# PROPERTIES OF SOLUTIONS OF PARABOLIC EQUATIONS AND INEQUALITIES

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**1. Introduction.** In this paper we shall be concerned with two problems: (i) the asymptotic behavior of solutions of parabolic inequalities and (ii) the uniqueness of the Cauchy problem for such inequalities when the data are prescribed on a portion of a time-like surface. The unifying feature of these rather separate problems is the employment of integral estimates of the same type in both cases.

We consider parabolic operators in self-adjoint form

$$L \equiv \frac{\partial}{\partial t} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial}{\partial x_j}\right), \qquad a_{ij} = a_{ji} \tag{1}$$

as well as the non-self-adjoint operator

$$M \equiv \frac{\partial}{\partial t} - \sum_{i,j=1}^{n} b_{ij}\frac{\partial^2}{\partial x_i \partial x_j}, \qquad b_{ij} = b_{ji}, \tag{2}$$

where the coefficients $a_{ij}(x, t) = a_{ij}\ (x_1, x_2, \ldots , x_n, t)$ are $C^1$ functions of $x$ and $t$ and the $b_{ij} = b_{ij}(x, t)$ are $C^2$ functions of $x$ and $t$. The portions of the operators

$$F \equiv \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial}{\partial x_j}\right)$$

$$G \equiv \sum_{i,j=1}^{n} b_{ij}\frac{\partial^2}{\partial x_i \partial x_j}$$

are assumed to be uniformly elliptic throughout the domain of definition.

To study asymptotic behavior we consider a bounded domain $D$ in $n$-dimensional euclidean space $E_n$ with boundary $\Gamma$. Denote by $I(T)$ the interval $0 \leqslant t \leqslant T$ and by $I$ the half-infinite interval $0 \leqslant t < \infty$. The $(n + 1)$-dimensional product domain $D \times I$ will be designated by $R$ while $S$ will be the portion of the boundary of $R$ consisting of $\Gamma \times I$.

We are interested in the growth of functions $u(x, t)$ which satisfy in $R$ differential inequalities of the form

$$(Lu)^2 \leqslant C_1(t)u^2 + C_2(t) \sum_{i=1}^{n} \left(\frac{\partial u}{\partial x_i}\right)^2 \tag{3}$$

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Received February 24, 1960. This research was supported by the United States Air Force through the Air Force Office of Scientific Research of the Air Research and Development Command under Contract No. AF 49(638)-398.

and

$$(4) \qquad (Mu)^2 \leqslant d_1(t)u^2 + d_2(t) \sum_{i=1}^{n} \left(\frac{\partial u}{\partial x_i}\right)^2$$

or more generally the same inequality in integrated form

$$\int_D (Lu)^2 \leqslant C_1(t) \int_D u^2 + C_2(t) \int_D \sum \left(\frac{\partial u}{\partial x_i}\right)^2$$

$$\int_D (Mu)^2 \leqslant d_1(t) \int_D u^2 + d_2(t) \int_D \sum \left(\frac{\partial u}{\partial x_i}\right)^2.$$

The further condition

$$(5) \qquad u = 0 \quad \text{on} \quad S$$

will be assumed throughout § 2. However the theorems of that section are applicable without change to the condition

$$\frac{\partial u}{\partial \nu} = 0 \qquad \text{on} \qquad S$$

where $\partial/\partial\nu$ is the co-normal derivative defined in the customary manner. In fact with suitable restrictions on $p(x, y)$, $q(x, y)$ the results apply with the more general condition

$$p(x, y) \frac{\partial u}{\partial \nu} + q(x, y)u = 0 \qquad \text{on} \qquad S.$$

We define the functions

$$A_0(t) = \sup_{x \in D} \left| \frac{\partial}{\partial t} a_{ij}(x, t) \right|, \quad i, j = 1, 2, \ldots, n$$

$$B_0(t) = \sup_{x \in D} \left| \frac{\partial}{\partial t} b_{ij}(x, t) \right|, \quad i, j = 1, 2, \ldots, n$$

$$B_1(t) = \sup_{x \in D} \left| \frac{\partial}{\partial x_j} (b_{ij}) \right|^2. \quad i, j = 1, 2, \ldots, n$$

The starting point of the investigation of asymptotic behaviour is the knowledge that solutions of the heat equation

$$\frac{\partial u}{\partial t} = \Delta u$$

which satisfy (5) decay as $e^{-\lambda t}$ for some positive $\lambda$ as $t \to \infty$. This result was extended considerably by Lax **(1)**, who showed that for abstract non-positive operators $N$ defined in a Hilbert space, and for functions $u$ satisfying (5) and an inequality of the form

$$\left|\left|\frac{\partial u}{\partial t} - Nu\right|\right| \leqslant C_1(t)||u||,$$

the rate of decay is again as $e^{-\lambda t}$, provided certain auxiliary conditions on the nature of the spectrum of $D$ and the function $C_1(t)$ are satisfied. Lees **(3)** also investigated the asymptotic behaviour of solutions of differential inequality (3) from the abstract point of view and his results apparently overlap with those given in § 2.

We shall show that under certain conditions on the functions $A_0(t)$, $B_0(t)$, $B_1(t)$, as well as on the functions $C_i(t)$, $d_i(t)$, $i = 1, 2$, solutions of either (3) or (4) decay as $\exp(-\lambda t^\eta)$ for some positive $\lambda$ and some $\eta \geqslant 1$ as $t \to \infty$. In case $u(x, t)$ satisfies the differential equation rather than the inequality, that is, if $C_i(t) = d_i(t) = 0$, $i = 1, 2$, then under natural hypotheses on the coefficients the solutions decay as $\exp(-\lambda t)$ for some positive $\lambda$. The methods employ $L_2$ estimates for functions with compact support in $t$ and kernels depending on $t$, but which merely satisfy (5) as functions of $x$. The estimates are in terms of parabolic operators (3), (4). These inequalities are a more or less natural development of those given in **(5)**, where the estimates are in terms of elliptic operators, and the subsequent ones derived in **(2)**, where the estimates are in terms of parabolic operators; but the functions are assumed to have compact support in $x$ and $t$.

In § 3 the problem of the uniqueness of the Cauchy problem for inequalities (3) or (4) is solved when the data are prescribed on a piece of a time-like surface. This question for parabolic equations was solved by Mizohata **(4)** using the Calderon–Zygmund method of singular integrals. Here the main tool consists of $L_2$ estimates (with a kernel depending on $x$) for functions with compact support in $x$ and $t$ in terms of operators (3), (4).

## 2. Asymptotic behavior.

Let $v(x,t) \equiv v(x_1, x_2, \ldots, x_n,t)$ be a $C^2$ function defined in $R$ and satisfying the conditions

$$v = 0 \quad \text{on} \quad S \tag{6}$$

$$v(x, t) = 0 \quad \text{for} \quad (x, t) \in D \times I(T_0) \tag{7}$$

for some $T_0 > 0$. Further it is supposed that for fixed $\eta > 1$, for every positive $\lambda > 0$ and for all $\beta$ the integral

$$\int_D t^{2\beta} e^{2\lambda t^\eta} \sum_{i=1}^{n} \left(\frac{\partial v}{\partial x_i}\right)^2 \to 0 \text{ as } t \to \infty. \tag{8}$$

Functions $v$ which satisfy (6), (7), and (8) are said to belong to *class* $C(\eta)$. We note that any function in $C(\eta)$ satisfies *a fortiori* the condition

$$\lim_{t\to\infty} \int_D t^{2\beta} e^{2\lambda t^\eta} v^2 = 0.$$

We define the function

$$K \equiv K(\beta, \lambda, \eta) \equiv t^{2\beta} e^{2\lambda t^\eta}.$$

Generally we shall employ the letter $m_0$ as a generic constant, depending only on $n$ and the ellipticity constants in the operators $F$ and $G$.

LEMMA 1. *If $v \in C(\eta)$ we have the inequality*

$$(9)\quad \int_R \left[\lambda\eta(\eta-1)t^{\eta-2} - \frac{\beta}{t^2}\right] K(\beta, \lambda, \eta)v^2$$
$$\leqslant \int_R K(\beta, \lambda, \eta)(Lv)^2 + m_0 \int_R A_0(t)K(\beta, \lambda, \eta) \sum_{i=1}^n \left(\frac{\partial v}{\partial x_i}\right)^2.$$

*Proof.* We define the function

$$z = \mathrm{K}(\tfrac{1}{2}\beta, \tfrac{1}{2}\lambda, \eta)v.$$

Then $z$ also satisfies conditions (6), (7), and (8) and hence is in $C(\eta)$. We have

$$\left(\frac{\partial v}{\partial t} - Fv\right)^2 = K(-\beta, -\lambda, \eta)\{Fz - [z_t - (\beta t^{-1} + \lambda\eta t^{\eta-1})z]\}^2$$

and

$$K(\beta, \lambda, \eta)(Lv)^2 = [Fz - z_t + (\beta t^{-1} + \lambda\eta t^{\eta-1})z]^2.$$

From the elementary inequality

$$(a+b+c)^2 \geqslant 2b(a+c)$$

we obtain

$$K(Lv)^2 \geqslant -2z_tFz - 2(\beta t^{-1} + \lambda\eta t^{\eta-1})zz_t.$$

Let $R(T)$ denote the domain $D \times I(T)$. Integrating this last inequality over the domain $R(T)$ we have

$$-2\int_{R(T)} z_t \sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial z}{\partial x_j}\right) - \int_{R(T)} (\beta t^{-1} + \lambda\eta t^{\eta-1})(z^2)_t \leqslant \int_{R(T)} K(Lv)^2.$$

An integration by parts yields

$$(10)\quad \int_{R(T)} [\lambda\eta(\eta-1)t^{\eta-2} - \beta t^{-2}]z^2 \leqslant \int_{R(T)} K(Lv)^2$$
$$+ \int_{R(T)} \sum_{i,j} \frac{\partial z}{\partial x_i}\frac{\partial z}{\partial x_j}\frac{\partial}{\partial t}(a_{ij}) + J$$

where $J$ consists of integrals taken over the boundary of $R(T)$. All such integrals vanish because of the boundary conditions except those taken over the portion where $t = T$. Since $z \in C(\eta)$ these integrals tend to zero as $t \to \infty$. Recalling the definition of $A_0(t)$ and noting that $K$ is independent of $x$ we have

$$\left|\int_{R(T)} \sum \frac{\partial z}{\partial x_i}\frac{\partial z}{\partial x_j}\frac{\partial}{\partial t}(a_{ij})\right| \leqslant m_0 \int_{R(T)} A_0(t)K \sum_{i=1}^n \left(\frac{\partial v}{\partial x_i}\right)^2.$$

Substituting this in (10), inserting $z$ in terms of $v$, and letting $T \to \infty$ we obtain (9).

LEMMA 2. *If $v \in C(\eta)$ we have the inequality*

$$(11)\quad \int_R K(\beta, \lambda, \eta) \sum_{i=1}^n \left(\frac{\partial v}{\partial x_i}\right)^2 \leqslant m_0 \int_R tK(Lv)^2 + m_0 \int_R (\lambda\eta t^{\eta-1} + |\beta|t^{-1})Kv^2.$$

*Proof.* We consider the identity

$$(12)\quad \int_{R(T)} KvLv = \frac{1}{2}\int_{R(T)} (Kv^2)_t - \beta \int_{R(T)} t^{-1}Kv^2 - \lambda\eta \int_{R(T)} t^{\eta-1}Kv^2$$
$$- \int_{R(T)} \sum_{i,j} \left(Ka_{ij}v \frac{\partial v}{\partial x_j}\right) + \int_{R_T} K \sum_{i,j} a_{ij} \frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j}.$$

The ellipticity of the operator $F$ asserts that there exist constants $\alpha_0$, $\alpha_1$ such that

$$\alpha_0 \sum_{i=1}^n \xi_i^2 \leqslant \sum_{i,j=1}^n \alpha_{ij}\xi_i\xi_j \leqslant \alpha_1 \sum_{i=1}^n \xi_i^2$$

for all real $n$-dimensional vectors $(\xi_1, \xi_2, \ldots, \xi_n)$. The uniform ellipticity simply means that $\alpha_0$, $\alpha_1$ are independent of $(x, t)$. Hence, integrating the identity (12) by parts and employing the above inequality, we find

$$\int_{R(T)} K \sum \left(\frac{\partial v}{\partial x_i}\right)^2 \leqslant m_0 \int_{R(T)} K|vLv| + m_0|\beta| \int_{R(T)} t^{-1}Kv^2$$
$$+ m_0\lambda\eta \int_{R(T)} t^{\eta-1}Kv^2 + J.$$

Again $J$ denotes surface integrals along $t = T$ which tend to zero as $T \to \infty$. We apply Cauchy's inequality to the first term on the right and obtain inequality (11) by letting $T$ tend to infinity.

Similar inequalities are obtained with respect to the operator $M$.

LEMMA 3. *If $v \in C(\eta)$ we have the inequality*

$$(13)\quad \int_R \left[\lambda\eta(\eta - 1)t^{\eta-2} - \frac{\beta}{t^2}\right] K(\beta, \lambda, \eta)v^2$$
$$\leqslant \int_R K(Mv)^2 + m_0 \int_R [B_0(t) + B_1(t)]K \sum_{i=1}^n \left(\frac{\partial v}{\partial x_i}\right)^2.$$

*Proof.* We define the function $z$ as in Lemma 1 and obtain

$$\left(\frac{\partial v}{\partial t} - Gv\right)^2 = K(-\beta, -\lambda, \eta)\{Gz - z_t + (\beta t^{-1} + \lambda\eta t^{\eta-1})z\}^2.$$

Using the elementary inequality

$$(a + b + c)^2 \geqslant b^2 + 2b(a + c)$$

we get

$$K(Mv)^2 \geqslant -2z_tGz - 2(\beta t^{-1} + \lambda\eta t^{\eta-1})zz_t + z_t^2.$$

Hence integrating over $R(T)$ we find after integrating by parts

$$\int_{R(T)} \left(\frac{\partial z}{\partial t}\right)^2 + \int_{R(T)} (\lambda\eta(\eta-1)t^{\eta-2} - \beta t^{-2})z^2 + 2\int_{R(T)} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}(b_{ij})\frac{\partial z}{\partial x_i}\frac{\partial z}{\partial t}$$
$$-\int_{R(T)} \sum_{i,j=1}^{n} \frac{\partial}{\partial t}(b_{ij})\frac{\partial z}{\partial x_i}\frac{\partial z}{\partial x_j} \leqslant \int_{R(T)} K(Mv)^2 + J,$$

where $J$ has its usual meaning. The last integral on the left is dominated by

$$m_0\int_{R(T)} B_0(t)K\sum_{i=1}^{n}\left(\frac{\partial v}{\partial x_i}\right)^2.$$

We also have the inequality

$$2\left|\int \sum \frac{\partial}{\partial x_i}(b_{ij})\frac{\partial z}{\partial x_i}\frac{\partial z}{\partial t}\right| \leqslant \frac{1}{2}\int\left(\frac{\partial z}{\partial t}\right)^2 + m_0\int B_1(t)K\sum_{i=1}^{n}\left(\frac{\partial v}{\partial x_i}\right)^2.$$

These inequalities combine to yield (13).

LEMMA 4. *If $v \in C(\eta)$ we have the inequality*

$$\int_R K(\beta, \lambda, \eta)\sum_{i=1}^{n}\left(\frac{\partial v}{\partial x_i}\right)^2 \leqslant m_0\int_R tK(Mv)^2 + m_0\int_R(\lambda\eta t^{\eta-1} + |\beta|t^{-1} + B_1(t))Kv^2. \tag{14}$$

*Proof.* We consider the identity

$$\int_{R(T)} KvMv = \frac{1}{2}\int_{R(T)}(Kv^2)_t - \beta\int_{R(T)} t^{-1}Kv^2 - \lambda\eta\int_{R(T)} t^{\eta-1}Kv^2$$
$$-\int_{R(T)}\sum_{i,j}\frac{\partial}{\partial x_i}\left(Kb_{ij}v\frac{\partial v}{\partial x_j}\right) + \int_{R(T)}\sum_{i,j=1}^{n} Kb_{ij}\frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j}$$
$$+\int_{R(T)}\sum_{i,j=1}^{n} K\frac{\partial}{\partial x_i}(b_{ij})v\frac{\partial v}{\partial x_j}.$$

From the ellipticity condition we have

$$m_0\int_{R(T)}\sum Kb_{ij}\frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j} \geqslant \int_{R(T)} K\sum_{i=1}^{n}\left(\frac{\partial v}{\partial x_i}\right)^2,$$

and from Cauchy's inequality

$$\left|\int_{R(T)} K\sum\frac{\partial}{\partial x_i}(b_{ij})v\frac{\partial v}{\partial x_j}\right| \leqslant \frac{1}{2}\int_{R(T)} K\sum_{i=1}^{n}\left(\frac{\partial v}{\partial x_i}\right)^2 + m_0\int_{R(T)} KB_1(t)v^2.$$

Hence, after an integration by parts, the above identity yields the inequality

$$\int_{R(T)} K\sum_{i=1}^{n}\left(\frac{\partial v}{\partial x_i}\right)^2 \leqslant \int_{R(T)} K|vMv| + m_0\int_{R(T)}(\lambda\eta t^{\eta-1} + |\beta|t^{-1} + B_1(t))Kv^2 + J.$$

Inequality (14) is obtained by letting $T \to \infty$.

LEMMA 5. *Let* $v \in C(\eta)$, $\eta > 1$ *and suppose* $A_0(t) = 0(t^{-1})$. *Let* $v \equiv 0$ *in* $D \times I(T^*)$ *where* $T^*$ *depends on* $A_0(t)$. *Then for sufficiently large* $\lambda$ *we have the inequality*

$$(15) \qquad \lambda \int_R t^{\eta-2} K(\beta, \lambda, \eta) v^2 + \int_R t^{-1} K \sum_{i=1}^n \left(\frac{\partial v}{\partial x_i}\right)^2 \leqslant m_0 \int_R K(Lv)^2.$$

*Proof.* From (9) for sufficiently large $\lambda$ and for $\eta > 1$, the expression on the left in (9) is dominated by

$$\lambda m_0 \int_R t^{\eta-2} K v^2$$

and we have the inequality

$$(16) \quad \lambda \int_R t^{\eta-2} K(\beta, \lambda, \eta) v^2 \leqslant m_0 \int_R K(\beta, \lambda, \eta)(Lv)^2 + m_0 \int_R A_0(t) K(\beta, \lambda, \eta) \sum_{i=1}^n \left(\frac{\partial v}{\partial x_i}\right)^2.$$

Replacing $\beta$ by $\beta + \frac{1}{2}$ in (16) we get

$$(17) \quad \lambda \int_R t^{\eta-1} K(\beta, \lambda, \eta) v^2 \leqslant m_0 \int_R t K(\beta, \lambda, \eta)(Lv)^2 + m_0 \int_R t A_0(t) K(\beta, \lambda, \eta) \sum_{i=1}^n \left(\frac{\partial v}{\partial x_i}\right)^2.$$

Similarly substituting $\beta - \frac{1}{2}(\eta - 1)$ for $\beta$ in (16) yields

$$(18) \qquad \lambda \int t^{-1} K v^2 \leqslant m_0 \int_R t^{1-\eta} K(Lv)^2 + m_0 \int_R t^{1-\eta} A_0(t) K \sum_{i=1}^n \left(\frac{\partial v}{\partial x_i}\right)^2.$$

If (17) and (18) are inserted into the right side of (11) we find

$$(19) \quad \int_R K(\beta, \lambda, \eta) \sum_{i=1}^n \left(\frac{\partial v}{\partial x_i}\right)^2 \leqslant m_0 \int_R t K(\beta, \lambda, \eta)(Lv)^2 + m_0 \int_R t A_0(t) K(\beta, \lambda, \eta) \sum_{i=1}^n \left(\frac{\partial v}{\partial x_i}\right)^2.$$

We now replace $\beta$ by $\beta - \frac{1}{2}$ in (19) and add the result to (16). This gives

$$\lambda \int_R t^{\eta-2} K v^2 + \int_R [t^{-1} - 2m_0 A_0(t)] K \sum_{i=1}^n \left(\frac{\partial v}{\partial x_i}\right)^2 \leqslant m_0 \int_R K(Lv)^2.$$

Since by hypothesis we have $A_0(t) = 0(t^{-1})$ we may select $T^*$ so large that $1 - 2m_0 A(t) \geqslant \frac{1}{2}$ for all $t \geqslant T^*$. With this choice of $T^*$ (15) follows at once.

THEOREM 1. *Let* $u(x, t)$ *satisfy inequality* (3):

$$(Lu)^2 \leqslant c_1(t) u^2 + c_2(t) \sum_{i=1}^n \left(\frac{\partial u}{\partial x_i}\right)^2$$

*in* $R$. *Let* $u$ *vanish on* $S$ *and suppose condition* (8):

$$\lim_{t\to\infty}\int_D t^{2\beta}e^{2\lambda t^\eta}\sum_{i=1}^{n}\left(\frac{\partial u}{\partial x_i}\right)^2 = 0$$

*holds* for some fixed* $\eta > 1$. *If* $c_1(t) = 0(t^{\eta-2})$, $c_2(t) = 0(t^{-1})$, *and* $A_0(t) = 0(t^{-1})$ *then* $u \equiv 0$ *in* $R$.

*Proof.* We define $\zeta = \zeta(t)$ as a monotone increasing smooth function of $t$ so that

$$\zeta = \begin{cases} 0\ , & 0 \leqslant t \leqslant T_1 \\ 0 < \zeta < 1, & T_1 \leqslant t \leqslant T_2\,. \\ 1\ , & T_2 \leqslant t < \infty \end{cases}$$

We select $T_1$ to satisfy two conditions. First, $T_1$ is selected larger than the quantity $T^*$ determined in Lemma 5. Second, $T_1$ is increased, if necessary, so that $m_0tc_2(t) \leqslant \frac{1}{2}$ for $t \geqslant T_1$ where $m_0$ is the constant in the right side of inequality (15). The function

$$v(x, t) = \zeta(t)u(x, t)$$

is in class $C(\eta)$ and inequality (15) is valid for $v$. We define $R(T_2 - T_1)$ to be the domain $D \times (I(T_2) - I(T_1))$ and $R(T_2)$ the domain $D \times (I - I(T_2))$. We have from (15) applied to $v$:

$$\lambda\int_{R(T_2)} t^{\eta-2}Ku^2 + \int_{R(T_2)} t^{-1}K\sum\left(\frac{\partial u}{\partial x_i}\right)^2 \leqslant m_0\int_{R(T_2-T_1)} K(Lv)^2$$
$$+ m_0\int_{R(T_2)} K(Lu)^2,$$

since the left side is decreased by omission of the integrals taken over the domain $R(T_2 - T_1)$. We substitute (3) into the last integral on the right and get

$$\int_{R(T_2)} [\lambda t^{\eta-2} - m_0c_1(t)]Ku^2 + \int_{R(T_2)} [t^{-1} - m_0c_2(t)]K\sum\left(\frac{\partial u}{\partial x_i}\right)^2$$
$$\leqslant m_0\int_{R(T_2-T_1)} (Lv)^2.$$

Since $t^{2-\eta}c_1(t)$ is bounded we select $\lambda$ so large that the coefficient of $Ku^2$ is dominated by $\frac{1}{2}\lambda t^{\eta-2}$. Further the integrals on the left are decreased if the range of integration is diminished to $R(T_3)$ for some $T_3 > T_2$. Hence

$$\int_{R(T_3)} \lambda t^{\eta-2}Ku^2 + \int_{R(T_3)} t^{-1}K\sum\left(\frac{\partial u}{\partial x_i}\right)^2 \leqslant 2m_0\int_{R(T_2-T_1)} K(Lu)^2.$$

From the definition of $K$, we obtain

$$T_3^{2\beta}e^{2\lambda T_3^\eta}\left[\lambda\int_{R(T_3)} u^2 + \int_{R(T_3)} t^{-1}\sum\left(\frac{\partial u}{\partial x_i}\right)^2\right] \leqslant 2m_0T_2^{2\beta}e^{2\lambda T_2^\eta}\int (Lv)^2.$$

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*If $c_2(t) \equiv 0$, the square of the gradient in (8) should be replaced by the square of the function $u$.

Letting $\lambda \to \infty$ we see at once that $u \equiv 0$ for $t \geqslant T_3$. Thus $u$ satisfies (3), vanishes on $S$, and vanishes for $t \geqslant T_3$. Theorem 1 of Lees and Protter **(2)** now applies, so we conclude that $u$ vanishes identically in $R$.

To prove the theorem corresponding to Theorem 1 for operators which are not self-adjoint we first establish the inequality analogous to (15).

LEMMA 6. *Let $v \in C(\eta)$, $n > 1$ and suppose $B_0(t) = o(t^{-1})$, $B_1(t) = o(t^{-1})$. Let $v \equiv 0$ in $D \times I(T^*)$ where $T^*$ depends on $B_0$, $B_1$. Then for sufficiently large $\lambda$ we have the inequality*

$$(20) \qquad \lambda \int_R t^{\eta-2} K(\beta, \lambda, \eta) v^2 + \int_R t^{-1} K \sum_{i=1}^n \left(\frac{\partial v}{\partial x_i}\right)^2 \leqslant m_0 \int_R K(Mv)^2$$

*Proof.* The establishment of (20) follows from Lemmas 3 and 4 in the same way that (15) was obtained from Lemmas 1 and 2. With the aid of Lemma 6 the proof of the following result parallels the proof of Theorem 1.

THEOREM 2. *Let $u(x, t)$ satisfy inequality* (4):

$$(Mu)^2 \leqslant d_1(t) u^2 + d_2(t) \sum_{i=1}^n \left(\frac{\partial u}{\partial x_i}\right)^2$$

*in $R$. Let $u$ vanish on $S$ and suppose condition* (8):

$$\lim_{t\to\infty} \int_D t^{2\beta} e^{2\lambda t^\eta} \sum_{i=1}^n \left(\frac{\partial u}{\partial x_i}\right)^2 = 0$$

*holds for some fixed $\eta > 1$. If $d_1(t) = 0(t^{\eta-2})$, $d_2(t) = 0(t^{-1})$, $B_0(t) = o(t^{-1})$, $B_1(t) = o(t^{-1})$ then $u \equiv 0$ in $R$.*

The basic inequalities of Lemmas 1 and 2 vary slightly for the case $\eta = 1$, that is, for solutions which decay as $e^{-\lambda t}$ for some positive $\lambda$. For this purpose we state the following inequalities.

LEMMA 7. *If $v \in C(1)$ we have the inequality*

$$-\beta \int_R t^{-2} K(\beta, \lambda, 1) v^2 \leqslant \int_R K(\beta, \lambda, 1)(Lv)^2 + m_0 \int_R A_0(t) K(\beta, \lambda, 1) \sum_{i=1}^n \left(\frac{\partial v}{\partial x_i}\right)^2$$

*valid for all $\beta$. This is obtained directly from Lemma 1 by setting $\eta \equiv 1$. For convenience we write $K(\beta, \lambda)$ for $K(\beta, \lambda, 1)$.*

LEMMA 8. *If $v \in C(1)$ we have the inequality*

$$\int_R K(\beta, \lambda) \sum_{i=1}^n \left(\frac{\partial v}{\partial x_i}\right)^2 \leqslant m_0 \int_R tK(Lv)^2 + m_0 \int_R (\lambda + |\beta| t^{-1}) K v^2.$$

This follows from Lemma 2 by setting $\eta = 1$. Combining Lemmas 7 and 8 we get:

LEMMA 9. *Let $v \in C(1)$ and suppose $A_0(t) = o(t^{-2})$. Let $v \equiv 0$ in $D \times I(T^*)$ where $T^*$ depends on $A_0(t)$. Then for sufficiently large $\lambda$ and $-\beta$ we have the inequality*

$$\int_R t^{-2}K(\beta, \lambda)v^2 + \int_R t^{-2}K \sum_{i=1}^{n} \left(\frac{\partial v}{\partial x_i}\right)^2 \leqslant \frac{\lambda}{\beta} m_0 \int_R K(Lv)^2.$$

This lemma is a consequence of Lemmas 7 and 8 in the same manner that Lemma 5 is derived from Lemmas 1 and 2. We thus obtain:

THEOREM 3. *Let $u(x, t)$ satisfy inequality* (3) *in $R$. Let $u$ vanish on $S$ and suppose condition* (8) *holds for $\eta = 1$. If $c_1(t)$, $c_2(t)$, and $A_0(t)$ are all $o(t^{-2})$ as $t \to \infty$ then $u \equiv 0$ in $R$. A similar result holds for inequality* (4) *pertaining to operators not in self-adjoint form.*

The results of this section are easily extended to operators of the form

$$M_1 \equiv \frac{\partial}{\partial t} - \sum_{i,j=1}^{n} b_{ij}(x, t) \frac{\partial^2}{\partial x_i \partial x_j} + e(x, t)$$

and the corresponding differential inequality

$$(M_1 u)^2 \leqslant d_1(t)u^2 + d_2(t) \sum_{i=1}^{n} \left(\frac{\partial u}{\partial x_i}\right)^2.$$

If the function $e(x, t)$ is bounded and satisfies the condition

$$\frac{\partial e}{\partial t} = 0(t^{\eta-2}), \qquad t \to \infty$$

then Theorem 2 is valid for operators $M_1$ with the proof unchanged. Similarly Theorem 1 holds for operators $L_1$ containing a zero order term. In particular if $e$ is independent of $t$ the above condition is automatically satisfied and merely boundedness suffices.

## 3. Cauchy problem with data on a time-like surface.

In this section we shall be concerned with the uniqueness of the Cauchy problem for the general inequality (4) with data given on a piece of time-like surface. In other words, we shall suppose that on a portion of the boundary surface $S$, say $S_0$, we prescribe

$$u = \frac{\partial u}{\partial n} = 0$$

where $\partial/\partial n$ is the derivative taken in a direction normal to $S$. From this we shall conclude that $u$ vanishes in the subregion of $R$ contained in the strip $T_1 \leqslant t \leqslant T_2$, where $T_1$ is the minimum value of $t$ in $S_0$ and $T_2$ is the maximum value of $t$ in $S_0$. The extension to the case where $S_0$ is any time-like surface is easily made. For this purpose we need two lemmas similar to ones established in **(2)**. We introduce Euclidean distance $r$ in $E_n$, that is,

$$r^2 = \sum_{i=1}^{n} x_i^2.$$

LEMMA 10. *Let $u \in C^2$ vanish outside the cylindrical domain $R(T): r_0 \leqslant r \leqslant r_1$, $0 \leqslant t \leqslant T$. Then for $r_1$ sufficiently small and for all sufficiently large $\beta$ we have*

$$(21) \quad \beta^4 \int_{R(T)} r^{-2\beta-2} e^{2r^{-\beta}} u^2 - m\beta r_0 \int_{R(T)} e^{2r^{-\beta}} \sum_{i=1}^{n} \left(\frac{\partial u}{\partial x_i}\right)^2 \leqslant m_0 \int_{R(T)} r^{\beta+2} (Mu)^2.$$

*Proof.* We select $r_1$ so small that

$$b_{ij}(0, t) = \delta_{ij} + b_{ij}^0$$

where

$$|b_{ij}^0| \leqslant m_0 r_1.$$

This can always be done by a change of independent variable. As before, we define

$$z = e^{r^{-\beta}} u$$

and consider the expression

$$r^{\beta+2} e^{2r^{-\beta}} (Mu)^2.$$

We have

$$(22) \quad r^{\beta+2} e^{2r^{-\beta}} (Gu - u_t)^2 = r^{\beta+2} \Big[ Gz + 2e^{r^{-\beta}} \sum_{i,j} b_{ij} \frac{\partial z}{\partial x_i} \frac{\partial}{\partial x_j} (e^{-r^{-\beta}}) + z\, e^{r^{-\beta}} G(e^{-r^{-\beta}}) - z_t \Big]^2.$$

We note that

$$e^{r^{-\beta}} \frac{\partial}{\partial x_j} (e^{-r^{-\beta}}) = \beta x_j r^{-\beta-2}$$

and use the elementary inequality

$$(a + b + c - d)^2 \geqslant (b - d)^2 + 2(b - d)(a + c).$$

Interpreting

$$a = Gz,$$

$$b = 2e^{r^{-\beta}} \sum b_{ij} \frac{\partial z}{\partial x_i} \frac{\partial}{\partial x_j} (e^{-r^{-\beta}}),$$

$$c = z\, e^{r^{-\beta}} G(e^{-r^{-\beta}}),$$

$$d = z_t$$

we get from (22)

$$r^{\beta+2} e^{2r^{-\beta}} (Mu)^2 \geqslant b^2 - 2bd + d^2 + 2ab + 2bc - 2ad - 2cd.$$

We now integrate throughout $(x, t)$ space. Each integral which contains $b_{ij}$ is further decomposed into integrals with $\delta_{ij}$, the principal part and $b_{ij}{}^0$, the residual part. Thus, for example, the principal part of $2ab$ is

$$2\beta\int\sum_{i=1}^{n} x_i\frac{\partial z}{\partial x_i}\Delta z$$

and this integral is non-negative. The residual part leads to an integral of the form

$$\beta\, m_0 r_1\int e^{2r^{-\beta}}\sum_{i=1}^{n}\left(\frac{\partial u}{\partial x_i}\right)^2.$$

The integrals $b^2 - 2bd + d^2 - 2ad$ yield a positive definite quadratic form for sufficiently large $\beta$. The principal part of the integral $2cd$ vanishes. The integral $2bc$ yields the term

$$\beta^4\int r^{-2\beta-2}e^{2r^{-\beta}}u^2.$$

These combine to give (21).

LEMMA 11. *Under the hypotheses of Lemma* 10 *we have*

$$(23)\quad \beta^4\int r^{-2\beta-2}e^{2r^{-\beta}}u^2 + \beta m_0\int e^{2r^{-\beta}}\sum\left(\frac{\partial u}{\partial x_1}\right)^2 \leqslant m_0\int r^{\beta+2}e^{2r^{-\beta}}(Mu)^2.$$

*Proof.* For functions $u$ with compact support and an arbitrary $C^2$ function, $a(x)$, independent of $t$, we have the identity

$$\int a\,u(u_t - Gu) = -\int a\,u\,Gu$$
$$= \int a\sum b_{ij}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} - \frac{1}{2}\int u^2\left[Gu + \sum a\frac{\partial^2 b_{ij}}{\partial x_i\partial x_j} + 2\frac{\partial a}{\partial x_i}\frac{\partial b_{ij}}{\partial x_j}\right].$$

Since $G$ is uniformly elliptic, when we select

$$a = e^{2r^{-\beta}},$$

we get

$$\int e^{2r^{-\beta}}\sum\left(\frac{\partial u}{\partial x_i}\right)^2 \leqslant m_0\int e^{2r^{-\beta}}|u\,Mu| + m_0\beta^2\int r^{-2\beta-2}e^{2r^{-\beta}}u^2.$$

We apply Cauchy's inequality to the first term on the right and obtain

$$\int e^{2r^{-\beta}}\sum\left(\frac{\partial u}{\partial x_i}\right)^2 \leqslant m_0 r^{\beta}\int r^{\beta+2}e^{2r^{-\beta}}(Mu)^2 + m_0\beta^2\int r^{-2\beta-2}e^{2r^{-\beta}}u^2.$$

We multiply this inequality by $\beta$ and add to (21). For $\beta$ sufficiently large and $r_1$ sufficiently small we deduce (23).

THEOREM 4. *Let $u$ satisfy inequality* (4) *in a region $R(T)$ and suppose that on a portion $S_0$ of the boundary $S$ the condition*

$$u = \frac{\partial u}{\partial n} = 0$$

*holds. Then $u \equiv 0$ in the subregion of $R(T)$: $T_1 \leqslant t \leqslant T_2$ where $T_1$ is the minimum and $T_2$ the maximum value of $t$ in $S_0$.*

*Proof.* We select the origin of our co-ordinate system outside of $R + S$ but so close to a point of $S_0$ that the distance $r_0$ of Lemma 10 is exterior to $R + S$ while the distance $r_1$ is interior to $R + S$.

We define the functions $\zeta_1(r)$, $\zeta_2(t)$ so that

$$\zeta_1(r) = \begin{cases} 1 & , \quad 0 \leqslant r \leqslant r_1 \\ 0 < \zeta_1 < 1 & , \quad r_1 \leqslant r \leqslant r_2 \\ 0 & , \quad r \geqslant r_2 \end{cases}$$

and

$$\zeta_2(t) = \begin{cases} 0 & , \quad t \geqslant T_3 \\ 0 < \zeta_2 < 1 & , \quad T_4 \leqslant t \leqslant T_3 \\ 1 & , \quad T_5 \leqslant t \leqslant T_4 \\ 0 < \zeta_2 < 1 & , \quad T_6 \leqslant t \leqslant T_5 \\ 0 & , \quad t \leqslant T_6 \end{cases}$$

where $T_3 < T_2$ and $T_6 > T_1$ and the functions $\zeta_1$, $\zeta_2$ are in $C^2$. In general we denote by $E(r_i, T_j, T_k)$ the region $0 \leqslant r \leqslant r_i$, $T_j \leqslant t \leqslant T_R$. We now define the function

$$v = \zeta_1(r)\zeta_2(t)u.$$

Then $v$ satisfies the conditions of Lemmas 10 and 11 so that (23) applied to $v$ yields

$$\beta^4 \int_{E(r_1,T_6,T_3)} r^{-2\beta-2} e^{2r^{-\beta}} v^2 + \beta m_0 \int_{E(r_1,T_6,T_3)} e^{2r^{-\beta}} \sum \left(\frac{\partial v}{\partial x_i}\right)^2 \leqslant m_0 \int_{E(r_2,T_6,T_3)} r^{\beta+2} e^{2r^{-\beta}} (Mv)^2.$$

Taking into account the fact that $\zeta_1$ and $\zeta_2$ are identically 1 in certain ranges of the variables we have

$$\begin{aligned} \beta^4 \int_{E(r_1,T_5,T_4)} r^{-2\beta-2} e^{2r^{-\beta}} u^2 + \beta m_0 \int_{E(r_1,T_5,T_4)} e^{2r^{-\beta}} \sum \left(\frac{\partial u}{\partial x_i}\right)^2 \\ + \beta^4 \int_{E(r_1,T_4,T_3)+E(r_1,T_6,T_5)} r^{-2\beta-2} e^{2r^{-\beta}} \zeta_2 u^2 \\ + \beta m_0 \int_{E(r_1,T_4,T_3)+E(r_1 T_6,T_5)} e^{2r^{-\beta}} \zeta_2 \sum \left(\frac{\partial u}{\partial x_i}\right)^2 \\ \leqslant m_0 \int_{E(r_1,T_6,T_3)} r^{\beta+2} e^{2r^{-\beta}} (M\zeta_2 u)^2 + m_0 \int_{E(r_2,T_6,T_3)-E(r_1,T_6,T_3)} r^{\beta+2} e^{2r^{-\beta}} (Mv)^2. \end{aligned}$$

We note that $(M\zeta_2 u)^2 = (\zeta_2'(t)u + \zeta_2(Mu))^2 \leqslant 2\zeta_2'^2 u^2 + 2\zeta_2^2 (Mu)^2$. Hence the first term on the right-hand side is dominated by

$$m_0 \int_{E(r_1,T_5,T_4)} r^{\beta+2} e^{2r^{-\beta}} (Mu)^2 + 2m_0 \int_{E(r_1 T_4 T_3)+E(r_1,T_6,T_5)} r^{\beta+2} e^{2r^{-\beta}} [\zeta_2'^2 u^2 + 2\zeta_2^2 (Mu)^2].$$

In these integrals we replace $(Mu)^2$ by larger quantities as given by (4) to obtain

$$\begin{aligned}\int_{E(r_1,T_5,T_4)} & \beta^4 r^{-2\beta-2} e^{2r^{-\beta}} u^2 + \beta\, m_0\, e^{2r^{-\beta}} \sum \left(\frac{\partial u}{\partial x_i}\right)^2 \\ & + \int_{E(r_1,T_4,T_3)+E(r_1,T_6,T_5)} \beta^4 r^{-2\beta-2} e^{2r^{-\beta}} \zeta_2 u^2 + \beta\, m_0\, e^{2r^{-\beta}} \zeta_2 \sum \left(\frac{\partial u}{\partial x_i}\right)^2 \\ & \leqslant m_0 \int_{E(r_1,T_5,T_4)} r^{\beta+2} e^{2r^{-\beta}} \left(u^2 + \sum \left(\frac{\partial u}{\partial x_i}\right)^2\right) \\ & + 2m_0 \int_{E(r_1,T_4,T_3)+E(r_1,T_6,T_5)} r^{\beta+2} e^{2r^{-\beta}} \left[\zeta_2'^2 u^2 + 2\zeta_2^2 u^2 + 2\zeta_2 \sum \left(\frac{\partial u}{\partial x_i}\right)^2\right] \\ & + m_0 \int_{E(r_2,T_6,T_3)-E(r_1,T_6,T_3)} r^{\beta+2} e^{2r^{-\beta}} (Mv)^2.\end{aligned}$$

For $\beta$ sufficiently large the first integral on the left dominates the first integral on the right and the second integral on the left dominates the second integral on the right. Thus we find

$$\begin{aligned}\int_{E(r_1,T_5,T_4)} & \beta^4 r^{-2\beta-2} e^{2r^{-\beta}} u^2 + \beta m_0\, e^{2r^{-\beta}} \sum \left(\frac{\partial u}{\partial x_i}\right)^2 \\ & \leqslant m_0 \int_{E(r_2,T_6,T_3)-E(r_1,T_6,T_3)} r^{\beta+2} e^{2r^{-\beta}} (Mv)^2.\end{aligned}$$

We now select $r_3 < r_1$ but sufficiently large so that the cylinder of radius $r_3$, axis along $x = 0$, intersects $R$. Then the above inequality is strengthened if the domain of integration on the left is reduced to $E(r_3, T_5, T_4)$. The above inequality may now be replaced by

$$\begin{aligned}\beta^4 r_3^{-2\beta-2} e^{2r_3^{-\beta}} & \int_{E(r_3,T_5,T_4)} u^2 + \beta m_0\, e^{2r_3^{-\beta}} \int_{E(r_3,T_5,T_4)} \sum \left(\frac{\partial u}{\partial x_i}\right)^2 \\ & \leqslant m_0 r_2^{\beta+2} e^{2\bar{r}^{-\beta}} \int_{E(r_2,T_6,T_3)-E(r_1,T_6,T_3)} (Mv)^2\end{aligned}$$

where $\bar{r}$ is the minimum value of $r$ in $E(r_2, T_6, T_3) - E(r_1, T_6, T_3)$. We note that from the manner in which the domains were determined the quantity $\bar{r}$ is larger than $r_3$. Now letting $\beta \rightarrow \infty$ we easily conclude that $u \equiv 0$ in $E(r_3, T_5, T_4)$. Proceeding step by step we conclude that $u \equiv 0$ for $T_1 \leqslant t \leqslant T_2$ and the proof is complete.

From the method of proof it is clear that the extension to zero Cauchy data given on a piece of an arbitrary time-like surface is immediate.

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