Adv. Appl. Prob. **22**, 488–490 (1990)
Printed in N. Ireland

A CONSTANT REGRESSION CHARACTERIZATION OF THE GAMMA LAW

JACEK WESOŁOWSKI,* *Politechnika Warszawska*

Abstract

A constant regression of the quotient on the sum of two i.i.d. non-degenerate positive random variables is a characteristic property of the gamma distribution.

In Lukacs (1955) the following nice characterization of the gamma law has been obtained: if random variables X, Y are independent non-degenerate and positive then the random variables X/Y and $X+Y$ are independent iff X and Y have gamma distributions. This result has been the starting point of numerous investigations. Many of them are listed in Wang (1981) and in Shanbhag's review of this paper in *Mathematical Reviews*. Some more recent contributions are by Khatri (1984), Dilip (1984), (1985), Sathe and Dixit (1985), Letac (1985), Sim (1986) and Wesołowski (1989).

Among regressional versions of Lukacs' theorem, considered by many authors, the most important seems to be the characterization using conditions

$$E(X \mid X+Y) = a(X+Y), \qquad E(X^2 \mid X+Y) = b(X+Y)^2,$$

where a, b are real numbers and X, Y are independent non-degenerate positive random variables.

In this note we replace Lukacs' independence condition by other regressional assumptions.

Theorem 1. Let X and Y be independent non-degenerate positive random variables such that $EX < \infty$ and $EX^{-1} < \infty$. If the conditions

$$E(X \mid X+Y) = a(X+Y), \tag{1}$$

$$E(X^{-1} \mid X+Y) = b(X+Y)^{-1} \tag{2}$$

hold for some real a, b, then $0 < b^{-1} < a < 1$ and X, Y have gamma distributions with the same scale parameter.

Proof. From the assumptions we get

$$(1-a)EX = aEY, \qquad EX^{-1} = bE(X+Y)^{-1},$$
$$EYEX^{-1} = b-1.$$

Consequently $0 < a < 1$, $b > 1$. The non-degeneracy of X implies $EXEX^{-1} > 1$ and $ab > 1$.

For $t < 0$, introduce $h(t) = EX^{-1}\exp(tX)$ and $f(t) = E\exp(tY)$; h, f and all their derivatives are strictly positive on $(-\infty, 0)$. Denoting $k = EX^{-1}$, the limits of h, h' and f are k, 1 and 1 when $t \uparrow 0$.

Received 8 December 1989; revision received 6 February 1990.

* Postal address: Instytut Matematyki, Politechnika Warszawska, Plac Jedności Robotniczej 1, 00-661 Warszawa, Poland.

From (1) and (2) we have, respectively,

$$(1-a)h''(t)f(t) = ah'(t)f'(t),$$
$$h(t)f'(t) = (b-1)h'(t)f(t).$$

Hence

(3) $$\frac{f'}{f} = (b-1)\frac{h'}{h} = \frac{(1-a)h''}{ah'}.$$

Denote $c = a(b-1)/(1-a)$. Observe that $c-1 = (ab-1)/(1-a) > 0$. From (3)

$$h'(t) = (k^{-1}h(t))^c,$$

which implies

$$(h(t))^{1-c} = k^{-c}(t(1-c)+k)$$

and consequently

$$E\exp(tX) = h'(t) = \left(1 - \frac{t(c-1)}{k}\right)^{-c/(c-1)}.$$

Now from the first equation in (3) we get

$$E\exp(tY) = \left(1 - \frac{t(c-1)}{k}\right)^{-c(1-a)/a(c-1)}.$$

Hence X and Y have gamma distributions with the same scale parameter $k/(c-1)$ and shape parameters

$$\frac{a(b-1)}{ab-1} \quad \text{and} \quad \frac{(b-1)(1-a)}{ab-1},$$

respectively.

As a simple corollary from Theorem 1 we have the following characterization for identically distributed random variables.

Theorem 2. Let X and Y be i.i.d. non-degenerate positive random variables such that $EX < \infty$ and $EX^{-1} < \infty$. If

(4) $$E(Y/X \mid X+Y) = c$$

for some real number c then $c > 1$ and the random variables X, Y have a gamma distribution.

Proof. The assumption of identical distributions yields (1) with $a = 1/2$. On the other hand (4) implies (2) with $b = c+1$. Consequently $1 > ab = (c+1)/2$ and thus the result follows from Theorem 1.

Acknowledgement

Application of Laplace transforms, suggested by the referee, instead of characteristic functions as used in the original version, eliminated various unnecessary subtleties.

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