the case overlooked in the proof is when the transition α is $puq \to ptutq$ and β is $ptutq \to pu'tstq$, where $u = u's$ in S). The proofs of Howie [3], Storrer [4] and Renshaw [5] involve nontrivial algebra. The proof of Philip [6], which completes Isbell's original proof [7], is topological in nature. We provide a short, elementary and intuitive proof of the monoid version of the zigzag theorem. Our proof is heavily based on the proposed proof of Higgins.

Recall [8] that if U is a submonoid of S, then the *dominion* of U in S is the set of all elements $s \in S$ such that, for all homomorphisms $h_1, h_2 : S \to T$ coinciding on U, $h_1(s) = h_2(s)$. Note that a homomorphism $h : U \to S$ is epi if and only if the dominion of $h(U)$ in S is S.

ISBELL'S ZIGZAG THEOREM. *Let U be a submonoid of a monoid S and s_* an element of S. Then s_* is in the dominion of U in S if and only if there exists a U-zigzag in S with value s_*, that is, a diagram of the form*
where $s_1, \ldots, s_n, s^1, \ldots, s^n \in S$, $u_1^0, u_1^1, u_2^1, u_2^2, \ldots, u_n^n, u_{n+1}^n \in U$, and all cells in the diagram commute, that is, $s_* = s_1 u_1^0$, $u_1^0 = u_1^1 s^1$, $s_1 u_1^1 = s_2 u_2^1$, $u_2^1 s^1 = u_2^2 s^2, \ldots,$ $s_n u_n^n = u_{n+1}^n$, $u_{n+1}^n s^n = s_*$.



A simple verification shows that indeed, if $h_1, h_2 : S \to T$ are homomorphisms coinciding on U and there is a U-zigzag with value s_*, then $h_1(s_*) = h_2(s_*)$, which proves that s_* is in the dominion.

So, assume now that s_* is in the dominion of U in S.

Let A consist of all elements s of the monoid S and of a new element $|$. Consider the set A^* of all finite words over the alphabet A, with ϵ denoting the empty word. On A^*, define three types of reductions:

refactorization $s_1 \ldots s_n \leftrightarrow s'_1 \ldots s'_k$ if $s_1 \cdot \ldots \cdot s_n = s'_1 \cdot \ldots \cdot s'_k$ holds in S ($n, k \geq 0$ and $s_1, \ldots, s_n, s'_1, \ldots, s'_k \in S$);

right/left shift $|u \leftrightarrow u|$ for $u \in U$;

creation/deletion $\epsilon \leftrightarrow ||$.

Let $\leftrightarrow$ be the reduction relation defined by the above reductions, and let $\leftrightarrow^+$ be its transitive closure; then $\leftrightarrow^+$ is a congruence on A^*, giving rise to a quotient monoid T.

Consider maps $\mu, \nu : S \to T$ given by $\mu(s) = s$ and $\nu(s) = |s|$ for all $s \in S$. Both μ and ν are monoid homomorphisms, and they coincide on U. Therefore we have $\mu(s_*) = \nu(s_*)$. In other words, $s_* \leftrightarrow^+ |s_*|$, which is equivalent to $s_*| \leftrightarrow^+ |s_*$. We will show that if a sequence q of reductions

$$s_*| = w_1 \leftrightarrow w_2 \leftrightarrow \ldots \leftrightarrow w_{n-1} \leftrightarrow w_n = |s_*$$

exists ($n \geq 2$), then there is a U-zigzag in S with value s_*.

The proof rests on two observations.

The first observation is that one may track particular *occurrences* of the symbol $|$. That is, any occurrence of $|$, if it is not the occurrence appearing in w_1, is first created by some reduction $w_{i-1} \leftrightarrow w_i$; then i is called its *birth* and the occurrence born with it is called its *birth pair*. The identity of an occurrence of $|$ may be tracked along any shifts it performs, as well as any reductions in which it does not take an active role. If the tracked occurrence is not the one appearing in w_n, then it must be deleted by some reduction $w_j \leftrightarrow w_{j+1}$; j is then called its *death* and the occurrence that died with it is called its *death pair*. Potential problems that could arise in reductions, such as $| \leftrightarrow |||$, where it is not clear whether the newly-born pair of occurrences appears to the right or

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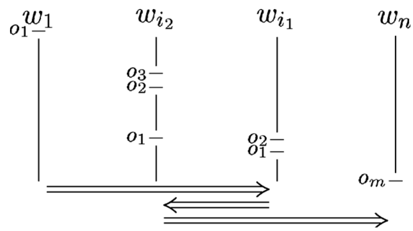

FIGURE 1. The sequence r of reductions.

to the left of the original occurrence, can be resolved by, for instance, always choosing births to happen at the left end of a word; the same remark applies to deaths.

The second observation is that if $w_i \leftrightarrow w_{i+1}$, and if w_i contains an occurrence of | that does not die at i, then also $v_i \leftrightarrow v_{i+1}$, where v_i and v_{i+1} arise from w_i and w_{i+1} by removing all occurrences of | other than the one under consideration. The reason is that a refactorization or a shift of the considered occurrence of | remain uninfluenced, while creations, deletions or shifts involving occurrences other than the considered one are replaced by trivial refactorizations.

Consider the following sequence r of reductions: start with w_1, and let o_1 be the original occurrence of | in w_1. Proceed as in q up to the point i_1 of o_1's death, removing all occurrences of | other than o_1; by the second observation, this leads to a correct reduction sequence. Let o_2 be the death pair of o_1. Observe that w_{i_1} with occurrences other than o_1 removed is the same as w_{i_1} with occurrences other than o_2 removed, since o_1 and o_2 are adjacent in w_{i_1}. So we may now proceed as in q, but in reverse, from i_1 to the point i_2 of o_2's birth, removing all occurrences of | other than o_2. Let o_3 be the birth pair of o_2. The above procedure may be repeated until an occurrence o_m appears that does not die at all; then o_m must coincide with the sole occurrence of | in w_n. By this procedure, depicted in Figure 1, we have built a sequence of reductions r starting with $s_*| = w_1$, ending with $w_n = |s_*$, and containing only words in which | appears exactly once; thus, r consists of refactorizations and shifts only.

For any word $w = s_1 \ldots s_n | s'_1 \ldots s'_k$ in the sequence r, consider the three-letter word $w^\# = (s_1 \cdot \ldots \cdot s_n)|(s'_1 \cdot \ldots \cdot s'_k)$. This gives rise to a sequence $r^\#$, which starts with $s_*|1$ and ends with $1|s_*$. If $w \leftrightarrow v$ was a refactorization in r, then $w^\# = v^\#$. If $w \leftrightarrow v$ was a right shift in r involving $u \in U$, then $w^\# = s|(u \cdot s')$ and $v^\# = (s \cdot u)|s'$ for some $s, s' \in S$; similarly for left shifts. Thus, $r^\#$ precisely corresponds to a U-zigzag in S with value s_*. This completes the proof.

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