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A CHARACTERIZATION OF THE COMMUTATOR SUBGROUP OF A GROUP

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ABSTRACT. An element a of a semigroup S is n-potent if there exist $a_1, a_2, \ldots, a_k \in S$ such that $a = a_1a_2 \cdots a_k$ and $a_1^n a_2^n \cdots a_k^n = a_1^{n+1} a_2^{n+1} \cdots a_k^{n+1}$. If S is a group, the set of n-potent elements is a normal subgroup of S and the set of 1-potent elements is the commutator subgroup of S.

1. Let S be a semigroup and let n be a positive integer. An element $a \in S$ is said to be *n-potent* if there exist $a_1, a_2, \ldots, a_k \in S$ such that

$$a = a_1a_2 \cdots a_k \quad \text{and} \quad a_1^n a_2^n \cdots a_k^n = a_1^{n+1} a_2^{n+1} \cdots a_k^{n+1}.$$

If it is not empty, the set S_n of the n-potent elements of a semigroup S is a subsemigroup of S and $(S_n)\alpha \subseteq S_n$ for every endomorphism α of S.

A semigroup is said to be *n-abelian*, $n>0$, if $ab^n = b^n a$ for all $a, b \in S$.

PROPOSITION 1. *If G_n is the set of the n-potent elements of a group G, then G_n is a normal subgroup of G and the quotient group G/G_n is n-abelian. If H is a normal subgroup of G such that G/H is n-abelian then $G_n \subseteq H$.*

Proof. It is immediate that G_n is a normal subgroup of G. Let $a, b \in G$. Then $x = ab^n a^{-1} b^{-n} = a \cdot b^{-1} \cdot b^{n+1} a^{-1} b^{-n}$ and $a^n \cdot (b^{-1})^n \cdot (b^{n+1} a^{-1} b^{-n})^n = a^{n+1} \cdot (b^{-1})^{n+1} \cdot (b^{n+1} a^{-1} b^{-n})^{n+1}$. Hence x is n-potent and $x \in G_n$. It follows then that G/G_n is n-abelian.

If $G_n \nsubseteq H$, then there exists $y \in G_n$ such that $y \notin H$. Since y is n-potent, then $y = y_1 y_2 \cdots y_k$ with $y_1^n y_2^n \cdots y_k^n = y_1^{n+1} y_2^{n+1} \cdots y_k^{n+1}$. The quotient group G/H being n-abelian, it follows then that

$$y_1^n \cdots y_k^n = y_1^{n+1} \cdots y_k^{n+1} \equiv y_1^n \cdots y_k^n y_1 \cdots y_k (\text{mod } H).$$

Therefore $y = y_1 y_2 \cdots y_k \in H$, a contradiction.

2. If $n=1$, then an element a of a semigroup S is 1-potent if and only if $a = a_1 a_2 \cdots a_k = a_1^2 a_2^2 \cdots a_k^2$. In particular, every idempotent element and every product of idempotent elements of S are 1-potent elements. If S is an abelian semigroup, then the 1-potent elements of S are the idempotent elements of S.

PROPOSITION 2. *Every semigroup can be embedded in a regular semigroup in which every element is* 1-*potent.*

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Proof. It has been proved by Howie [2] that every semigroup can be embedded in a regular semigroup generated by idempotent elements. The proposition is therefore an immediate consequence of this result.

For the case of 1-potent elements, the Proposition 1 takes the following special form:

PROPOSITION 3. *The commutator subgroup of a group G is the set of the* 1*-potent elements of G.*

3. An n-potent element of θ a semigroup S is said to be of *degree k*, if k is the least positive integer such that $a=a_1a_2\cdots a_k$ with $a_1^na_2^n\cdots a_k^n=a_1^{n+1}a_2^{n+1}\cdots a_k^{n+1}$.

An element is 1-potent of degree 1 if and only if it is idempotent. In a group, the only n-potent element of degree 1 is the identity element. Furthermore a group has no 1-potent element of degree 2 because $a=bc=b^2c^2$ implies $bc=e$ (the identity of the group). The n-potent elements of a group of degree 3 are the elements of the form $ab^na^{-1}b^{-n}$ that are different from the identity.

A semigroup can contain 1-potent elements of degree 2, but no idempotent elements. An example of a class of such semigroups is given by the class of Baer-Levi semigroups. (See Clifford-Preston [1].) If S is a Baer-Levi semigroup, then S is a right cancellative right simple semigroup without idempotent. If $a \in S$, then there exists $c \in S$ such that $a^2c=a$. Therefore $b=ac=a^2c^2$ is a 1-potent element of degree 2.

4. Let G be a group and let e be the identity of G. An element $a \in G$ is said to be an *even n-potent* element if either $a=e$ or $a=a_1a_2\cdots a_k$ and $a_1^na_2^n\cdots a_k^n=a_1^{n+1}a_2^{n+1}\cdots a_k^{n+1}$, where k is an even integer and $a_i\neq e$ for $i=1, 2, \ldots, k$.

PROPOSITION 4. *The set E_n of the even n-potent elements of a group G is a normal subgroup of G contained in the subgroup G_n of the n-potent elements of G and the index of E_n in G_n is either* 1 *or* 2.

Proof. Immediate.

If $G=\{\pm 1, \pm i, \pm j, \pm k\}$ is the group of quaternions, then $G_1=\{1, -1\}$ and $E_1=\{1\}$.

If G is a non-abelian simple group, then $G=G_1=E_1$. If $n\geq 2$, then either $G=G_n=E_n$ or $G_n=E_n=\{e\}$.

REFERENCES

1. A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups*, Vol. **2**, Math. Survey 7, Amer. Math. Soc. 1967.

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