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C-FLOWS ON A LIE GROUP FOR EULER EQUATIONS

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§ 0. Introduction

The purpose of this paper is to determine left-invariant vector fields on a Lie group G with a left-invariant Riemannian metric which induces C-flows on G.

In his paper [1], V.I. Arnol'd has obtained a differential equation (Euler equation) analogous to that of motion of a rigid body about a fixed point under no forces. Let $\{g_t;\, t\in R\}$ be a geodesic on a Lie group G with a left-invariant Riemannian metric and let $\mathfrak{G}$ be its Lie algebra. Then

$$X_t = (L_{g_t}^{-1})_* \dot{g}_t \in \mathfrak{G} \tag{1}$$

satisfies the following differential equation (Euler equation),

$$\dot{X}_t = B(X_t, X_t), \tag{2}$$

where B is a bilinear map of $\mathfrak{G}\times\mathfrak{G}$ into $\mathfrak{G}$ depending on the Riemannian metric.

In particular, Arnol'd has paid attention to the geodesic expressed as a one-parameter subgroup of G, which is analogous to a closed (periodic) geodesic, to study the stability of the stationary points X $(B(X,X)=0)$ of the equation (2) in $\mathfrak{G}$ rather than the stability of the corresponding geodesics of one-parameter subgroups.

In connection with ergodic theory, we are interesed in the exact form of the Euler equation with a particular choice of the Lie group. Namely, we consider the Euler equation on the projective special linear group PSL $(2.R)$ of 2-nd order over R with the natural left-invariant Riemannian metric, and determine its instable stationary points. Then we shall see that each instable stationary point induces a C-flow on PSL $(2.R)$ in our sense (Cor. 3.1.).

Being inspired by the remarkable property above, we shall determine left-invariant vector fields, each of which induces a C-flow on a Lie group.

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Our main results read as follows. We employ the usual notations such as Ad and ad to denote the adjoint representation of a Lie group and the adjoint representation of its Lie algebra.

THEOREM 2. *Let G be an n-dimensional oriented, connected real Lie group with the unit element e, and let $\mathfrak{G}$ be its Lie algebra. Assume that $\mathfrak{G}$ contains an element X satisfying the following conditions:*

1) *$ad(X)$ is diagonal on the complexification $\mathfrak{G}^c$ of $\mathfrak{G}$.*

2) *The multiplicity of the eigenvalue 0 is exactly equal to 1.*

3) *The rest of the eigenvalues of $ad(X)$ is divided into two parts, call them $\lambda_1, \cdots, \lambda_k; \mu_1 \cdots \mu_l$, in such a way that*

$$Re\,\lambda_i > 0, \quad i = 1, \cdots, k,$$
$$Re\,\mu_i < 0, \quad i = 1, \cdots, l,$$

where $k \geq 1$, $l \geq 1$ and $k + l + 1 = n$.

4) *$Tr\ ad(X) = 0$.*

Then the one-parameter group of the diffeomorphisms $\text{Expt}X$ *of G is a C-flow with respect to any left-invariant Riemannian metric ds^2 on G.*

If we specify the Lie group to be semi-simple, we shall have the following

THEOREM 4. *The semi-simple real Lie algebra $\mathfrak{G}$ is isomorphic to $\mathfrak{sl}(2.R)$, if $\mathfrak{G}$ contains an element satisfying the conditions* 1), 2), 3), 4) *of Theorem* 2.

Observing the above-mentioned situation, we are led to pay specific attention to $PSL(2.R)$ and to transform C-flows on $PSL(2.R)$ formed as above to the unitary tangent bundle T_1L of the Lobachevsky-plane L. There we shall see an interesting result:almost all C-flows thus transformed can not be geodesic flows on the upper half-plane respect to *any* Riemannian metric (Theorem 5).

Finally, the author hopes that our approach to the stability of a stationary point of the equation (2) will be useful in the investigation of the stability of a stationary current for Euler equation appearing in the hydrodynamics on a Riemannian manifold (see Arnol'd [1]).

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§1. Notations and Definitions

We shall list some notations and give definitions used in the later sections.

Let G be a real connected Lie group and let $\mathfrak{G}$ be its Lie algebra over real numbers R with the bracket $[\ ,\ ]$. Assume that G has a left-invariant Riemannian metric ds^2 i.e. the pull-back $L_g^* ds^2$ of ds^2 by any left-translation $L_g (g \in G)$ is equal to ds^2. Then an inner product $\langle\ ,\ \rangle$ is naturally induced in $\mathfrak{G}$ from ds^2. By means of this inner product, we define a bilinear map B:

$$B: \mathfrak{G} \times \mathfrak{G} \ni (X, Y) \longrightarrow B(X, Y) \in \mathfrak{G}$$

of the product space $\mathfrak{G} \times \mathfrak{G}$ into $\mathfrak{G}$ as follows:

(1) $$\langle B(X, Y), Z\rangle = \langle [Y, Z], X\rangle \quad \text{for} \quad X, Y, Z \in \mathfrak{G}.$$

Now let us consider the following differentisl equation for C^1-curve $\{X(t) \in \mathfrak{G};\ t \in R\}$:

(2) $$\frac{d}{dt} X(t) \Big|_{t=s} = B(X(s),\ X(s))$$

This equation is called the *Euler equation* on G (or on $\mathfrak{G}$) associated with the left-invariant Riemannian metric ds^2.

Next let us consider the following differential equation in the n-dimensional vector space R^n:

(3) $$\frac{d}{dt} X(t) \Big|_{t=s} = f(X(s)),\ \ X(s) \in R^n.$$

Then we can think of the equation (3) as the expression of the infinitesimal transformation on R^n. A point $X_0 \in R^n$ is said to be *stationary* if $f(X_0) = 0$. A stationary point X_0 is said to be *stable* if X_0 satisfies the following conditions: Let $X(t)$ be the solution of

$$\frac{d}{dt} X(t) \Big|_{t=s} = f(X(s))$$
$$X(0) = X.$$

Then, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|X(t) - X_0\| < \varepsilon \text{ for any } t \geq 0 \text{ and any } X \in R^n \text{ with } \|X - X_0\| < \delta,$$

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where

$$\|(x_1, \cdots, x_n)\| = \max\{|x_i|;\ i = 1, \cdots, n\}.$$

If a stationary point is not stable, it is said to be *instable*.

Remark. The uniqueness of the solution of the equation (2) is guaranteed in our case. It should be noted that we are always given the solutoin $X(t)$ for which (2) holds for *all* s.

§ 2. Euler equation on the projective special linear group *PSL* (2. *R*) of second order

In this section we construct Euler equation on the Lie group $PSL(2. R)$ with some natural left-invariant Riemannian metric, and determine the stability of its stationary points.

Let us list some notations.

The Lobachevsky-plane is denoted by (L, ds_1^2) i.e.

$$L = \{(x, y) | x \in R,\ y > 0\},\ ds_1^2 = \frac{dx^2 + dy^2}{y^2}.$$

A tangent vector v at a point $(x, y) \in L$ is expressex as follows:

$$v = -hy\sin\theta\left(\frac{\partial}{\partial x}\right)_{(x,y)} + hy\cos\theta\left(\frac{\partial}{\partial y}\right)_{(x,y)},$$

where $h \geq 0$ and $-\pi \leq \theta < \pi$.

The vector v_0 always means

$$v_0 = \left(\frac{\partial}{\partial y}\right)_{(0.1)}.$$

We denote by g an isometry on L induced from the element $g \in PSL(2. R)$ under the natural identification of $PSL(2. R)$ with the isometries on L. Under this identification, we define a diffeomorphism Φ of the unitary tangent bundle T_1L (see [2] for definition) of L onto $PSL(2. R)$ as follows:

$$\Phi(g_* v_0) = g \in PSL(2. R),$$

where g_* denotes the differential of the isometry g on L. By means of the following unique decomposition of elements of $PSL(2. R)$, we parametrize $PSL(2. R)$

$$(x, y, \theta) = \pm \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{y} & 0 \\ 0 & \frac{1}{\sqrt{y}} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix},$$

where

$$x \in R, \ y > 0, \ -\frac{\pi}{2} \leq \theta < \frac{\pi}{2}.$$

With these notions the following propositions can easily be proved.

PROPOSITION 2.1. *Making use of the parameters* (x, y, θ) *in* T_1L *and the parameters* (x, y, θ) *in* $PSL(2.R)$, *the diffeomorphism* Φ *of* T_1L *onto* $PSL(2.R)$ *is expressed as follows:*

$$\Phi: T_1L \ni (x, y, \theta) \longrightarrow \left(x, y, \frac{\theta}{2}\right) \in PSL(2.R).$$

PROPOSITION 2.2. *The Riemannian metric* ds_2^2 *on* T_1L *induced from* ds_1^2 *on* T_1L *in the usual manner is expressed in the form*

$$ds_2^2 = \frac{2}{y^2} dx^2 + \frac{1}{y^2} dy^2 + \frac{2}{y} dx d\theta + d\theta^2.$$

PROPOSITION 2.3. *The Riemannian metric* ds_2^2 *on* T_1L *induces by* Φ^{-1} *the following Riemannian metric* ds^2 *on* $PSL(2.R)$

$$ds^2 = (\Phi^{-1})^* ds_2^2 = \frac{2}{y^2} dx^2 + \frac{1}{y^2} dy^2 + \frac{4}{y} dx d\theta + 4d\theta^2.$$

PROPOSITION 2.4. *The above Riemannian metric* ds^2 *on* $PSL(2.R)$ *is left-invariant:*

$$L_g^* ds^2 = ds^2 \quad \textit{for} \quad g \in PSL(2.R).$$

(*Proof*) Let g, h be elements of $PSL(2.R)$ with the unit element e. Then by the definition of Φ,

$$g_* h_* v_0 = (g \cdot h)_* v_0 = \Phi^{-1}(g \cdot h) = \Phi^{-1} L_{g \cdot h} e$$
$$= \Phi^{-1} L_g \Phi \Phi^{-1} L_h e = \Phi^{-1} L_g \Phi h_* v_0,$$

that is,

$$\Phi^{-1} L_g = g_* \Phi^{-1}.$$

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This implies

$$\Phi_*^{-1}(L_g)_* = (g_*)_*\Phi_*^{-1} \quad \text{for} \quad g \in PSL(2.R).$$

Therefore, denoting by $\|\cdot\|$ the norn given by ds^2 and by $\|\cdot\|'$ the norm given by ds_2^2, the following equalities hold for any tangent vector v of $PSL(2.R)$ and any element $g \in PSL(2.R)$:

$$\|(L_g)_*v\| = \|\Phi_*^{-1}(L_g)_*v\|' = \|(g_*)_*\Phi_*^{-1}v\|' = \|\Phi_*^{-1}v\|' = \|v\|,$$

which complete the proof. Q.E.D.

The Lie algebra of $PSL(2.R)$, as is well known, coincides with $\mathfrak{sl}(2.R)$. We now introduce the following left-invariant vector fields X_1, X_2, X_3 given by

$$(X_1)_e = \left(\frac{\partial}{\partial x}\right)_e, \quad (X_2)_e = 2\left(\frac{\partial}{\partial y}\right)_e, \quad (X_3)_e = \left(\frac{\partial}{\partial \theta}\right)_e.$$

Then we can easily prove the following

PROPOSITION 2.5. *The commutation-relations for X_1, X_2, X_3 are expressed in thr form*

$$[X_1, X_2] = -2X_1,$$
$$[X_2, X_3] = -2X_3 + 4X_1,$$
$$[X_3, X_1] = X_2.$$

By the definition of the inner product and with the choice of these X_i's we are given the following formula

$$\langle \xi_1X_1 + \xi_2X_2 + \xi_3X_3,\ \eta_1X_1 + \eta_2X_2 + \eta_3X_3 \rangle$$
$$= 2\xi_1\eta_1 + 4\xi_2\eta_2 + 4\xi_3\eta_3 + 2\xi_1\eta_3 + 2\xi_3\eta_1,$$

where ξ_1, ξ_2, ξ_3, η_1, η_2, η_3 are all real numbers.

By virtue of the propositions above, we obtain

THEOREM 1. *The Euler equation* (2) *on* $\mathfrak{sl}(2.R)$ *associated with the left-invariant Riemannian metric ds^2 is expressed as follows:*

$$\begin{cases} \dot{\xi}_1(t) = 4\xi_2(t)\,(\xi_1(t) + 2\xi_3(t)) \\ \dot{\xi}_2(t) = -\xi_1(t)\,(\xi_1(t) + 2\xi_3(t)) \\ \dot{\xi}_3(t) = -2\xi_2(t)\,(\xi_1(t) + 2\xi_3(t)), \end{cases}$$

where

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$$X(t) = \xi_1(t)X_1 + \xi_2(t)X_2 + \xi_3(t)X_3.$$

The stationary points X *of this Euler equation are expressed in the form*

$$X = \xi_3 X_3 \qquad (\xi_3 \in R),$$

or

$$X = -2\xi_3 X_1 + \xi_2 X_2 + \xi_3 X_3 \quad (\xi_2, \xi_3 \in R).$$

Further, the stationary points $X = \xi_3 X_3$ *are stable, while the stationary points* $X = -2\xi_3 X_1 + \xi_2 X_2 + \xi_3 X_3$ $(\xi_2^2 + \xi_3^2 \neq 0)$ *are instable.*

(*Proof*) Let X, X' be elements of $\mathfrak{sl}(2.R)$:

$$X = \xi_1 X_1 + \xi_2 X_2 + \xi_3 X_3$$
$$X' = \xi_1' X_1 + \xi_2' X_2 + \xi_3' X_3.$$

Then by Proposition 2.5,

$$[X, X'] = 2(\xi_2\xi_1' - \xi_1\xi_2' + 2\xi_2\xi_3' - 2\xi_3\xi_2')X_1 + (\xi_3\xi_1' - \xi_1\xi_3')X_2 + 2(\xi_3\xi_2' - \xi_2\xi_3')X_3.$$

Hence we have by Proposition 2.5,

$$\langle [X, X'], X\rangle = 4(\xi_1 + 2\xi_3)\,\xi_2\xi_1' - 4(\xi_1 + 2\xi_3)\xi_1\xi_2'.$$

Set

$$B(X, X) = \eta_1 X_1 + \eta_2 X_2 + \eta_3 X_3,$$

then we have

$$\langle B(X, X), X'\rangle = 2(\eta_1 + \eta_3)\xi_1' + 4\eta\xi_2' + 2(\eta_1 + 2\eta_3)\xi_3'.$$

By the definition of the map B we have

$$\langle B(X, X), X'\rangle = \langle [X, X'], X\rangle.$$

Hence

$$\begin{cases} \eta_1 = 4\xi_2(\xi_1 + 2\xi_3) \\ \eta_2 = -\xi_1(\xi_1 + 2\xi_3) \\ \eta_3 = -2\xi_2(\xi_1 + 2\xi_3). \end{cases}$$

Consequently, we obtain the following expression:

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$$\begin{cases} \dot{\xi}_1(t) = 4\xi_2(t)\,(\xi_1(t) + 2\xi_3(t)) \\ \dot{\xi}_2(t) = -\,\xi_1(t)\,(\xi_1(t) + 2\xi_3(t)) \\ \dot{\xi}_3(t) = -\,2\xi_2(t)\,(\xi_1(t) + 2\xi_3(t)), \end{cases}$$

where

$$X(t) = \xi_1(t)X_1 + \xi_2(t)X_2 + \xi_3(t)X_3.$$

Now let us introduce a new variable ζ by

$$\zeta = \xi_1 + 2\xi_3.$$

Then the Euler equation turns out to be

$$(*) \quad \begin{cases} \dot{\xi}_1(t) = 4\xi_2(t)\zeta(t) \\ \dot{\xi}_2(t) = -\,\xi_1(t)\zeta(t) \\ \dot{\zeta}(t) = 0. \end{cases}$$

This implies

$$\begin{cases} \xi_1(t) = \xi_1(0)\cos 2\zeta(0)t + 2\xi_2(0)\sin 2\zeta(0)t \\ \xi_2(t) = -\dfrac{\xi_1(0)}{2}\sin 2\zeta(0)t + \xi_2(0)\cos 2\zeta(0)t \\ \zeta(t) = \zeta(0). \end{cases}$$

Hence

$$\{(\xi_1 = 0, \xi_2 = 0, \zeta = w);\ w \in R\} \cup \{(\xi_1 = u, \xi_2 = v, \zeta = 0);\ u^2 + v^2 \neq 0,\ u, v \in R\}$$

is the set of all the stationary points of the equation $(*)$, and further it is easily seen that the stationary point $(\xi_1 = 0,\ \xi_2 = 0,\ \zeta = w)$ is stable for the equation $(*)$, while the stationary point $(\xi_1 = u,\ \xi_2 = v, \zeta = 0)$ $(u^2 + v^2 \neq 0)$ is inatsble for the equation $(*)$.

The following linear map of 3-dimensional vector space R^3 into itself is regular,

$$R^3 \ni (\xi_1,\ \xi_2,\ \xi_3) \longrightarrow (\xi_1,\ \xi_2,\ \xi_1 + 2\xi_3) \in R^3.$$

Summing up the above results, we obtain the conclusions. (Q.E.D.)

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§3. The left-invariant vector fields which induce C-flows on a connected real Lie group

Let G be a connected real Lie group with a left-invariant Riemannian metric. We give a sufficient condition for a left-invariant vector field on G to induce a C-flow on it in our sense (Theorem 2). We prove that if, in particular, the Lie group $\mathfrak{G}$ is semi-simple, then its Lie algebra $\mathfrak{G}$ is isomorphic to $\mathfrak{sl}(2, R)$ (Theorem 4).

We begin with the definitions of C-flows in our sense.

DEFINITION. Let $\{\varphi_t;\ t \in R\}$ be a one-parameter group of C^2-diffeomorphisms of n-dimensional oriented, connected C^∞-manifold M with a Riemannian metric ds^2.

If the following conditions are satisfied, φ_t is called a C-flow:

1) The infinitesimal transformation X of φ_t vanishes nowhere, and the divergence of X vanishes everywhere.

2) The tangent vector space TM_x at $x \in M$ splits into a direct sum:

$$TM_x = A_x \oplus B_x \oplus C_x,$$

where A_x and B_x are vector subspaces with $\dim A_x = k \geq 1$, $\dim B_x = l \geq 1$, and where C_x is the 1-dimentional subspace spanned by X_x.

3) For any $v \in A_x$

$$\|(\varphi_t)_* v\| \geqq a e^{\lambda t} \|v\|,\ t \geq 0,$$

$$\|(\varphi_t)_* v\| \leqq b e^{\lambda t} \|v\|,\ t \leq 0;$$

and for any $v \in B_x$

$$\|(\varphi_t)_* v\| \leqq b e^{-\lambda t} \|v\|,\ t \geq 0,$$

$$\|(\varphi_t)_* v\| \geqq a e^{-\lambda t} \|v\|,\ t \leq 0,$$

where $\|\cdot\|$ denotes the norm given by ds^2 and where a, b, λ are positive constants.

We denote by Ad and ad the adjoint representation of a Lie group and the adjoint representation of its Lie algebra respectively.

THEOREM 2. *Let G be a n-dimensional oriented connected real Lie group with the unti element e, and let $\mathfrak{G}$ be its Lie algebra. Assume that $\mathfrak{G}$ contains an element X satisfying the following conditions:*

The term "C-flow" is usually used only in the case where the manifold is compact.

1) *$ad(X)$ is diagonal on the complexification $\mathfrak{G}^c$ of $\mathfrak{G}$.*

2) *The multiplicity of the eigenvalue 0 is exactly equal to 1.*

3) *The rest of the eigenvalues of $ad(X)$ is divided into two parts, call them $\lambda_1, \cdots, \lambda_k$; $\mu_1, \cdots, \mu_l$, in such a way that*

$$Re\,\lambda_i > 0,\ i = 1, \cdots, k,$$

$$Re\,\mu_i < 0,\ i = 1, \cdots, l,$$

where $k \geq 1$, $l \geq 1$ and $k + l + 1 = n$.

4) *$Tr\,ad(X) = 0$.*

Then the one-parameter group of the diffeomorphisms $Exp\,tX$ of G is a C-flow with respect to any left-invariant Riemannian metric ds^2 on G.

(*Proof*) 1^0. By the assumption 3) we can express $\{\lambda_i\}$ in the form

$$\lambda_i = \alpha_i + \sqrt{-1}\beta_i,\ i = 1, \cdots, p,\ p \geqq 0$$

$$\lambda_{i+p} = \alpha_i - \sqrt{-1}\beta_i,\ \alpha_i > 0,\ \beta_i > 0$$

$$\lambda_i > 0,\ i = 2p + 1, \cdots, k.$$

Since $ad(X)$ is diagonal on $\mathfrak{G}^c$, there exist $Z_i(\neq 0)$ in $\mathfrak{G}$ such that

$$ad(X)Z_i = -\lambda_i Z_i,\ i = 2p + 1, \cdots, k,$$

that is,

$$\exp(-t\,ad(X))Z_i = e^{\lambda_i t}Z_i\ \ i = 2p + 1, \cdots, k.$$

Similarly there exist $X_i(\neq 0)$, $Y_i(\neq 0)$ in $\mathfrak{G}$ such that

$$ad(X)(X_i + \sqrt{-1}Y_i) = -(\alpha_i + \sqrt{-1}\beta_i)(X_i + \sqrt{-1}Y_i)\ \ i = 1, \cdots, p.$$

Therefore we have

$$\exp(-t\ ad(X))(X_i + \sqrt{-1}Y_i) = e^{(\alpha_i + \sqrt{-1}\beta_i)t}(X_i + \sqrt{-1}Y_i),$$

which implies

$$\exp(-t\ ad(X))X_i = e^{\alpha_i t}(\cos\beta_i t X_i - \sin\beta_i t Y_i)$$

$$\exp(-t\ ad(X))Y_i = e^{\alpha_i t}(\cos\beta_i t Y_i + \sin\beta_i t X_i).$$

On the other hand, since $\beta_i \neq 0, X_i$ and Y_i are R-lineraly independent. Hence the collection $\{X_i, Y_i, Z_i\}$ spans a k-dimensional subspace A of $\mathfrak{G}$.

Now for any combination of real numbers

$$\{x_i,\ y_i;\ i=1,\cdots,p\}\cup\{z_i;\ i=2p+1,\cdots,k\},$$

we define $Y\in\mathfrak{G}$ by

$$Y=\sum_{i=1}^{p}x_iX_i+\sum_{i=1}^{p}y_iY_i+\sum_{i=2p+1}^{k}z_iZ_i.$$

Then we have

$$\exp(-t\ ad(X))Y=\sum_{i=1}^{p}e^{\alpha_i t}\rho_i(\cos(\beta_i t+\theta_i)X_i+\sin(\beta_i t+\theta_i)Y_i)$$
$$+\sum_{i=2p+1}^{k}z_ie^{\lambda_i t}Z_i,$$

where ρ_i, θ_i $(i=1,\cdots,p)$ are determined by the following formulas,

$$x_i=\rho_i\cos\theta_i,\ \ y_i=\rho_i\cos\theta_i,\ \ \rho_i\geqq 0,\ \ 0\leqq\theta_i<2\pi.$$

For a moment, we introduce an inner product in A so that $\{X_i, Y_i, Z_i\}$ forms a complete orthonormal system in A, and denote by $\|\cdot\|'$ the norm given by this inner product. Further we define $l(t)$ by

$$l(t)^2=\|\exp(-t\ ad(X))Y\|'^2.$$

Then it holds that

$$l(t)^2=\sum_{i=1}^{p}e^{2\alpha_i t}\rho_i^2+\sum_{i=2p+1}^{k}e^{2\lambda_i t}z_i^2.$$

Hence we have

$$l(t)^2\geqq e^{2\nu t}l(0)^2,\ \ t\geqq 0,$$
$$l(t)^2\leqq e^{2\nu t}l(0)^2,\ \ t\leqq 0,$$

where

$$\nu=\min\{Re\ \lambda_i;\ i=1,\cdots,k\}.$$

It is easily seen that there exist positive numbers a_1, b_1 for which the following inequalities hold:

$$\|\exp(-t\ ad(X)Y\|\geqq a_1e^{\nu t}\|Y\|,\ \ t\geqq 0,$$
$$\|\exp(-t\ ad(X)Y\|\leqq b_1e^{\nu t}\|Y\|,\ t\leqq 0,\ \text{for any } Y\in A,$$

where $\|\cdot\|$ denotes the original norm.

Denoting by B the l-dimensional subspace of $\mathfrak{G}$ corresponding to the eigenvalues $\mu_1, \cdots, \mu_l$, we can, in a similar manner, find numbers a_2, b_2 such that

$$\|\exp(-t\ ad(X))Y\| \leqq b_2 e^{-\mu t}\|Y\|,\ t \geqq 0,$$
$$\|\exp(-t\ ad(X))Y\| \geqq a_2 e^{-\mu t}\|Y\|,\ t \leqq 0,\ \text{for any } Y \in B,$$

where

$$-\mu = \max\{Re\,\mu_i;\ i = 1, \cdots, l\}.$$

Consequently there exist positive numbers a, b such that the following two pairs of inequalities hold simultaneously:

$$\|\exp(-t\ ad(X))Y\| \geqq a e^{\lambda t}\|Y\|,\ t \geqq 0,$$
$$\|\exp(-t\ ad(X))Y\| \leqq b e^{\lambda t}\|Y\|,\ t \leqq 0,\ \text{for any } Y \in A,$$

and

$$\|\exp(-t\ ad(X))Y\| \leqq b e^{-\lambda t}\|Y\|,\ t \geqq 0,$$
$$\|\exp(-t\ ad(X))Y\| \geqq a e^{-\lambda t}\|Y\|,\ t \leqq 0,\ \text{for any } Y \in B,$$

where

$$\lambda = \min\{\nu, \mu\}.$$

2^0. We are given by ds^2 the inner product $\langle\ ,\ \rangle_h$ and norm $\|\cdot\|_h$ (or simply $\langle\ ,\ \rangle$, $\|\cdot\|$) in the tangent vector space TG_h at $h \in G$. Let Y be an element of A, and let h be an element of G. Then, by the left-invariance of ds^2, we obtain the following formulas:

$$\begin{aligned}\|(\mathrm{Exp}\,tX)_* Y_h\| &= \|(L_h^{-1})_*(R_{\exp tX})_* Y_h\| = \|(R_{\exp tX})_* Y_e\| \\ &= \|(L_{\exp(-tX)})_*(R_{\exp tX})_* Y_e\| = \|Ad_{\exp(-tX)} Y\| \\ &= \|\exp(-t\ ad(X))Y\|.\end{aligned}$$

Hence we have

$$\|(\mathrm{Exp}\,tX)_* Y_h\| \geqq a e^{\lambda t}\|Y\| = a e^{\lambda t}\|Y_h\|,\ t \geqq 0$$
$$\|(\mathrm{Exp}\,tX)_* Y_h\| \leqq b e^{\lambda t}\|Y\| = b e^{\lambda t}\|Y_h\|,\ t \leqq 0 \text{ for any } Y \in A.$$

For any element $h \in G$, we can also prove the following formulas:

$$\|(\mathrm{Exp}\,tX)_* Y_h\| \leqq b e^{-\lambda t}\|Y_h\|,\ t \geqq 0,$$
$$\|(\mathrm{Exp}\,tX)_* Y_h\| \geqq a e^{-\lambda t}\|Y_h\|,\ t \leqq 0 \text{ for any } Y \in B.$$

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3^0 For a vector field Z on G we define a one-form ω_Z on G as follows:

$$\omega_Z(\cdot) = \langle \cdot, Z \rangle.$$

Let $\{W_1, \cdots, W_n\}$ be the orthonormal basis in $\mathfrak{G}$. Then

$$\Omega = \omega_{W_1} \wedge \cdots \wedge \omega_{W_n}$$

is a volume element on G.

Now let g, h be elements of G, and let v be an element of TG_h. Then we have

$$(L_g^* \omega_{W_i})_h(v) = (\omega_{W_i})_{g \cdot h}(L_{g*}v) = \langle (W_i)_{g \cdot h}, L_{g*}v \rangle$$
$$= \langle L_{g*}^{-1}(W_i)_{g \cdot h}, v \rangle = \langle (W_i)_h, v \rangle = (\omega_{W_i})_h(v),$$

that is,

$$L_g^* \omega_{W_i} = \omega_{W_i}.$$

Hence, for any element $Y \in \mathfrak{G}$ it holds that

$$((\mathrm{Exp}\, tX)^* \omega_{W_i})(Y) = (L^*_{\exp(-tX)} R^*_{\exp tX} \omega_{W_i})(Y)$$
$$= \langle W_i, Ad_{\exp(-tX)} Y \rangle = \langle W_i, \exp(-t\ ad(X)) Y \rangle.$$

Hence we have

$$(L_X \omega_{W_i})(Y) = \lim_{t \to 0} \frac{(\mathrm{Exp}\, tX)^* \omega_{W_i} - \omega_{W_i}}{t}(Y)$$

$$= \langle W_i, \lim_{t \to 0} \frac{\exp(-t\ ad(X)) - 1}{t} Y \rangle$$
$$= - \langle W_i, ad(X) Y \rangle = - \langle B(W_i, X), Y \rangle,$$

where L_X is the Lie derivative with respect to X. Namely, we have proved

$$L_X \omega_{W_i} = - \omega_{B(W_i, X)}.$$

Let $\{C^i_{jk};\ i, j, k = 1, \cdots, n\}$ be the structure-constants of G with respect to the basis $\{W_1, \cdots, W_n\}$. Then

$$B(W_i, X) = \sum_{j,k=1}^{n} \alpha_j C^i_{jk} W_k,$$

where

$$X = \sum_{j=1}^{n} \alpha_j W_j.$$

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By the expression of Ω we see that

$$-L_X\Omega = \omega_{B(W_1,X)}\wedge\cdots\wedge\omega_{W_n}+\cdots+\omega_{W_1}\wedge\cdots\wedge\omega_{B(W_n,X)}$$
$$=(\sum_{i,j=1}^{n}\alpha_j C^i_{ji})\Omega = (Tr\ ad(X))\Omega.$$

Hence by the assumption 4) we obtain

$$\operatorname{div} X = 0.$$

Thus the proof is completed. (G.E.D.)

THEOREM 3. *The left-invariant vector field X on PSL(2.R) which induces a C-flow for left-invariant Riemannian metric on PSL(2.R) is expressed in the form*

$$X = \xi_1 X_1 + \xi_2 X_2 + \xi_3 X_3,\quad \xi_2^2 - \xi_3^2 - \xi_1\xi_3 > 0,$$

where X_1, X_2, X_3 are left-invariant vector fields defined in § 2.

(*Proof*) Let X be an element of the Lie algebra of $PSL(2.R)$;

$$X = \xi_1 X_1 + \xi_2 X_2 + \xi_3 X_3.$$

Then

$$ad(X) = \begin{pmatrix} 2\xi_2 & -2\xi_1 - 4\xi_3 & 4\xi_2 \\ \xi_3 & 0 & -\xi_1 \\ 0 & 2\xi_3 & -2\xi_2 \end{pmatrix}$$

Hence the characteristic equation is expressed in the form

$$\det(ad(X) - \lambda E) = -\lambda^3 - 4(\xi_3\xi_1 - \xi_2^2 + \xi_3^2)\lambda = 0,$$

where E is the unit matrix. Hence, for X to satisty the conditions 1), 2), 3) of Theorem 2,

$$\xi_2^2 - \xi_3^2 - \xi_1\xi_3 > 0$$

is necessary and sufficient.

It is noted that, since $PSL(2.R)$ is a simple Lie group, the condition 4) holds for any left-invariant vector field on $PSL(2.R)$. Thus the theorem is proved. (Q.E.D.)

COROLLARY 3.1. *Each instable stationary point described in Theorem* 1 *induces a C-flow on PSL(2.R) for any left-invariant Riemannian metric.*

THEOREM 4. *The semi-semple real Lie algebra* $\mathfrak{G}$ *which contains an element satisfying the conditions* 1), 2), 3), 4) *of Theorem* 2 *is isomorphic to* $\mathfrak{sl}(2.R)$.

(*Proof*) The condition 1) implies that

$$\text{rank } \mathfrak{G} = 1.$$

Hence the complexification $\mathfrak{G}^c$ of $\mathfrak{G}$ is a simple Lie algebra with rank 1. Appealing to Cartan's classification of simple Lie algebras over complex numbers C, we see that $\mathfrak{G}^c$ is isomorphic to $\mathfrak{sl}(2.C)$. Hence $\mathfrak{G}$ is a 3-dimensional simple Lie algebra, and therefore $\mathfrak{G}$ is isomorphic to $\mathfrak{sl}(2.R)$ or $\mathfrak{so}(3.R)$. On the other hand, the fact that the group is non-compact shows that $\mathfrak{G}$ can not be isomorphic to $\mathfrak{so}(3.R)$. Therefore, from Theorem 3, $\mathfrak{G}$ must be isomorphic to $\mathfrak{sl}(2.R)$. (Q.E.D.)

§ 4. Further discussions on the C-flows on the unitary tangent bundle T_1L of the Lobachevsky-plane L.

In this section we discuss what movement is given on T_1L by the C-flow described in Theorem 3.

Let us denote by $\{\varphi_t;\ t\in R\}$ the geodesic flow on the Lobachevsky-plane L. Then we have

PROPOSITION 4.1. *Let* Φ *be the diffeomorphism of* T_1L *onto* $PSL(2.R)$ *described in* § 2. *Then we have*

$$\Phi\cdot\varphi_t\cdot\Phi^{-1} = Exp\frac{t}{2}X_2,$$

where X_2 *is the element of the Lie algebra* $\mathfrak{sl}(2.R)$ *given in* § 2.

(*Proof*) Recall the definition of v_0, and define $\{g_t\in PSL(2.R);\ t\in R\}$ as follows:

$$\Phi(\varphi_t v_0) = g_t.$$

Then, we have for t, $s\in R$,

$$\begin{aligned} g_{t+s} &= \Phi(\varphi_{t+s}v_0) = \Phi(\varphi_t\varphi_s v_0) = \Phi(\varphi_t g_{s*}v_0) \\ &= \Phi(g_{s*}\varphi_t v_0) = \Phi(g_{s*}g_{t*}v_0) = g_s\cdot g_t. \end{aligned}$$

Hence there exists $X\in\mathfrak{sl}(2.R)$ such that

$$g_t = \exp tX.$$

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For any element $g \in PSL(2.R)$, the relations

$$\Phi \cdot \varphi_t \cdot \Phi^{-1}(g) = \Phi(\varphi_t g_* v_0) = \Phi(g_* g_{t*} v_0) = g \cdot g_t = R_{g_t}(g)$$

prove that

$$\Phi \cdot \varphi_t \cdot \Phi^{-1} = \operatorname{Exp} tX.$$

On the other hand, the following formulas are easily obtained:

$$x(\varphi_t v_0) = 0, y(\varphi_t v_0) = e^t, \theta(\varphi_t v_0) = 0,$$

where (x, y, θ) is the local coordinate in T_1L. Hence by Proposition 2.1,

$$\Phi(\varphi_t v_0) = g_t = (0, e^t, 0),$$

or equivalently,

$$X_e = \left(\frac{\partial}{\partial y}\right)_e = \frac{1}{2}(X_2)_e.$$

Consequently we obtain

$$\Phi \cdot \varphi_t \cdot \Phi^{-1} = \operatorname{Exp} \frac{t}{2} X_2. \qquad \text{(Q.E.D.)}$$

Let us denote by $T_\alpha (0 \leq \alpha < 2\pi)$ the diffeomorphism of T_1L onto itself given by

$$T_\alpha;\ T_1L \ni (x, y, \theta) \longrightarrow (x, y, \theta + \alpha) \in T_1L,$$

and denote by H the upper half-plane. Then we obtain

THEOREM 5.

i) *Assume that*

$$2\xi_3 + \xi_1 \neq 0,\ \xi_2^2 - \xi_3^2 - \xi_1\xi_3 > 0.$$

Then $\operatorname{Exp} t\Phi_*^{-1}(\xi_1 X_1 + \xi_2 X_2 + \xi_3 X_3)$ *is a C-flow on* T_1L, *but it cannot be a geodesic flow with respect to any Riemannian metric on* H.

ii) *Assume that*

$$2\xi_3 + \xi_1 = 0,\ \xi_2^2 + \frac{1}{4}\xi_1^2 > 0.$$

Then we have

$$\operatorname{Exp} t\Phi_*^{-1}(\xi_1 X_1 + \xi_2 X_2 + \xi_3 X_3) = T_\alpha \cdot \varphi_{\rho t} \cdot T_{-\alpha},$$

where

$$\xi_1 = \rho \sin\alpha, \;\; 2\xi_2 = \rho\cos\alpha, \;\; \rho > 0, \;\; 0 \leqq \alpha < 2\pi.$$

(*Proof*) Let us now express X_1, X_2, X_3 in terms of the local coordinates (x, y, θ) in T_1L;

$$(*) \quad \begin{cases} X_1 = y\cos 2\theta \dfrac{\partial}{\partial x} + 2y\sin 2\theta \dfrac{\partial}{\partial y} + \sin^2\theta \dfrac{\partial}{\partial \theta} \\ X_2 = -2y\sin 2\theta \dfrac{\partial}{\partial x} + 2y\cos 2\theta \dfrac{\partial}{\partial y} + \sin 2\theta \dfrac{\partial}{\partial \theta} \\ X_3 = \dfrac{\partial}{\partial \theta}. \end{cases}$$

By Proposition 2.1, $\Phi_*^{-1}(\xi_1X_1 + \xi_2X_2 + \xi_3X_3)$ induces the following differential equations on T_1L;

$$\begin{cases} \dot{x} = \xi_1 y\cos\theta - 2\xi_2 y\sin\theta \\ \dot{y} = \xi_1 y\sin\theta + 2\xi_2 y\cos\theta \\ \dot{\theta} = 2\left(\xi_1 \sin^2 \dfrac{\theta}{2} + \xi_2 \sin\theta + \xi_3\right), \end{cases}$$

which imply

$$\begin{cases} \ddot{x} - \dfrac{2\dot{x}\dot{y}}{y} = -(2\xi_3 + \xi_1)\dot{y} \\ \ddot{y} + \dfrac{\dot{x}^2 - \dot{y}^2}{y} = (2\xi_3 + \xi_1)\dot{x}. \end{cases}$$

These prove the case i).

We now assume

$$2\xi_3 + \xi_1 = 0.$$

Then follows

$$\xi_2^2 - \xi_3^2 - \xi_1\xi_3 = \xi_2^2 + \frac{1}{4}\xi_1^2 > 0.$$

Let us introduce new parameters ρ, α as follows:

$$2\xi_2 = \rho\cos\alpha, \;\; \xi_1 = \rho\sin\alpha, \;\; \rho > 0, \;\; 0 \leq \alpha < 2\pi.$$

Then, by the formulas (*) we have

$$\Phi_*^{-1}(\xi_1 X_1 + \xi_2 X_2 + \xi_3 X_3) = -\rho y \sin(\theta - \alpha)\frac{\partial}{\partial x} + \rho y \cos(\theta - \alpha)\frac{\partial}{\partial y} + \rho \sin(\theta - \alpha)\frac{\partial}{\partial \theta}.$$

On the other hand, by Proposition 4.1. the infinitesimal transformation of φ_t is expressed in the form

$$\Phi_*^{-1}X = \frac{1}{2}\Phi_*^{-1}X_2 = -y\sin\theta\frac{\partial}{\partial x} + y\cos\theta\frac{\partial}{\partial y} + \sin\theta\frac{\partial}{\partial \theta}.$$

Therefore we have

$$\Phi_*^{-1}(\xi_1 X_1 + \xi_2 X_2 + \xi_3 X_3) = T_{\alpha *}\rho\Phi_*^{-1}X.$$

We exponentiate both sides to obtain

$$\begin{aligned} \operatorname{Exp} t\Phi_*^{-1}(\xi_1 X_1 + \xi_2 X_2 + \xi_3 X_3) &= \operatorname{Exp} tT_{\alpha *} o\Phi_*^{-1}X \\ &= T_\alpha \cdot \operatorname{Exp} \rho t\Phi_*^{-1}X \cdot T_\alpha^{-1} \\ &= T_\alpha \cdot \varphi_{\rho t} \cdot T_{-\alpha}, \end{aligned}$$

which proves the case ii). (Q.E.D.)

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