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FREE ALGEBRAS IN VARIETIES OF BL-ALGEBRAS GENERATED BY A BL$_n$-CHAIN

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Abstract

Free algebras with an arbitrary number of free generators in varieties of BL-algebras generated by one BL-chain that is an ordinal sum of a finite MV-chain $\mathbf{L}_n$ and a generalized BL-chain **B** are described in terms of weak Boolean products of BL-algebras that are ordinal sums of subalgebras of $\mathbf{L}_n$ and free algebras in the variety of basic hoops generated by **B**. The Boolean products are taken over the Stone spaces of the Boolean subalgebras of idempotents of free algebras in the variety of MV-algebras generated by $\mathbf{L}_n$.

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Introduction

Basic Fuzzy Logic (BL for short) was introduced by Hájek (see [19] and the references given there) to formalize fuzzy logics in which the conjunction is interpreted by a continuous t-norm on the real segment [0, 1] and the implication by its corresponding adjoint. He also introduced BL-algebras as the algebraic counterpart of these logics. BL-algebras form a variety (or equational class) of residuated lattices [19]. More precisely, they can be characterized as *bounded basic hoops* [1, 7]. Subvarieties of the variety of BL-algebras are in correspondence with axiomatic extensions of BL. Important examples of subvarieties of BL-algebras are MV-algebras (that correspond to Łukasiewicz many-valued logics, see [14]), linear Heyting algebras (that correspond to the superintuitionistic logic characterized by the axiom $(P \Rightarrow Q) \vee (Q \Rightarrow P)$, see [25] for a historical account about this logic), PL-algebras (that correspond to the logic determined by the t-norm given by the ordinary product on [0, 1], see [15]), and also Boolean algebras (that correspond to classical logic).

Since the propositions under BL equivalence form a free BL-algebra, descriptions of free algebras in terms of functions give concrete representations of these propositions. Such descriptions are known for some subvarieties of BL-algebras. The best known example is the representation of classical propositions by Boolean functions. Free MV-algebras have been described in terms of continuous piecewise linear functions by McNaughton [22] (see also [14]). Finitely generated free linear Heyting algebras were described by Horn [20], and a description of finitely generated free PL-algebras was given in [15]. Linear Heyting algebras and PL-algebras are examples of varieties of BL-algebras satisfying the *Boolean retraction property*. Free algebras in these varieties were completely described in [17].

In [10] the first author described the finitely generated free algebras in the varieties of BL-algebras generated by a single BL-chain which is an ordinal sum of a finite MV-chain $\mathbf{L}_n$ and a generalized BL-chain $\mathbf{B}$. We call these chains BL$_n$-chains. The aim of this paper is to extend the results of [10] considering the case of infinitely many free generators. The results of [10] were heavily based on the fact that the Boolean subalgebras of finitely generated algebras in the varieties generated by BL$_n$-chains are finite. Therefore the methods of [10] cannot be applied to the general case.

As a preliminary step we characterize the Boolean algebra of idempotent elements of a free algebra in $\mathcal{MV}_n$, the variety of MV-algebras generated by the finite MV-chain $\mathbf{L}_n$. It is the free Boolean algebra over a poset which is the cardinal sum of chains of length $n-1$. In the proof of this result a central role is played by the Moisil algebra reducts of algebras in $\mathcal{MV}_n$.

Free algebras in varieties of BL-algebras generated by a single BL$_n$-chain $\mathbf{L}_n \uplus \mathbf{B}$ are then described in terms of weak Boolean products of BL-algebras that are ordinal sums of subalgebras of $\mathbf{L}_n$ and free algebras in the variety of basic hoops generated by $\mathbf{B}$. The Boolean products are taken over the Stone spaces of the Boolean algebras of idempotent elements of free algebras in $\mathcal{MV}_n$. An important intermediate step is the characterization of the variety of generalized BL-algebras generated by $\mathbf{B}$ (Corollary 3.5).

The paper is organized as follows. In the first section we recall, for further reference, some basic notions on BL-algebras and on the varieties $\mathcal{MV}_n$. We also recall some facts about the representation of free algebras in varieties of BL-algebras as weak Boolean products. The only new result is given in Theorem 1.5. In Section 2, after giving the necessary background on Moisil algebra reducts of algebras in $\mathcal{MV}_n$, we characterize the Boolean algebras of idempotent elements of free algebras in $\mathcal{MV}_n$. These results are used in Section 3 to give the mentioned description of free algebras in the varieties of BL-algebras generated by a BL$_n$-chain. Finally in Section 4 we give some examples and we compare our results with those of [10] and [17].

1. Preliminaries

1.1. BL-algebras: basic notions A *hoop* [7] is an algebra $\mathbf{A} = (A, *, \to, \top)$ of type $(2, 2, 0)$, such that $(A, *, \top)$ is a commutative monoid and for all $x, y, z \in A$:

(1) $x \to x = \top$,
(2) $x * (x \to y) = y * (y \to x)$,
(3) $x \to (y \to z) = (x * y) \to z$.

A *basic hoop* [1] or a *generalized BL-algebra* [18], is a hoop that satisfies the equation

$$(((x \to y) \to z) * ((y \to x) \to z)) \to z = \top. \tag{1.1}$$

It is shown in [1] that generalized BL-algebras can be characterized as algebras $\mathbf{A} = (A, \wedge, \vee, *, \to, \top)$ of type $(2, 2, 2, 2, 0)$ such that

(1) $(A, *, \top)$, is an commutative monoid,
(2) $\mathbf{L}(\mathbf{A}) := (A, \wedge, \vee, \top)$, is a lattice with greatest element $\top$,
(3) $x \to x = \top$,
(4) $x \to (y \to z) = (x * y) \to z$,
(5) $x \wedge y = x * (x \to y)$,
(6) $(x \to y) \vee (y \to x) = \top$.

A *BL-algebra* or *bounded basic hoop* is a bounded generalized BL-algebra, that is, it is an algebra $\mathbf{A} = (A, \wedge, \vee, *, \to, \bot, \top)$ of type $(2, 2, 2, 2, 0, 0)$ such that $(A, \wedge, \vee, *, \to, \top)$ is a generalized BL-algebra, and $\bot$ is the lower bound of $\mathbf{L}(\mathbf{A})$. In this case, we define the unary operation $\neg$ by the equation $\neg x = x \to \bot$. The BL-algebra with only one element, that is, $\bot = \top$, is called the *trivial BL-algebra*. The varieties of BL-algebras and of generalized BL-algebras will be denoted by $\mathcal{BL}$ and $\mathcal{GBL}$, respectively.

In every generalized BL-algebra $\mathbf{A}$ we denote by $\leq$ the (partial) order defined on A by the lattice $\mathbf{L}(\mathbf{A})$, that is, for $a, b \in A$, $a \leq b$ if and only if $a = a \wedge b$ if and only if $b = a \vee b$. This order is called the *natural order* of $\mathbf{A}$. When this natural order is total (that is, for each $a, b, \in A$, $a \leq b$ or $b \leq a$), we say that $\mathbf{A}$ is a *generalized BL-chain* (*BL-chain* in case $\mathbf{A}$ is a BL-algebra). The following theorem makes obvious the importance of BL-chains and can be easily derived from [19, Lemma 2.3.16].

THEOREM 1.1. *Each BL-algebra is a subdirect product of BL-chains.*

In every BL-algebra $\mathbf{A}$ we define a binary operation $x \oplus y = \neg(\neg x * \neg y)$. For each positive integer k, the operations x^k and $k\,x$ are inductively defined as follows:

(a) $x^1 = x$ and $x^{k+1} = x^k * x$,
(b) $1x = x$ and $(k+1)x = (kx) \oplus x$.

MV-algebras, the algebras of Łukasiewicz infinite-valued logic, form a subvariety of $\mathcal{BL}$, which is characterized by the equation $\neg\neg x = x$ (see [19]). The variety of MV-algebras is denoted by $\mathcal{MV}$. Totally ordered MV-algebras are called *MV-chains*. For each BL-algebra $\mathbf{A}$, the set

$$MV(\mathbf{A}) := \{x \in A : \neg\neg x = x\}$$

is the universe of a subalgebra $\mathbf{MV}(\mathbf{A})$ of $\mathbf{A}$ which is an MV-algebra (see [18]).

A *PL-algebra* is a BL-algebra that satisfies the two axioms:

(1) $(\neg\neg z * ((x * z) \to (y * z))) \to (x \to y) = \top$,
(2) $x \wedge \neg x = \bot$.

PL-algebras correspond to *product fuzzy logic*, see [15] and [19].

It follows from Theorem 1.1 that for each BL-algebra $\mathbf{A}$ the lattice $\mathbf{L}(\mathbf{A})$ is distributive. The complemented elements of $L(\mathbf{A})$ form a subalgebra $\mathbf{B}(\mathbf{A})$ of $\mathbf{A}$ which is a Boolean algebra. Elements of $B(\mathbf{A})$ are called *Boolean elements* of $\mathbf{A}$.

1.2. Implicative filters

DEFINITION 1.2. An *implicative filter* of a BL-algebra $\mathbf{A}$ is a subset $F \subseteq A$ satisfying the conditions

(1) $\top \in F$.
(2) If $x \in F$ and $x \to y \in F$, then $y \in F$.

An implicative filter is called *proper* provided that $F \neq A$. If W is a subset of a BL-algebra $\mathbf{A}$, the implicative filter generated by W will be denoted by $\langle W \rangle$. If U is a filter of the Boolean subalgebra $\mathbf{B}(\mathbf{A})$, then the implicative filter $\langle U \rangle$ is called *Stone filter of* $\mathbf{A}$. An implicative filter F of a BL-algebra $\mathbf{A}$ is called *maximal* if and only if it is proper and no proper implicative filter of $\mathbf{A}$ strictly contains F.

Implicative filters characterize congruences in BL-algebras. Indeed, if F is an implicative filter of a BL-algebra $\mathbf{A}$ it is well known (see [19, Lemma 2.3.14]), that the binary relation $\equiv_F$ on A defined by

$$x \equiv_F y \quad \text{if and only if} \quad x \to y \in F \quad \text{and} \quad y \to x \in F$$

is a congruence of $\mathbf{A}$. Moreover, $F = \{x \in A : x \equiv_F \top\}$. Conversely, if $\equiv$ is a congruence relation on A, then $\{x \in A : x \equiv \top\}$ is an implicative filter, and $x \equiv y$ if and only if $x \to y \equiv \top$ and $y \to x \equiv \top$. Therefore, the correspondence $F \mapsto \equiv_F$ is a bijection from the set of implicative filters of $\mathbf{A}$ onto the set of congruences of $\mathbf{A}$.

LEMMA 1.3 (see [17]). *Let* $\mathbf{A}$ *be a BL-algebra, and let* F *be a filter of* $\mathbf{B}(\mathbf{A})$. *Then* $(\equiv_F) = \{(a, b) \in A \times A : a \wedge c = b \wedge c$ *for some* $c \in F\}$ *is a congruence relation on* $\mathbf{A}$ *that coincides with the congruence relation given by the implicative filter* $\langle F \rangle$ *generated by* F.

1.3. MV$_n$-algebras For $n \geq 2$, we define:

$$L_n = \left\{ \frac{0}{n-1}, \frac{1}{n-1}, \frac{2}{n-1}, \dots, \frac{n-1}{n-1} \right\}.$$

The set L_n equipped with the operations $x * y = \max(0, x + y - 1)$, $x \to y = \min(1, 1 - x + y)$, and with $\perp = 0$ defines a finite MV-algebra, which shall be denoted by $\mathbf{L}_n$. Clearly $B(\mathbf{L}_n) = \{0, 1\}$.

A BL-algebra $\mathbf{A}$ is said to be *simple* provided it is nontrivial and the only proper implicative filter of $\mathbf{A}$ is the singleton $\{\top\}$. In [14], it is proved that $\mathbf{L}_n$ is a simple MV-algebra for each integer n.

We shall denote by $\mathcal{MV}_n$ the subvariety of $\mathcal{MV}$ generated by $\mathbf{L}_n$. The elements of $\mathcal{MV}_n$ are called *MV$_n$-algebras*. A finite MV-chain $\mathbf{L}_m$ belongs to $\mathcal{MV}_n$ if and only if $m - 1$ is a divisor of $n - 1$. Therefore it is not hard to corroborate that every MV$_n$-algebra is a subdirect product of a family of algebras $(\mathbf{L}_{m_i}, i \in I)$ where $m_i - 1$ divides $n - 1$ for each $i \in I$.

It can be deduced from [14, Corollary 8.2.4 and Theorem 8.5.1] that $\mathcal{MV}_n$ is the proper subvariety of $\mathcal{MV}$ characterized by the equations

$$(\alpha_n) \qquad x^{(n-1)} = x^n,$$

and if $n \geq 4$, for every integer $p = 2, \dots, n-2$ that does not divide $n - 1$

$$(\beta_n) \qquad (p\, x^{p-1})^n = n\, x^p.$$

If $\mathbf{A}$ is an MV$_n$-algebra, it is not hard to verify that for each $x \in A \setminus \{\top\}$, $x^n = \perp$ and for each $y \in A \setminus \{\perp\}$, $n\, y = \top$.

1.4. Ordinal sum and decomposition of BL-chains Let $\mathbf{R} = (R, *_{\mathbf{R}}, \to_{\mathbf{R}}, \top)$ and $\mathbf{S} = (S, *_{\mathbf{S}}, \to_{\mathbf{S}}, \top)$ be two hoops such that $R \cap S = \{\top\}$. Following [7] we can define the *ordinal sum* $\mathbf{R} \uplus \mathbf{S}$ of these two hoops as the hoop given by $(R \cup S, *, \to, \top)$ where the operations $(*, \to)$ are defined as follows:

$$x * y = \begin{cases} x *_{\mathbf{R}} y & \text{if } x, y \in R, \\ x *_{\mathbf{S}} y & \text{if } x, y \in S, \\ x & \text{if } x \in R \setminus \{\top\} \text{ and } y \in S, \\ y & \text{if } y \in R \setminus \{\top\} \text{ and } x \in S. \end{cases}$$

$$x \to y = \begin{cases} \top & \text{if } x \in R \setminus \{\top\},\ y \in S, \\ x \to_{\mathbf{R}} y & \text{if } x, y \in R, \\ x \to_{\mathbf{S}} y & \text{if } x, y \in S, \\ y & \text{if } y \in R \setminus \{\top\} \text{ and } x \in S. \end{cases}$$

If $R \cap S \neq \{\top\}$, $\mathbf{R}$ and $\mathbf{S}$ can be replaced by isomorphic copies whose intersection is $\{\top\}$, thus their ordinal sum can be defined. When $\mathbf{R}$ is a generalized BL-chain and $\mathbf{S}$ is a generalized BL-algebra, the hoop resulting from their ordinal sum satisfies equation (1.1). Thus $\mathbf{R} \uplus \mathbf{S}$ is a generalized BL-algebra. Moreover, if $\mathbf{R}$ is a BL-chain, then $\mathbf{R} \uplus \mathbf{S}$ is a BL-algebra, where $\bot = \bot_{\mathbf{R}}$. If S is totally ordered it is obvious that the chain $\mathbf{R} \uplus \mathbf{S}$ is subdirectly irreducible if and only if $\mathbf{S}$ is subdirectly irreducible. Notice also that for any generalized BL-algebra $\mathbf{S}$, $\mathbf{L_2} \uplus \mathbf{S}$ is the BL-algebra that arises from adjoining a bottom element to $\mathbf{S}$.

Given a BL-algebra $\mathbf{A}$, we can consider the set $D(\mathbf{A}) := \{x \in \mathbf{A} : \neg x = \bot\}$. It is shown in [18], that $\mathbf{D}(\mathbf{A}) = (D(\mathbf{A}), \wedge, \vee, *, \to, \top)$ is a generalized BL-algebra.

THEOREM 1.4 (see [10]). *For each BL-chain* $\mathbf{A}$, *we have that* $\mathbf{A} \cong \mathbf{MV}(\mathbf{A}) \uplus \mathbf{D}(\mathbf{A})$.

THEOREM 1.5. *Let* $\mathbf{A}$ *be a BL-algebra such that* $\mathbf{MV}(\mathbf{A}) \cong \mathbf{L}_n$ *for some integer* n. *Then* $\mathbf{A} \cong \mathbf{MV}(\mathbf{A}) \uplus \mathbf{D}(\mathbf{A}) \cong \mathbf{L}_n \uplus \mathbf{D}(\mathbf{A})$.

PROOF. From Theorem 1.1, we can think of each non trivial BL-algebra $\mathbf{A}$ as a subdirect product of a family $(\mathbf{A}_i,\ i \in I)$ of non trivial BL-chains, that is, there exists an embedding $e : \mathbf{A} \to \prod_{i \in I} \mathbf{A}_i$, such that $\pi_i(e(\mathbf{A})) = \mathbf{A}_i$ for each $i \in I$, where π_i denotes each projection. We shall identify $\mathbf{A}$ with $e(\mathbf{A})$. Then each element of A is a tuple $\mathbf{x}$ and coordinate i is $x_i \in A_i$. With this notation we have that for each $\mathbf{x} \in A$, $\pi_i(\mathbf{x}) = x_i$. We will prove the following items:

(1) *For each* $i \in I$, $\mathbf{MV}(\mathbf{A}_i)$ *is isomorphic to* $\mathbf{L}_n$.

Since for each $i \in I$, π_i is a homomorphism and $\pi_i(MV(\mathbf{A})) \subseteq A_i$, we have that $\pi_i(MV(\mathbf{A})) \subseteq MV(\mathbf{A}_i)$. Then $\pi_i(\mathbf{MV}(\mathbf{A}))$ is a subalgebra of $\mathbf{MV}(\mathbf{A}_i)$. On the other hand, given $i \in I$, let $x_i \in MV(\mathbf{A}_i)$. Then $\neg\neg x_i = x_i$ and there exists an element $\mathbf{x} \in A$ such that $\pi_i(\mathbf{x}) = x_i$. Taking $\mathbf{y} = \neg\neg\mathbf{x} \in MV(\mathbf{A})$ we have that $\pi_i(\mathbf{y}) = x_i$ and $x_i \in \pi_i(MV(\mathbf{A}))$. Hence $MV(\mathbf{A}_i) \subseteq \pi_i(MV(\mathbf{A}))$.

In conclusion $\mathbf{MV}(\mathbf{A}_i) = \pi_i(\mathbf{MV}(\mathbf{A})) = \pi_i(\mathbf{L}_n) = \mathbf{L}_n$, because $\mathbf{L}_n$ is simple.

(2) *If* $\mathbf{x} \in A$, *then* $\mathbf{x} \in MV(\mathbf{A}) \cup D(\mathbf{A})$.

Let $\mathbf{x} \in A$ and let $\mathbf{y} = n(\neg\mathbf{x})$. If $x_i \in L_n \setminus \{\top\}$, then $\neg x_i \in L_n \setminus \{\bot\}$. From equation (α_n) we obtain that $y_i = n(\neg x_i) = \top$. On the other hand, if $\neg x_i = \bot$, then $y_i = n(\neg x_i) = \bot$. Now let $\mathbf{z} = (\neg\neg\mathbf{x})^n$. If $x_i \in L_n \setminus \{\top\}$, then $z_i = \bot$, but if $\neg\neg x_i = \top$, then $z_i = \top$.

Suppose there exists $\mathbf{x} \in A$ such that $\mathbf{x} \notin MV(\mathbf{A})$ and $\mathbf{x} \notin D(\mathbf{A})$. It follows from Theorem 1.4 that for each $i \in I$, $\mathbf{A}_i = \mathbf{MV}(\mathbf{A}_i) \uplus \mathbf{D}(\mathbf{A}_i)$. Then there exist $i, j \in I$, such that $x_i \in MV(\mathbf{A}_i) \setminus \{\top\} = L_n \setminus \{\top\}$ and $x_j \in D(\mathbf{A}_j) \setminus \{\top\}$.

Let $\mathbf{y} = n(\neg\mathbf{x})$. Then $y_i = \top$, $y_j = \bot$, and $y_k \in \{\bot, \top\}$ for each $k \in I \setminus \{i, j\}$. Now let $\mathbf{z} = (\neg\neg\mathbf{x})^n$. We have that $z_j = \top$, $z_i = \bot$, and $z_k \in \{\bot, \top\}$ for each

$k \in I \setminus \{i, j\}$. It follows that $\mathbf{y}$ and $\mathbf{z}$ are elements in the chain $MV(\mathbf{A}) = L_n$, which are not comparable, and this is a contradiction.

(3) *If* $\mathbf{x} \in MV(\mathbf{A}) \setminus \{\top\}$ *and* $\mathbf{y} \in D(\mathbf{A})$, *then* $\mathbf{x} < \mathbf{y}$.

The statement is clear if $x_i \in MV(\mathbf{A}_i) \setminus \{\top\}$ for every $i \in I$ or if $y_i = \top$ for each $i \in I$. Otherwise, suppose $x_i = \top$ for some $i \in I$. Since $\mathbf{x} \neq \top$, there must exist $j \in I$ such that $x_j \neq \top$. If $y_i = \top$ for each $i \in I$ such that $x_i = \top$, then $\mathbf{x} < \mathbf{y}$. If not, let $\mathbf{z} = \mathbf{x} \wedge \mathbf{y}$. Since operations are coordinatewise, $z_j \in MV(\mathbf{A}_j) \setminus \{\top\}$ and $z_i \in D(\mathbf{A}_i) \setminus \{\top\}$, for some $i \in I$. Hence $\mathbf{z} \notin MV(\mathbf{A})$ and $\mathbf{z} \notin D(\mathbf{A})$, contradicting the previous item.

(4) *If* $\mathbf{x} \in MV(\mathbf{A}) \setminus \{\top\}$ *and* $\mathbf{y} \in D(\mathbf{A})$, *then* $\mathbf{y} \to \mathbf{x} = \mathbf{x}$ *and* $\mathbf{y} * \mathbf{x} = \mathbf{x}$.

Since $\neg\mathbf{y} = \bot$ we have that

$$\begin{aligned}\mathbf{y} \to \mathbf{x} &= \mathbf{y} \to \neg\neg\mathbf{x} = \mathbf{y} \to (\neg\mathbf{x} \to \bot) = \neg\mathbf{x} \to (\mathbf{y} \to \bot)\\ &= \neg\mathbf{x} \to \bot = \neg\neg\mathbf{x} = \mathbf{x},\end{aligned}$$

and

$$\mathbf{x} = \mathbf{y} \wedge \mathbf{x} = \mathbf{y} * (\mathbf{y} \to \mathbf{x}) = \mathbf{y} * \mathbf{x}.$$

From the previous items it follows that $\mathbf{A} \cong \mathbf{MV}(\mathbf{A}) \uplus \mathbf{D}(\mathbf{A}) = \mathbf{L}_n \uplus \mathbf{D}(\mathbf{A})$. □

1.5. Free algebras in varieties of BL-algebras generated by a BL$_n$-chain Recall that an algebra $\mathbf{A}$ in a variety $\mathcal{K}$ is said to be *free over a set* Y if and only if for every algebra $\mathbf{C}$ in $\mathcal{K}$ and every function $f : Y \to \mathbf{C}$, f can be uniquely extended to a homomorphism of $\mathbf{A}$ into $\mathbf{C}$. Given a variety $\mathcal{K}$ of algebras, we denote by $\mathbf{Free}_{\mathcal{K}}(X)$ the free algebra in $\mathcal{K}$ over X. As mentioned in the introduction, we define a *BL$_n$-chain* as a BL-chain that is an ordinal sum of the MV-chain $\mathbf{L}_n$ and a generalized BL-chain. Once we fixed the generalized BL-chain $\mathbf{B}$, we study the free algebra $\mathbf{Free}_{\mathcal{V}}(X)$, where $\mathcal{V}$ is the variety of BL-algebras generated by the BL$_n$-chain

$$\mathbf{T}_n := \mathbf{L}_n \uplus \mathbf{B}.$$

Notice that $\mathbf{MV}(\mathbf{T}_n) \cong \mathbf{L}_n$ and if $x \notin MV(\mathbf{T}_n) \setminus \{\top\}$, then $x \in D(\mathbf{T}_n) = B$.

Recall that a *weak Boolean product* of a family $(A_y, y \in Y)$ of algebras over a Boolean space Y is a subdirect product $\mathbf{A}$ of the given family such that the following conditions hold:

(1) If $a, b \in A$, then $[a = b] = \{y \in Y : a_y = b_y\}$ is open.
(2) If $a, b \in A$ and Z is a clopen in X, then $a|_Z \cup b|_{X \setminus Z} \in A$.

Since the variety $\mathcal{BL}$ is congruence distributive, it has the Boolean Factor Congruence property. Therefore each nontrivial BL-algebra can be represented as a weak Boolean product of directly indecomposable BL-algebras (see [5] and [23]). The explicit representation of each BL-algebra as a weak Boolean product of directly indecomposable algebras is given in [17] by the following lemma.

LEMMA 1.6. *Let* $\mathbf{A}$ *be a BL-algebra and let* $\operatorname{Sp}\mathbf{B}(\mathbf{A})$ *be the Boolean space of ultrafilters of the Boolean algebra* $\mathbf{B}(\mathbf{A})$. *The correspondence*

$$a \mapsto (a/\langle U\rangle)_{U\in \operatorname{Sp}\mathbf{B}(\mathbf{A})}$$

gives an isomorphism of $\mathbf{A}$ *onto the weak Boolean product of the family*

$$(\mathbf{A}/\langle U\rangle) : U \in \operatorname{Sp}\mathbf{B}(\mathbf{A})$$

over the Boolean space $\operatorname{Sp}\mathbf{B}(\mathbf{A})$. *This representation is called the Pierce representation. Any other representation of* $\mathbf{A}$ *as a weak Boolean product of a family of directly indecomposable algebras is equivalent to the Pierce representation.*

Therefore, to describe $\mathbf{Free}_{\mathcal{V}}(X)$ we need to describe $\mathbf{B}(\mathbf{Free}_{\mathcal{V}}(X))$ and the quotients $\mathbf{Free}_{\mathcal{V}}(X)/\langle U\rangle$ for each $U \in \operatorname{Sp}\mathbf{B}(\mathbf{Free}_{\mathcal{V}}(X))$.

In Section 2 we obtain a characterization of the Boolean algebra $\mathbf{B}(\mathbf{Free}_{\mathcal{V}}(X))$. Once this aim is achieved, we consider the quotients $\mathbf{Free}_{\mathcal{V}}(X)/\langle U\rangle$.

2. $\mathbf{B}(\mathbf{Free}_{\mathcal{V}}(X))$

The next two results can be found in [18].

THEOREM 2.1. *For each BL-algebra* $\mathbf{A}$, $\mathbf{B}(\mathbf{A}) \cong \mathbf{B}(\mathbf{MV}(\mathbf{A}))$.

THEOREM 2.2. *For each variety* $\mathcal{K}$ *of BL-algebras and each set* X

$$\mathbf{MV}(\mathbf{Free}_{\mathcal{K}}(X)) \cong \mathbf{Free}_{\mathcal{MV}\cap\mathcal{K}}(\neg\neg X).$$

THEOREM 2.3. $\mathcal{V} \cap \mathcal{MV}$ *is the variety* $\mathcal{MV}_n$.

PROOF. Since $\mathbf{L}_n \cong \mathbf{MV}(\mathbf{T_n})$ is in $\mathcal{V}\cap\mathcal{MV}$, we have that $\mathcal{MV}_n \subseteq \mathcal{V}\cap\mathcal{MV}$. On the other hand, let $\mathbf{A}$ be an MV-algebra in $\mathcal{V}\cap\mathcal{MV}$. Suppose $\mathbf{A}$ is not in $\mathcal{MV}_n$. Then there exists an equation $e(x_1,\dots,x_p) = \top$ that is satisfied by $\mathbf{L}_n$ and is not satisfied by $\mathbf{A}$, that is, there exist elements $a_1,\dots,a_p$ in A such that $e(a_1,\dots,a_p) \neq \top$. Since $(\neg\neg b_1,\dots,\neg\neg b_p)$ is in $(L_n)^p$, for each tuple $(b_1,\dots,b_p)$ in $(T_n)^p$, the equation $e'(x_1,\dots,x_p) = e(\neg\neg x_1,\dots,\neg\neg x_p) = \top$ is satisfied in $\mathcal{V}$. Since $\mathbf{A} \in \mathcal{V}\cap\mathcal{MV}$, it follows that $\top = e'(a_1,\dots,a_p) = e(\neg\neg a_1,\dots,\neg\neg a_p) = e(a_1,\dots,a_p) \neq \top$, a contradiction. Hence $\mathcal{MV}_n = \mathcal{V}\cap\mathcal{MV}$. □

From these results we obtain the following theorem.

THEOREM 2.4. $\mathbf{B}(\mathbf{Free}_{\mathcal{V}}(X)) \cong \mathbf{B}(\mathbf{Free}_{\mathcal{MV}_n}(\neg\neg X))$.

2.1. n-valued Moisil algebras Boolean elements of $\mathbf{Free}_{\mathcal{MV}_n}(\neg\neg X)$ depend on some operators that can be defined on each MV_n-algebra. Such operators provide each MV_n-algebra with an n-valued Moisil algebra structure, in the sense of the following definition.

DEFINITION 2.5. For each integer $n \geq 2$, an *n-valued Moisil algebra* ([8, 11]) or *n-valued Łukasiewicz algebra* ([4, 12, 13]) is an algebra

$$\mathbf{A} = (A, \wedge, \vee, \neg, \sigma_1^n, \dots, \sigma_{n-1}^n, 0, 1)$$

of type $(2, 2, 1, \dots, 1, 0, 0)$ such that $(A, \wedge, \vee, 0, 1)$ is a distributive lattice with unit 1 and zero 0, and $\neg, \sigma_1^n, \dots, \sigma_{n-1}^n$ are unary operators defined on A that satisfy the following conditions:

(1) $\neg\neg x = x$,
(2) $\neg(x \vee y) = \neg x \wedge \neg y$,
(3) $\sigma_i^n(x \vee y) = \sigma_i^n x \vee \sigma_i^n y$,
(4) $\sigma_i^n x \vee \neg\sigma_i^n x = 1$,
(5) $\sigma_i^n \sigma_j^n x = \sigma_j^n x$, for $i, j = 1, 2, \dots n-1$,
(6) $\sigma_i^n(\neg x) = \neg(\sigma_{n-i}^n x)$,
(7) $\sigma_i^n x \vee \sigma_{i+1}^n x = \sigma_{i+1}^n x$, for $i = 1, 2, \dots, n-2$,
(8) $x \vee \sigma_{n-1}^n x = \sigma_{n-1}^n x$,
(9) $(x \wedge \neg\sigma_i^n x \wedge \sigma_{i+1}^n y) \vee y = y$, for $i = 1, 2, \dots, n-2$.

Properties and examples of n-valued Moisil algebras can be found in [4] and [8]. The variety of n-valued Moisil algebras will be denoted $\mathcal{M}_n$.

THEOREM 2.6 (see [11]). *Let* $\mathbf{A}$ *be in* $\mathcal{M}_n$. *Then* $x \in B(\mathbf{A})$ *if and only if*

$$\sigma_{n-1}^n(x) = x.$$

Furthermore,

$$\sigma_{n-1}^n(x) = \min\{b \in B(\mathbf{A}) : x \leq b\} \quad \text{and} \quad \sigma_1^n(x) = \max\{a \in B(\mathbf{A}) : a \leq x\}.$$

DEFINITION 2.7. For each integer $n \geq 2$, a *Post algebra of order n* is a system

$$\mathbf{A} = (A, \wedge, \vee, \neg, \sigma_1^n, \dots, \sigma_{n-1}^n, e_1, \dots, e_{n-1}, 0, 1)$$

such that $(A, \wedge, \vee, \neg, \sigma_1^n, \dots, \sigma_{n-1}^n, 0, 1)$ is an n-valued Moisil algebra and $e_1, \dots, e_{n-1}$ are constants that satisfy the equations:

$$\sigma_i^n(e_j) = \begin{cases} 0 & \text{if } i + j < n, \\ 1 & \text{if } i + j \geq n. \end{cases}$$

For every $n \geq 2$, we can define one-variable terms $\sigma_1^n(x), \ldots, \sigma_{n-1}^n(x)$ in the language $(\neg, \rightarrow, \top)$ such that evaluated on the algebras $\mathbf{L}_n$ give

$$\sigma_i^n\left(\frac{j}{(n-1)}\right) = \begin{cases} 1 & \text{if } i+j \geq n, \\ 0 & \text{if } i+j < n, \end{cases}$$

for $i = 1, \ldots, n-1$ (see [13] or [24]). It is easy to check that

$$\mathbf{M}(\mathbf{L}_n) = (L_n, \wedge, \vee, \neg, \sigma_1^n, \ldots, \sigma_{n-1}^n, 0, 1)$$

is a n-valued Moisil algebra. Since these algebras are defined by equations and $\mathbf{L}_n$ generates the variety $\mathcal{MV}_n$, we have that each $\mathbf{A} \in \mathcal{MV}_n$ admits a structure of an n-valued Moisil algebra, denoted by $\mathbf{M}(\mathbf{A})$. The chain $\mathbf{M}(\mathbf{L}_n)$ plays a very important role in the structure of n-valued Moisil algebras, since each n-valued Moisil algebra is a subdirect product of subalgebras of $\mathbf{M}(\mathbf{L}_n)$ (see [4] or [12]). If we add to the structure $\mathbf{M}(\mathbf{L}_n)$ the constants $e_i = i/(n-1)$, for $i = 1, \ldots, n-1$, then $\mathbf{PT}(\mathbf{L}_n) = (L_n, \wedge, \vee, \neg, \sigma_1^n, \ldots, \sigma_{n-1}^n, e_1, \ldots, e_{n-1}, 0, 1)$ is a Post algebra.

Not every n-valued Moisil algebra has a structure of MV_n-algebra (see [21]). For example, a subalgebra of $\mathbf{M}(\mathbf{L}_n)$ may not be a subalgebra of $\mathbf{L}_n$ as MV_n-algebra. For instance, the set

$$C = \left\{\frac{0}{4}, \frac{1}{4}, \frac{3}{4}, \frac{4}{4}\right\}$$

is the universe of a subalgebra of $\mathbf{M}(\mathbf{L}_5)$, but not the universe of a subalgebra of $\mathbf{L}_5$. On the other hand, every Post algebra has a structure of MV_n-algebra (see [24, Theorem 10]).

The next example will play an important role in what follows.

EXAMPLE 2.8. Let $\mathbf{C} = (C, \wedge, \vee, \neg, 0, 1)$ be a Boolean algebra. We define

$$C^{[n]} := \{\mathbf{z} = (z_1, \ldots, z_{n-1}) \in C^{n-1} : z_1 \leq z_2 \leq \ldots \leq z_{n-1}\}.$$

For each $\mathbf{z} = (z_1, \ldots, z_{n-1}) \in C^{[n]}$, we define

$$\begin{aligned} \neg_n \mathbf{z} &= (\neg z_{n-1}, \ldots, \neg z_1), \\ \mathbf{0} &= (0, \ldots, 0), \\ \mathbf{1} &= (1, \ldots, 1), \\ \sigma_i^n(\mathbf{z}) &= (z_i, z_i, \ldots, z_i) \quad \text{for } i = 1, \ldots, n-1. \end{aligned}$$

With $\wedge$ and $\vee$ defined coordinatewise, $\mathbf{C}^{[n]} = (C^{[n]}, \wedge, \vee, \neg_n, \sigma_1^n, \ldots, \sigma_{n-1}^n, \mathbf{0}, \mathbf{1})$ is an n-valued Moisil algebra (see [8, Chapter 3, Example 1.10]). If we define $\mathbf{e_j} = (e_{j,1}, \ldots, e_{j,n-1})$ by

$$e_{j,i} = \begin{cases} 0 & \text{if } i < j, \\ 1 & \text{if } i \geq j, \end{cases}$$

then $\mathbf{C}^{[n]} = (C^{[n]}, \wedge, \vee, \neg_n, \sigma_1^n, \ldots, \sigma_{n-1}^n, \mathbf{e}_1, \ldots, \mathbf{e}_{n-1}, \mathbf{0}, \mathbf{1})$ is a Post algebra. Consequently, $\mathbf{C}^{[n]}$ has a structure of MV$_n$-algebra.

It is easy to see that for each MV$_n$-algebra $\mathbf{A}$, $\mathbf{B}(\mathbf{A}) = \mathbf{B}(\mathbf{M}(\mathbf{A}))$.

We need to show that the Boolean elements of the MV$_n$-algebra generated by a set G coincide with the Boolean elements of the n-valued Moisil algebra generated by the same set. In order to prove this result it is convenient to consider the following operators on each n-valued Moisil algebra $\mathbf{A}$. For each $i = 0, \ldots, n-1$,

$$J_i(x) = \sigma_{n-i}^n(x) \wedge \neg\sigma_{n-i-1}^n(x),$$

where $\sigma_0^n(x) = 0$ and $\sigma_n^n(x) = 1$. In $\mathbf{M}(\mathbf{L}_n)$ we have

$$J_i\left(\frac{j}{(n-1)}\right) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

LEMMA 2.9. *Let* $\mathbf{A}$ *be an* MV_n*-algebra, and let* $G \subset A$. *If* $\langle G\rangle_{\mathcal{MV}_n}$ *is the subalgebra of* $\mathbf{A}$ *generated by the set* G *and* $\langle G\rangle_{\mathcal{M}_n}$ *is the subalgebra of* $\mathbf{M}(\mathbf{A})$ *generated by* G, *then* $\mathbf{B}(\langle G\rangle_{\mathcal{MV}_n}) = \mathbf{B}(\langle G\rangle_{\mathcal{M}_n})$.

PROOF. Since $\langle G\rangle_{\mathcal{M}_n}$ is always a subalgebra of $\mathbf{M}(\langle G\rangle_{\mathcal{MV}_n})$, we have that $\mathbf{B}(\langle G\rangle_{\mathcal{M}_n})$ is a subalgebra of $\mathbf{B}(\langle G\rangle_{\mathcal{MV}_n})$.

We will see that $B(\langle G\rangle_{\mathcal{MV}_n}) \subseteq B(\langle G\rangle_{\mathcal{M}_n})$. The case $G = \emptyset$ is clear. Suppose that G is a finite set of cardinality $p \geq 1$. Since MV$_n$-algebras are locally finite (see [9, Chapter II, Theorem 10.16]), we obtain that $\langle G\rangle_{\mathcal{MV}_n}$ is a finite MV$_n$-algebra. Since finite MV$_n$-algebras are direct products of simple algebras, there exists a finite $k \geq 1$ such that $\langle G\rangle_{\mathcal{MV}_n} = \prod_{i=1}^k \mathbf{L}_{m_i}$, where each $m_i - 1$ divides $n-1$, for each $i = 1, \ldots, k$. If $k = 1$, then $\langle G\rangle_{\mathcal{M}_n}$ and $\langle G\rangle_{\mathcal{MV}_n}$ are finite chains whose only Boolean elements are their extremes. Otherwise, we can think of the elements of $\langle G\rangle_{\mathcal{MV}_n}$ as k-tuples, that is, if $\mathbf{x} \in \langle G\rangle_{\mathcal{MV}_n}$, then $\mathbf{x} = (x_1, \ldots, x_k)$. We shall denote by $\mathbf{1}^j$ the k-tuple given by

$$(\mathbf{1}^j)_i = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

It is clear that for each $j = 1, \ldots, k$, $\mathbf{1}^j$ is in $\langle G\rangle_{\mathcal{MV}_n}$. From this it follows that for every pair $i \neq j$, $i, j \in \{1, \ldots, k\}$, there exists an element $\mathbf{x} \in G$ such that $x_j \neq x_i$. Indeed, suppose on the contrary that there exist $i, j \leq k$ such that $x_i = x_j$, for every $\mathbf{x} \in G$. Then for every $\mathbf{z} \in \langle G\rangle_{\mathcal{MV}_n}$ we would have $z_j = z_i$ contradicting the fact that $\mathbf{1}^i$ is in $\langle G\rangle_{\mathcal{MV}_n}$.

To see that every Boolean element in $\langle G\rangle_{\mathcal{MV}_n}$ is also in $\langle G\rangle_{\mathcal{M}_n}$, it is enough to prove that $\mathbf{1}^j$ is in $\langle G\rangle_{\mathcal{M}_n}$ for every $j = 1, \ldots, k$. For a fixed j, for each $i \neq j$,

$i = 1, \dots k$, we choose $\mathbf{x}^i \in G$ such that $x^i_j \neq x^i_i$. Let j_i be the numerator of $x^i_j \in L_n$. It is not hard to verify that

$$\mathbf{1}^j = \bigwedge_{i=1, i\neq j}^{k} J_{j_i}(\mathbf{x}^i).$$

Therefore $\mathbf{1}^j \in \langle G\rangle_{\mathcal{M}_n}$ and $B(\langle G\rangle_{\mathcal{MV}_n}) \subseteq B(\langle G\rangle_{\mathcal{M}_n})$.

If G is not finite, let $\mathbf{y}$ be a Boolean element in $\langle G\rangle_{\mathcal{MV}_n}$. Hence, there exists a finite subset $G_{\mathbf{y}}$ of G such that $\mathbf{y}$ belongs to the subalgebra of $\langle G\rangle_{\mathcal{MV}_n}$ generated by $G_{\mathbf{y}}$. Therefore, since $\mathbf{y}$ is Boolean, $\mathbf{y}$ belongs to the subalgebra of $\langle G\rangle_{\mathcal{M}_n}$ generated by $G_{\mathbf{y}}$, and we conclude that $B(\langle G\rangle_{\mathcal{MV}_n}) \subseteq B(\langle G\rangle_{\mathcal{M}_n})$ for all sets G. □

Given an algebra $\mathbf{A}$ in a variety $\mathcal{K}$, a subalgebra $\mathbf{S}$ of $\mathbf{A}$, and an element $x \in A$, we shall denote by $\langle \mathbf{S}, x\rangle_{\mathcal{K}}$ the subalgebra of $\mathbf{A}$ generated by the set $S \cup \{x\}$ in $\mathcal{K}$.

LEMMA 2.10. *Let* $\mathbf{C}$ *be in* $\mathcal{M}_n$ *and* $x \in C$. *Let* $\mathbf{S}$ *be a subalgebra of* $\mathbf{C}$ *such that* $\sigma^n_i(x)$ *belongs to* $B(\mathbf{S})$ *for each* $i = 1, \dots, n-1$. *Then* $\mathbf{B}(\langle \mathbf{S}, x\rangle_{\mathcal{M}_n}) = \mathbf{B}(\mathbf{S})$.

PROOF. Clearly $\mathbf{B}(\mathbf{S})$ is a subalgebra of $\mathbf{B}(\langle \mathbf{S}, x\rangle_{\mathcal{M}_n})$. It is left to check that $B(\langle \mathbf{S}, x\rangle_{\mathcal{M}_n}) \subseteq B(\mathbf{S})$. To achieve this aim, we shall study the form of the elements in $\langle \mathbf{S}, x\rangle_{\mathcal{M}_n}$. We define for each $s \in S$,

$$\begin{aligned}
\alpha(s) &= s \wedge x,\\
\beta(s) &= s \wedge \neg x,\\
\gamma_i(s) &= s \wedge \sigma^n_i(x), \qquad \text{for } i = 1, \dots n-1,\\
\delta_i(s) &= s \wedge \neg\sigma^n_i(x), \quad \text{for } i = 1, \dots n-1.
\end{aligned}$$

For all $s \in S$ we have that $\gamma_i(s)$ and $\delta_i(s)$ are in S for $i = 1, \dots, n-1$. Let

$$M := \left\{ y = \bigvee_{j=1}^{k_y} \bigwedge_{i=1}^{p_j} f_i(s_i) : f_i \in \{\alpha, \beta, \gamma_1, \delta_1, \dots \gamma_{n-1}, \delta_{n-1}\} \text{ and } s_i \in S \right\}.$$

We shall see that $\langle \mathbf{S}, x\rangle_{\mathcal{M}_n} = \mathbf{M} = (M, \wedge, \vee, \neg, \sigma^n_1, \dots, \sigma^n_{n-1}, 0, 1)$. Indeed, for all $s \in S$, $s = \gamma_1(s) \vee \delta_1(s)$, and then $S \subseteq M$. Besides, $x \in M$ because $x = \alpha(1)$. Lastly, it is easy to see that M is closed under the operations of n-valued Moisil algebra. Thus $\langle \mathbf{S}, x\rangle_{\mathcal{M}_n}$ is a subalgebra of $\mathbf{M}$. From the definition of M, it is obvious that $M \subseteq \langle \mathbf{S}, x\rangle_{\mathcal{M}_n}$, and the equality follows.

Now let $z \in B(\langle \mathbf{S}, x\rangle_{\mathcal{M}_n})$. By Theorem 2.6, $\sigma^n_{n-1}(z) = z$ and $z = \bigvee_{j=1}^{k_z} \bigwedge_{i=1}^{p_j} f_i(s_i)$ with $f_i \in \{\alpha, \beta, \gamma_1, \delta_1, \dots, \gamma_{n-1}, \delta_{n-1}\}$ and $s_i \in S$. Then we have

$$z = \sigma^n_{n-1}(z) = \sigma^n_{n-1}\left(\bigvee_{j=1}^{k_z} \bigwedge_{i=1}^{p_j} f_i(s_i)\right) = \bigvee_{j=1}^{k_z} \bigwedge_{i=1}^{p_j} \sigma^n_{n-1}(f_i(s_i)),$$

is in $B(\mathbf{S})$ because $\sigma_{n-1}^n(f_i(s_i)) = \gamma_k(\sigma_{n-1}^n(s_i))$ or $\sigma_{n-1}^n(f_i(s_i)) = \delta_k(\sigma_{n-1}^n(s_i))$, for some $k = 1, \ldots, n-1$. □

THEOREM 2.11. *Let* $\mathbf{C}$ *be an* MV_n*-algebra and* $x \in \mathbf{C}$. *Let* $\mathbf{S}$ *be a subalgebra of* $\mathbf{C}$ *such that* $\sigma_i^n(x)$ *belongs to* $B(\mathbf{S})$ *for each* $i = 1, \ldots, n-1$. *Then*

$$\mathbf{B}(\langle \mathbf{S}, x\rangle_{\mathcal{MV}_n}) = \mathbf{B}(\mathbf{S}).$$

PROOF. By Lemma 2.9 and Lemma 2.10 we obtain $\mathbf{B}(\langle \mathbf{S}, x\rangle_{\mathcal{MV}_n}) = \mathbf{B}(\langle \mathbf{S}, x\rangle_{\mathcal{M}_n}) = \mathbf{B}(\mathbf{S})$. □

2.2. Boolean elements in $\mathbf{Free}_{\mathcal{MV}_n}(Z)$ Recall that a Boolean algebra $\mathbf{B}$ is said to be *free over a poset* Y if for each Boolean algebra $\mathbf{C}$ and for each non-decreasing function $f : Y \to \mathbf{C}$, f can be uniquely extended to a homomorphism from $\mathbf{B}$ into $\mathbf{C}$.

THEOREM 2.12. $\mathbf{B}(\mathbf{Free}_{\mathcal{MV}_n}(Z))$ *is the free Boolean algebra over the poset* $Z' := \{\sigma_i^n(z) : z \in Z, i = 1, \ldots, n-1\}$.

PROOF. Let $\mathbf{S}$ be the subalgebra of $\mathbf{B}(\mathbf{Free}_{\mathcal{MV}_n}(Z))$ generated by Z'. Let $\mathbf{C}$ be a Boolean algebra and let $f : Z' \to \mathbf{C}$ be a non-decreasing function. The monotonicity of f implies that the prescription

$$f'(z) = (f(\sigma_1^n(z)), \ldots, f(\sigma_{n-1}^n(z)))$$

defines a function $f' : Z \to \mathbf{C}^{[n]}$. Since $\mathbf{C}^{[n]} \in \mathcal{MV}_n$, there is a unique homomorphism $h' : \mathbf{Free}_{\mathcal{MV}_n}(Z) \to \mathbf{C}^{[n]}$ such that $h'(z) = f'(z)$ for every $z \in Z$. Let $\pi : \mathbf{C}^{[n]} \to \mathbf{C}$ be the projection over the first coordinate. The composition $\pi \circ h'$ restricted to $\mathbf{S}$ is a homomorphism $h : \mathbf{S} \to \mathbf{C}$, and for $y = \sigma_j^n(z) \in Z'$ we have

$$\begin{aligned} h(y) &= \pi(h'(\sigma_j^n(z))) = \pi(\sigma_j^n(h'(z))) = \pi(\sigma_j^n(f'(z))) \\ &= \pi(\sigma_j^n(f(\sigma_1^n(z)), \ldots, f(\sigma_{n-1}^n(z)))) \\ &= \pi(f(\sigma_j^n(z)), \ldots, f(\sigma_j^n(z))) = f(\sigma_j^n(z)) = f(y). \end{aligned}$$

Hence $\mathbf{S}$ is the free Boolean algebra over the poset Z'. However, since $\sigma_j^n(z)$ is in $\mathbf{S}$ for all $z \in Z$ and $j = 1, \ldots n-1$, Theorem 2.11 asserts that

$$\mathbf{S} = \mathbf{B}(\mathbf{S}) = \mathbf{B}(\langle \mathbf{S}, z\rangle_{\mathcal{MV}_n})$$

for every $z \in Z$. From the fact that $\mathbf{S}$ is a subalgebra of $\mathbf{B}(\mathbf{Free}_{\mathcal{MV}_n}(Z))$ we obtain

$$\mathbf{S} = \mathbf{B}(\langle \mathbf{S}, Z\rangle_{\mathcal{MV}_n}) = \mathbf{B}(\mathbf{Free}_{\mathcal{MV}_n}(Z))$$

that is, $\mathbf{B}(\mathbf{Free}_{\mathcal{MV}_n}(Z))$ is the free Boolean algebra generated by the poset Z'. □

From Theorem 2.4 we obtain the follwoing result.

COROLLARY 2.13. $\mathbf{B}(\mathbf{Free}_{\mathcal{V}}(X))$ *is the free Boolean algebra generated by the poset* $Y := \{\sigma_i^n(\neg\neg x) : x \in X, i = 1, \ldots, n-1\}$.

REMARK 2.14. If $n = 2$, that is, the variety considered $\mathcal{V}$ is generated by a BL_2-chain, then $\sigma_1^2(x) = x$ for each $x \in X$. Therefore, in this case, $Y = \{\neg\neg x : x \in X\}$, and the cardinality of Y equals the cardinality of X. It follows that $\mathbf{B}(\mathbf{Free}_{\mathcal{V}}(X))$ is the free Boolean algebra over the set Y.

3. $\mathbf{Free}_{\mathcal{V}}(X)/\langle U\rangle$

Following the program established at the end of Section 2, our next aim is to describe $\mathbf{Free}_{\mathcal{V}}(X)/\langle U\rangle$ for each ultrafilter U in the free Boolean algebra generated by $Y = \{\sigma_i^n(\neg\neg x) : x \in X, i = 1, \ldots, n-1\}$, where $\langle U\rangle$ is the BL-filter generated by the Boolean filter U.

The plan is to prove that $\mathbf{MV}(\mathbf{Free}_{\mathcal{V}}(X)/\langle U\rangle)$ is a subalgebra of $\mathbf{L}_n$ and then, using Theorem 1.5, decompose each quotient $\mathbf{Free}_{\mathcal{V}}(X)/\langle U\rangle$ into an ordinal sum. To accomplish this we need the following results.

THEOREM 3.1. *Let* $\mathbf{A}$ *be a BL-algebra and* $U \in Sp\,\mathbf{B}(\mathbf{A})$. *Then*

$$\mathbf{MV}(\mathbf{A}/\langle U\rangle) \cong \mathbf{MV}(\mathbf{A})/(\langle U\rangle \cap \mathbf{MV}(\mathbf{A})).$$

PROOF. Let $V =: \langle U\rangle \cap \mathbf{MV}(\mathbf{A})$ and let $f : \mathbf{MV}(\mathbf{A})/V \to \mathbf{MV}(\mathbf{A}/\langle U\rangle)$ be given by $f(a/V) = a/\langle U\rangle$, for each $a \in MV(\mathbf{A})$. It is easy to see that f is a homomorphism into $\mathbf{MV}(\mathbf{A}/\langle U\rangle)$. We have that

(1) *f is injective.*

Let $a/\langle U\rangle = b/\langle U\rangle$, with $a, b \in MV(\mathbf{A})$. From Lemma 1.3, we know that there exists $u \in U$ such that $a \wedge u = b \wedge u$. Since $U \subseteq MV(\mathbf{A})$, we have that $u \in V$. From the fact that u is Boolean (see [17, Lemma 2.2]), we have that $a * u = a \wedge u = b \wedge u \leq b$, thus $u \leq a \to b$ and similarly $u \leq b \to a$. Then $a \to b$ and $b \to a$ are in V and this means that $a/V = b/V$.

(2) *f is surjective.*

Let $a/\langle U\rangle \in MV(\mathbf{A}/\langle U\rangle)$. Then $a/\langle U\rangle = \neg\neg(a/\langle U\rangle) = \neg\neg a/\langle U\rangle$, and since $\neg\neg a \in MV(\mathbf{A})$ we obtain that $f(\neg\neg a/V) = a/\langle U\rangle$. □

By Theorem 2.4, if $U \in \mathrm{Sp}\,\mathbf{B}(\mathbf{Free}_{\mathcal{V}}(X))$, then U is an ultrafilter in

$$\mathbf{B}(\mathbf{Free}_{\mathcal{MV}_n}(\neg\neg X)).$$

Moreover, $\langle U\rangle \cap \mathbf{MV}(\mathbf{Free}_{\mathcal{V}}(X)) = \langle U\rangle \cap \mathbf{Free}_{\mathcal{MV}_n}(\neg\neg X)$ is the Stone ultrafilter of $\mathbf{Free}_{\mathcal{MV}_n}(\neg\neg X)$ generated by U. From [14, Chapter 6.3], we have that

$$\langle U\rangle \cap \mathbf{Free}_{\mathcal{MV}_n}(\neg\neg X)$$

is a maximal filter of $\mathbf{Free}_{\mathcal{MV}_n}(\neg\neg X)$. It follows from [14, Corollary 3.5.4] that the MV-algebra $\mathbf{MV}(\mathbf{Free}_{\mathcal{V}}(X))/(\langle U\rangle \cap \mathbf{MV}(\mathbf{Free}_{\mathcal{V}}(X)))$ is an MV-chain in $\mathcal{MV}_n$, thus from Theorem 3.1 we have the follwoing result.

THEOREM 3.2. $\mathbf{MV}(\mathbf{Free}_{\mathcal{V}}(X)/\langle U\rangle) \cong \mathbf{L}_s$ *with* $s-1$ *dividing* $n-1$.

From Theorems 1.5 and 3.2 we obtain the next result.

THEOREM 3.3. *For each* $U \in \operatorname{Sp}\mathbf{B}(\mathbf{Free}_{\mathcal{V}}(X))$, *we have that*

$$\mathbf{Free}_{\mathcal{V}}(X)/\langle U\rangle \cong \mathbf{L}_s \uplus \mathbf{D}(\mathbf{Free}_{\mathcal{V}}(X)/\langle U\rangle)$$

for some $s-1$ *dividing* $n-1$.

In order to complete the description of $\mathbf{Free}_{\mathcal{V}}(X)$ we have to find a description of $\mathbf{D}(\mathbf{Free}_{\mathcal{V}}(X)/\langle U\rangle)$ for each $U \in \operatorname{Sp}\mathbf{B}(\mathbf{Free}_{\mathcal{V}}(X))$. This last description depends on the characterization of the variety $\mathcal{W}$ of generalized BL-algebras generated by the generalized BL-chain $\mathbf{B}$. Therefore, we shall firstly consider such variety.

3.1. The subvariety of $\mathcal{GBL}$ generated by B We recall that $\mathcal{V}$ is the variety of BL-algebras generated by the BL-chain $\mathbf{T}_n = \mathbf{L}_n \uplus \mathbf{B}$. Let $\mathcal{W}$ be the variety of generalized BL-algebras generated by the chain $\mathbf{B}$.

Let $\{e_i,\ i \in I\}$ be the set of equations that define $\mathcal{MV}_n$ as a subvariety of $\mathcal{BL}$, and $\{d_j,\ j \in J\}$ be the set of equations that define $\mathcal{W}$ as a subvariety of $\mathcal{GBL}$. For each $i \in I$, let e_i' be the equation that results from substituting $\neg\neg x$ for each variable x in e_i, and for each $j \in J$, let d_j' be the equation that results from substituting $\neg\neg x \to x$ for each variable x in the equation d_j. Let $\mathcal{V}'$ denote the variety of BL-algebras characterized by the equations of BL-algebras plus the equations $\{e_i',\ i \in I\} \cup \{d_j',\ j \in J\}$.

THEOREM 3.4. $\mathcal{V}' \subseteq \mathcal{V}$.

PROOF. Let $\mathbf{A}$ be a subdirectly irreducible BL-algebra in $\mathcal{V}'$. From Theorem 1.1, $\mathbf{A}$ is a BL-chain, and by Theorem 1.4, $\mathbf{A} = \mathbf{MV}(\mathbf{A}) \uplus \mathbf{D}(\mathbf{A})$. Since for each $x \in MV(\mathbf{A})$, we have $\neg\neg x = x$, $\mathbf{MV}(\mathbf{A})$ satisfies equations $\{e_i,\ i \in I\}$. Then $\mathbf{MV}(\mathbf{A})$ is a chain in $\mathcal{MV}_n$, that is, $\mathbf{MV}(\mathbf{A}) \cong \mathbf{L}_s$, with $s-1$ dividing $n-1$. Moreover, since for each $x \in D(\mathbf{A})$, we have $\neg\neg x \to x = x$, $\mathbf{D}(\mathbf{A})$ satisfies equations $\{d_j,\ j \in J\}$. Hence

$\mathbf{D}(\mathbf{A}) = \mathbf{C}$ is a generalized BL-chain in $\mathcal{W}$. Since $\mathbf{A}$ is subdirectly irreducible, $\mathbf{C}$ is also subdirectly irreducible, and since $\mathcal{GBL}$ is a congruence distributive variety, we can apply Jónsson's Lemma (see [9]) to conclude that $\mathbf{C} \in \mathbf{HSP}_u(\mathbf{B})$. Hence there is a set $J \neq \emptyset$ and an ultrafilter U over J such that $\mathbf{C}$ is a homomorphic image of a subalgebra of $\mathbf{B}^J/U$. From the proof of [2, Proposition 3.3] it follows that $(\mathbf{L}_n \uplus \mathbf{B})^J/U = \mathbf{L}_n^J/U \uplus \mathbf{B}^J/U$, and since $\mathbf{L}_n$ is finite, $\mathbf{L}_n^J/U \cong \mathbf{L}_n$. Now it is easy to see that $\mathbf{A} = \mathbf{L}_s \uplus \mathbf{C} \in \mathbf{HSP}_u(\mathbf{L}_n \uplus \mathbf{B}) \subseteq \mathcal{V}$. □

The next corollary states the main result of this section.

COROLLARY 3.5. *The variety $\mathcal{W}$ of generalized BL-algebras generated by* $\mathbf{B}$ *consists of the generalized BL-algebras* $\mathbf{C}$ *such that* $\mathbf{L}_n \uplus \mathbf{C}$ *belongs to* $\mathcal{V}$.

PROOF. Given $\mathbf{C} \in \mathcal{W}$, $\mathbf{L}_n \uplus \mathbf{C} \in \mathcal{V}' \subseteq \mathcal{V}$. On the other hand, if $\mathbf{C}$ is a generalized BL-algebra such that $\mathbf{L}_n \uplus \mathbf{C} \in \mathcal{V}$, then the elements of C satisfy equations d'_j for each $j \in J$ and since $\neg\neg x \to x = x$ for each $x \in C$, the elements of C satisfy equations d_j for each $j \in J$. Hence $\mathbf{C}$ is in $\mathcal{W}$. □

3.2. $\mathbf{D}(\mathbf{Free}_{\mathcal{V}}(X)/\langle U\rangle)$ We know that the ultrafilters of a Boolean algebra are in bijective correspondence with the homomorphisms from the algebra into the two elements Boolean algebra, $\mathbf{2}$. Since every upwards closed subset of the poset $Y = \{\sigma_i^n(\neg\neg x) : x \in X, i = 1, \ldots, n-1\}$ is in correspondence with an increasing function from Y onto $\mathbf{2}$, and every increasing function from Y can be extended to a homomorphism from $\mathbf{B}(\mathbf{Free}_{\mathcal{V}}(X))$ into $\mathbf{2}$, the ultrafilters of $\mathbf{B}(\mathbf{Free}_{\mathcal{V}}(X))$ are in correspondence with the upwards closed subsets of Y. This is summarized in the following lemma.

LEMMA 3.6. *Consider the poset* $Y = \{\sigma_i^n(\neg\neg x) : x \in X,\ i = 1, \ldots, n-1\}$. *The correspondence that assigns to each upwards closed subset* $S \subseteq Y$ *the Boolean filter* U_S *generated by the set* $S \cup \{\neg y : y \in Y \setminus S\}$, *defines a bijection from the set of upwards closed subsets of* Y *onto the ultrafilters of* $\mathbf{B}(\mathbf{Free}_{\mathcal{V}}(X))$.

We shall refer to each member of $\operatorname{Sp}\mathbf{B}(\mathbf{Free}_{\mathcal{V}}(X))$ by U_S making explicit reference to the upwards closed subset S that corresponds to it.

LEMMA 3.7. *Let* $\mathbf{F}_S$ *be the subalgebra of the generalized BL-algebra*

$$\mathbf{D}(\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S\rangle)$$

generated by the set $X_S := \{x/\langle U_S\rangle : x \in X, \neg\neg x \in \langle U_S\rangle\}$. *Then*

$$\mathbf{F}_S = \mathbf{D}(\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S\rangle).$$

PROOF. $\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S\rangle$ is the BL-algebra generated by the set $Z_S = \{x/\langle U_S\rangle : x \in X\}$. From Theorem 3.3, there exists an integer m such that

$$\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S\rangle = \mathbf{L}_m \uplus \mathbf{D}(\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S\rangle).$$

Hence each element of Z_S is either in $L_m \setminus \{\top\}$ or it is in $D(\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S\rangle)$.

If $X_S = \emptyset$, then $F_S = D(\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S\rangle) = \{\top\}$. So let us suppose $X_S \neq \emptyset$. Let $y \in D(\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S\rangle)$. Recalling that $\mathbf{F}_S$ is the generalized BL-algebra generated by X_S, we will check that y is in F_S. Since $y \in \mathbf{Free}_{\mathcal{V}}(X)/\langle U_S\rangle$, y is given by a term on the elements $x/\langle U_S\rangle \in Z_S$. By induction on the complexity of y, we have:

- If y is a generator, that is, $y = x/\langle U_S\rangle$ for some $x/\langle U_S\rangle \in Z_S$, since $y \in D(\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S\rangle)$, we have that $\top = \neg\neg y = \neg\neg(x/\langle U_S\rangle) = (\neg\neg x)/\langle U_S\rangle$. This happens only if $\neg\neg x \in \langle U_S\rangle$.
- Suppose that for each element $z \in D(\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S\rangle)$ of complexity less than k, z can be written as a term in the variables $x/\langle U_S\rangle$ in X_S. Let $y \in D(\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S\rangle)$ be an element of complexity k. The possible cases are the following:
  (1) $y = a \to b$ for some elements a, b of complexity $< k$. In this case the possibilities are
    (a) $a \leq b$. This means $a \to b = \top$ and y can be written as $x/\langle U_S\rangle \to x/\langle U_S\rangle$ for any $x/\langle U_S\rangle \in X_S$, and thus $y \in F_S$,
    (b) $a \not\leq b$. Since $y = a \to b$ is in $D(\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S\rangle)$, the only possibility is that $a, b \in D(\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S\rangle)$ and by inductive hypothesis y is in F_S.
  (2) $y = a * b$ for some elements a, b of complexity $< k$. In this case necessarily $a, b \in D(\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S\rangle)$ and by inductive hypothesis y is in F_S.

Then for each $y \in D(\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S\rangle)$, y can be written as a term on the elements of X_S. Therefore $y \in F_S$ and we conclude that $\mathbf{F}_S = \mathbf{D}(\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S\rangle)$. □

With the notation of the previous lemma, we have the following theorem.

THEOREM 3.8. *For each U_S in* $\operatorname{Sp}\mathbf{B}(\mathbf{Free}_{\mathcal{V}}(X))$,

$$\mathbf{D}(\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S\rangle) \cong \mathbf{Free}_{\mathcal{W}}(X_S).$$

PROOF. From Theorem 2.6 and Lemma 3.6 we can deduced that $\neg\neg x \in \langle U_S\rangle$ if and only if $\sigma_1^n(\neg\neg x) \in S$ if and only if $\sigma_i^n(\neg\neg x) \in S$ for $i = 1, \ldots, n-1$. Hence if $\neg\neg x \notin \langle U_S\rangle$ there is a j such that $\sigma_j^n(\neg\neg x) \notin S$. We define, for each $x \in X$,

$$j_x = \begin{cases} \bot & \text{if } \neg\neg x \in \langle U_S\rangle, \\ \max\{i \in \{1, \ldots, n-1\} : \sigma_i^n(\neg\neg x) \notin S\} & \text{otherwise.} \end{cases}$$

Let $\mathbf{C} \in \mathcal{W}$ and let $\mathbf{C}' = \mathbf{L}_n \uplus \mathbf{C}$. From Theorem 3.5, $\mathbf{C}'$ is in $\mathcal{V}$. Given a function $f : X_S \to \mathbf{C}$, define $\hat{f} : X \to \mathbf{C}'$ by the prescriptions:

$$\hat{f}(x) = \begin{cases} f(x/\langle U_S \rangle) & \text{if } \neg\neg x \in \langle U_S \rangle, \\ (n - j_x - 1)/(n-1) & \text{otherwise.} \end{cases}$$

There is a unique homomorphism $\hat{h} : \mathbf{Free}_{\mathcal{V}}(X) \to \mathbf{C}'$ such that $\hat{h}(x) = \hat{f}(x)$ for each $x \in X$. We have that $U_S \subseteq \hat{h}^{-1}(\{\top\})$. Indeed, if $\neg\neg x \in \langle U_S \rangle$, then $\hat{h}(\sigma_i^n(\neg\neg x)) = \sigma_i^n(\neg\neg(\hat{h}(x)) = \sigma_i^n(\neg\neg f(x/\langle U_S \rangle)) = \sigma_i^n(\top) = \top$. If $\neg\neg x \notin \langle U_S \rangle$, then

$$\hat{h}(\sigma_i^n(\neg\neg x)) = \sigma_i^n\left(\neg\neg\frac{n - j_x - 1}{n-1}\right) = \sigma_i^n\left(\frac{n - j_x - 1}{n-1}\right) = \begin{cases} \bot & \text{if } i \leq j_x, \\ \top & \text{otherwise.} \end{cases}$$

Hence there is a unique homomorphism $h_1 : \mathbf{Free}_{\mathcal{V}}(X)/\langle U_S \rangle \to \mathbf{C}'$ such that $h_1(a/\langle U_S \rangle) = \hat{h}(a)$ for all $a \in \mathbf{Free}_{\mathcal{V}}(X)$. By Lemma 3.7, $\mathbf{D}(\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S \rangle)$ is the algebra generated by X_S. Then the restriction h of h_1 to $\mathbf{D}(\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S \rangle)$ is a homomorphism $h : \mathbf{D}(\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S \rangle) \to \mathbf{C}$, and for each x such that $\neg\neg x \in \langle U_S \rangle$,

$$h(x/\langle U_S \rangle) = h_1(x/\langle U_S \rangle) = \hat{h}(x) = \hat{f}(x) = f(x/\langle U_S \rangle).$$

Therefore we conclude that $\mathbf{D}(\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S \rangle) \cong \mathbf{Free}_{\mathcal{W}}(X_S)$. □

THEOREM 3.9. *The free BL-algebra* $\mathbf{Free}_{\mathcal{V}}(X)$ *can be represented as a weak Boolean product of the family* $(\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S \rangle) : U_S \in \mathrm{Sp}\,\mathbf{B}(\mathbf{Free}_{\mathcal{V}}(X))$, *where* $\mathbf{B}(\mathbf{Free}_{\mathcal{V}}(X))$ *is the free Boolean algebra over the poset* $Y = \{\sigma_i^n(\neg\neg x) : x \in X, i = 1, \dots, n-1\}$. *Moreover, for each* $U_S \in \mathrm{Sp}\,\mathbf{B}(\mathbf{Free}_{\mathcal{V}}(X))$, *there exists* $m \geq 2$ *such that* $m-1$ *divides* $n-1$ *and*

$$\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S \rangle = \mathbf{L}_m \uplus \mathbf{Free}_{\mathcal{W}}(X_S),$$

where $X_S := \{x/\langle U_S \rangle : \neg\neg x \in \langle U_S \rangle\}$ *and* $\mathcal{W}$ *is the variety of generalized BL-algebras generated by* $\mathbf{B}$.

4. Examples

4.1. PL-algebras Let $\mathbf{G}$ be a lattice-ordered abelian group (ℓ-group), and $G^- = \{x \in G : x \leq 0\}$ its negative cone. For each pair of elements $x, y \in G^-$, we define the following operators:

$$x * y = x + y \quad \text{and} \quad x \to y = 0 \wedge (y - x).$$

Then $\mathbf{G}^- = (G^-, \wedge, \vee, *, \to, 0)$ is a generalized BL-algebra. The following result can be deduced from [3] (see also [6] and [15]).

THEOREM 4.1. *The following conditions are equivalent for a generalized BL-algebra* $\mathbf{A}$:

(1) $\mathbf{A}$ *is a cancellative hoop.*
(2) *There is an* ℓ*-group* $\mathbf{G}$ *such that* $\mathbf{A} \cong \mathbf{G}^-$.
(3) $\mathbf{A}$ *is in the variety of generalized BL-algebras generated by* $\mathbf{Z}^-$, *where* $\mathbf{Z}$ *denotes the additive group of integers with the usual order.*

Let us consider $\mathcal{W}$, the variety of generalized BL-algebras generated by $\mathbf{Z}^-$, that is, the variety of cancellative hoops. In [16] a description of $\mathbf{Free}_{\mathcal{W}}(X)$ is given for any set X of free generators. Therefore we can have a complete description of free algebras in varieties of BL-algebras generated by the ordinal sum

$$\mathbf{PL}_n = \mathbf{L}_n \uplus \mathbf{Z}^-.$$

Indeed, if we denote by $\mathcal{PL}_n$ the variety of BL-algebras generated by $\mathbf{PL}_n$, from Theorem 3.9 we obtain that $\mathbf{Free}_{\mathcal{PL}_n}(X)$ is a weak Boolean product of algebras of the form $\mathbf{L}_s \uplus \mathbf{Free}_{\mathcal{W}}(X')$ with $s-1$ dividing $n-1$ and some set X' of cardinality less or equal than X. Therefore, in the present case, the BL-algebra $\mathbf{Free}_{\mathcal{PL}_n}(X)$ can be completely described as a weak Boolean product of ordinal sums of two known algebras.

From [15, Theorem 2.8], $\mathcal{PL}_2$ is the variety of PL-algebras $\mathcal{PL}$. From Remark 2.14, $\operatorname{Sp}\mathbf{B}(\mathbf{Free}_{\mathcal{PL}}(X))$ is the Cantor space $\mathbf{2}^{|X|}$. From Theorem 3.9, the free PL-algebra over a set X can be describe as a weak Boolean product over the Cantor space $\mathbf{2}^{|X|}$ of algebras of the form $\mathbf{L}_2 \uplus \mathbf{Free}_{\mathcal{W}}(X')$ for some set X' of cardinality less or equal than X.

Given a BL-algebra $\mathbf{A}$, the radical $R(\mathbf{A})$ of $\mathbf{A}$ is the intersection of all maximal implicative filters of $\mathbf{A}$. We have that $\mathbf{r}(\mathbf{A}) = (R(\mathbf{A}), *, \to, \wedge, \vee, \top)$ is a generalized BL-algebra. Let

$$\mathcal{PL}^r = \{\mathbf{R} : \mathbf{R} = \mathbf{r}(\mathbf{A}) \text{ for some } \mathbf{A} \in \mathcal{PL}\}.$$

$\mathcal{PL}^r$ is a variety of generalized BL-algebras. In [17] a description of $\mathbf{Free}_{\mathcal{PL}}(X)$ is given. From Example 4.7 and Theorem 5.7 in the mentioned paper we obtain that $\mathbf{Free}_{\mathcal{PL}}(X)$ is the weak Boolean product of the family $(\mathbf{L}_2 \uplus \mathbf{Free}_{\mathcal{PL}^r}(S) : S \subseteq \mathbf{2}^{|X|})$ over the Cantor space $\mathbf{2}^{|X|}$. In order to check that our description and the one given in [17] coincide it is only left to check that $\mathcal{PL}^r = \mathcal{W}$. From Corollary 3.5 we have that $\mathcal{W}$ consist on the generalized BL-algebras $\mathbf{C}$ such that $\mathbf{L}_2 \uplus \mathbf{C} \in \mathcal{PL}$.

THEOREM 4.2. $\mathcal{PL}^r = \mathcal{W}$.

PROOF. Let $\mathbf{C} \in \mathcal{PL}^r$. Then there exists a BL-algebra $\mathbf{A} \in \mathcal{PL}$ such that $\mathbf{r}(\mathbf{A}) = \mathbf{C}$. It is not hard to check that $\mathbf{L}_2 \uplus \mathbf{C}$ is a subalgebra of $\mathbf{A}$, thus $\mathbf{L}_2 \uplus \mathbf{C}$ is in $\mathcal{PL}$. It follows that $\mathbf{C} \in \mathcal{W}$. On the other hand, let $\mathbf{C} \in \mathcal{W}$. Then $\mathbf{L}_2 \uplus \mathbf{C}$ is in $\mathcal{PL}$, and $\mathbf{C} \in \mathcal{PL}^r$. □

4.2. Finitely generated free algebras As we mentioned in the introduction, when the set of generators X is finite, let us say of cardinality k, the algebra $\mathbf{Free}_{\mathcal{V}}(X)$ is described in [10] as a direct product of algebras of the form $\mathbf{L}_s \uplus \mathbf{Free}_{\mathcal{W}}(X')$, with $s-1$ that divides $n-1$ and some set X' of cardinality less than or equal to the cardinality of X, where $\mathcal{W}$ is again the subvariety of $\mathcal{GBL}$ generated by $\mathbf{B}$. The method used to describe the algebras strongly relies on the fact that the Boolean elements of $\mathbf{Free}_{\mathcal{V}}(X)$ form a finite Boolean algebra. Indeed, $\mathbf{Free}_{\mathcal{V}}(X)$ is a direct product of n^k algebras obtained by taking the quotients by the implicative filters generated by the atoms of $\mathbf{B}(\mathbf{Free}_{\mathcal{V}}(X))$. In this case, once you know the form of the atom that generates the ultrafilter U you also know the number s such that $\mathbf{MV}((\mathbf{Free}_{\mathcal{V}}(X))/\langle U\rangle) = \mathbf{L}_s$.

When the set X of generators is finite, of cardinality k, then $Y = \{\sigma_i^n(\neg\neg x) : x \in X,\ i = 1, \ldots, n-1\}$ is the cardinal sum of k chains of length $n-1$. Therefore the number of upwards closed subsets of Y is n^k. Since weak Boolean products over discrete finite spaces coincide with direct products, Theorem 3.9 asserts that $\mathbf{Free}_{\mathcal{V}}(X)$ is a direct product of n^k BL-algebras of the form $\mathbf{L}_s \uplus \mathbf{Free}_{\mathcal{W}}(Y)$, with $s-1$ that divides $n-1$ and some set Y of cardinality less than or equal to k.

Therefore the description given in the present paper coincides with the one in [10]. However, the description given in [10], based on a detailed analysis of the structure of the atoms of $\mathbf{B}(\mathbf{Free}_{\mathcal{V}}(X))$ for a finite X, is more precise because it gives the number of factors of each kind appearing in the direct product representation.

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