

Elliptic $K3$ Surfaces with Geometric Mordell–Weil Rank 15

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Abstract. We prove that the elliptic surface $y^2 = x^3 + 2(t^8 + 14t^4 + 1)x + 4t^2(t^8 + 6t^4 + 1)$ has geometric Mordell–Weil rank 15. This completes a list of Kuwata, who gave explicit examples of elliptic $K3$ -surfaces with geometric Mordell–Weil ranks $0, 1, \dots, 14, 16, 17, 18$.

1 Introduction

The Mordell–Weil rank r of a Jacobian elliptic surface $\pi: X \rightarrow C$ is defined as the rank of the group of sections of π . If X is a $K3$ surface, then it follows easily that $C = \mathbf{P}^1$. If one works over a field of characteristic 0, then it is well known that $0 \leq r \leq 18$. (In positive characteristic we know that $0 \leq r \leq 20$.)

By a result of Cox [4], there exists a Jacobian elliptic $K3$ surface defined over \mathbf{C} with any given Mordell–Weil rank r , with r an integer, $0 \leq r \leq 18$. Actually, using reasoning similar to [4], one can show there are infinitely many $18 - r$ -dimensional families of Jacobian elliptic $K3$ surfaces defined over \mathbf{C} , with Mordell–Weil rank r . The examples constructed in the proof of Cox are not explicit: the existence of such examples follows from properties of the period map.

Kuwata [10] has given a list of explicit Weierstrass equations for elliptic $K3$ surfaces defined over \mathbf{Q} with Mordell–Weil rank r (over $\overline{\mathbf{Q}}$) for any r between 0 and 18, except for the case $r = 15$.

The aim of this paper is to complete this list by producing an explicit example of an elliptic $K3$ surface with Mordell–Weil rank 15. This is achieved in two steps. In Section 3 we prove the following.

Theorem 1.1 *Let K be an algebraically closed field, with $\text{char}(K) \neq 2, 3$. Let $a, b, c \in K$. Let $E_{a,b,c}/K(s)$ be the curve given by the Weierstrass equation*

$$y^2 = x^3 + A_{a,b,c}(s)x + B_{a,b,c}(s),$$

with

$$A_{a,b,c}(s) = 4a^3b^3((b-a)cs^8 + (4ab - 2ac - 2bc)s^4 + (b-a)c),$$

$$B_{a,b,c} = 16a^5b^5s^2((a-b)s^8 + 2(b+a)s^4 + (a-b)).$$

For a general $(a, b, c) \in K^3$ this defines an elliptic $K3$ surface with 24 fibers of type I_1 and Mordell–Weil rank at least 15. In case $K = \mathbf{C}$, a generic member of this family has Mordell–Weil rank 15.

Received by the editors February 28, 2005.

AMS subject classification: Primary: 14J27 14J28; secondary: 11G05.

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The strategy of our proof is the following. We start with a Jacobian elliptic $K3$ surface $\pi: Y \rightarrow \mathbf{P}^1$ such that there are exactly 15 components of fibers of π not intersecting the zero-section. The Shioda–Tate formula (Theorem 2.4) implies that $\rho(Y) := \text{rank } NS(Y) \geq 17$.

The particular examples $\pi: X \rightarrow \mathbf{P}^1$ presented here allow a degree 8 base-change of π such that its associated relatively minimal model $\varphi: X \rightarrow \mathbf{P}^1$ has only irreducible fibers and X is a $K3$ surface as well. One can show in different ways (either in a direct and elaborate way or, if $K = \mathbf{C}$, using a powerful result in Hodge theory) that $\rho(X) = \rho(Y)$. The Shioda–Tate formula (Theorem 2.4) implies that the Mordell–Weil rank of φ is at least 15.

In Section 5 we give an explicit example.

Theorem 1.2 *Assume that $K = \overline{\mathbf{Q}}$. The elliptic $K3$ surface $\pi: X \rightarrow \mathbf{P}^1$ with Weierstrass equation $y^2 = x^3 + 2(t^8 + 2t^4 + 1)x - 4t^2(t^8 - 6t^4 + 1)$ has Mordell–Weil rank 15.*

The surface X is isomorphic to the surface obtained by choosing $(a, b, c) = (2, 4, 2)$ in the equations of Theorem 1.1.

We use the following strategy to prove Theorem 1.2. The proof of Theorem 1.1 yields that the Mordell–Weil rank of π is at least 15. To prove equality we do the following. It follows from the Shioda–Tate formula (Theorem 2.4) that it suffices to prove $\rho(X) \leq 17$. Since elliptic $K3$ surfaces over finite fields satisfy the Tate conjectures, one can determine $\rho(X \bmod q)$ from the characteristic polynomial $P_2(t)$ of Frobenius on $H_{\text{ét}}^2(X \bmod q, \mathbf{Q}_\ell)$. The polynomial $P_2(t)$ can be easily determined using the Lefschetz fixed point formula and counting the number of points on $X \bmod q$. To prove $\rho(X) \leq 17$ we find two prime numbers p_1, p_2 of good reduction such that the reduction $X \bmod p_1$ and $X \bmod p_2$ have Picard number 18. This is the best possible bound one can hope for by only considering $\rho(X \bmod q)$, since $\rho(X \bmod q)$ is even. From the fact $\rho(X) \leq \rho(X \bmod q)$ for any prime q of good reduction it follows $\rho(X) \leq 18$.

We now use a refinement of the Tate conjectures to prove that $\rho(X) \leq 17$. Let G_q be the Gram matrix of the intersection pairing on $NS(X \bmod q)$. If $\rho(X)$ would be 18, then $\det(G_{p_1})$ and $\det(G_{p_2})$ would differ by a square. We use the Artin–Tate conjecture (which is both a refinement of and equivalent to the Tate conjecture, hence it holds for our $K3$ surfaces $X \bmod p_i$) to determine $\det(G_{p_1})$ and $\det(G_{p_2})$ up to squares. Since they differ, we deduce that $\rho(X) \leq 17$.

2 Definitions and Notation

Definition 2.1 *An elliptic surface is a triple (π, X, C) with X a smooth projective surface, C a smooth projective curve and π a morphism $X \rightarrow C$, such that almost all fibers are irreducible genus 1 curves and X is relatively minimal, i.e., no fiber of π contains an irreducible rational curve D with $D^2 = -1$.*

We denote by $j(\pi): C \rightarrow \mathbf{P}^1$ the rational function such that $j(\pi)(P)$ equals the j -invariant of $\pi^{-1}(P)$, whenever $\pi^{-1}(P)$ is non-singular.

A *Jacobian elliptic surface* is an elliptic surface together with a section $\sigma_0: C \rightarrow X$ of π . The set of sections of π is an abelian group, with σ_0 as the identity element. Denote this group by $MW(\pi)$.

Let $NS(X)$ be the group of divisors on X modulo algebraic equivalence, called the *Néron–Severi group* of X . The *Picard number* $\rho(X)$ is by definition the rank of the Néron–Severi group of X .

Remark 2.2 Suppose we are working over a field not of characteristic 2 or 3. If P is a point on C , such that $\pi^{-1}(P)$ is singular, then $j(\pi)(P)$ and the number of components of $\pi^{-1}(P)$ are as in Table 1. For proofs of these facts see [1, Section V.7] (or [2, Section V.7]), [22, Theorem IV.8.2], [23, p. 46] or [16, Lecture 1].

Kodaira type of fiber over P	$j(\pi)(P)$	number of geometric components
I_0^*	$\neq \infty$	5
I_ν ($\nu > 0$)	∞	$\nu + 1$
I_ν^* ($\nu > 0$)	∞	$\nu + 5$
II	0	1
IV	0	3
IV*	0	7
II*	0	9
III	1728	2
III*	1728	8

Table 1: Classification of singular fibers

Definition 2.3 Let X be a surface, let C and C_1 be curves. Let $\varphi: X \rightarrow C$ and $f: C_1 \rightarrow C$ be two morphisms. Then we denote by $\widetilde{X \times_C C_1}$ the smooth relatively minimal model of the fiber product of X and C_1 over C .

Recall the following theorem.

Theorem 2.4 (Shioda-Tate [20, Theorem 1.3; Corollary 5.3]) *Let $\pi: X \rightarrow C$ be a Jacobian elliptic surface, such that π has at least one singular fiber. Then the Néron–Severi group of X is generated by the classes of $\sigma_0(C)$, a non-singular fiber, the components of the singular fibers not intersecting $\sigma_0(C)$, and the generators of the Mordell–Weil*

group. Moreover, let S be the set of points P such that $\pi^{-1}(P)$ is singular. Let $m(P)$ be the number of irreducible components of $\pi^{-1}(P)$. Then

$$\rho(X) = 2 + \sum_{P \in S} (m(P) - 1) + \text{rank}(MW(\pi)).$$

The following result will be used several times. It is a direct consequence of the Shioda–Tate formula.

Theorem 2.5 ([20, Theorem 10.3]) *Let $\pi: X \rightarrow \mathbf{P}^1$ be a rational Jacobian elliptic surface. Then the rank of the Mordell–Weil group is 8 minus the number of irreducible components of singular fibers not intersecting the zero section.*

Given a Jacobian elliptic surface $\pi: X \rightarrow C$ over a field K , we can associate an elliptic curve in $\mathbf{P}_{K(C)}^2$ corresponding to the generic fiber of π . This induces a bijection between isomorphism classes of Jacobian elliptic surfaces and isomorphism classes of elliptic curves over $K(C)$.

Two elliptic curves E_1 and E_2 are isomorphic over $K(C)$ if and only if $j(E_1) = j(E_2)$ and the quotient of the discriminants of $E_1/K(C)$ and $E_2/K(C)$ is a 12-th power (in $K(C)^*$).

Assume that E_1, E_2 are elliptic curves over $K(C)$ with $j(E_1) = j(E_2) \neq 0, 1728$. Then one easily shows that the quotient of the discriminants of E_1 and E_2 equals u^6 for some $u \in K(C)^*$. Hence E_1 and E_2 are isomorphic over $K(C)(\sqrt{u})$. We call E_2 the twist of E_1 by u , denoted by $E_1^{(u)}$. Actually, we are not interested in the function u , but in the places at which the valuation of u is odd.

Definition 2.6 Let $\pi: X \rightarrow C$ be a Jacobian elliptic surface. Fix $2n$ points $P_i \in C(\overline{K})$. Let $E/K(C)$ be the Weierstrass model of the generic fiber of π .

A Jacobian elliptic surface $\pi': X' \rightarrow C$ is called a (quadratic) twist of π by (P_1, \dots, P_n) if the Weierstrass model of the generic fiber of π' is isomorphic to $E^{(f)}$, where $E^{(f)}$ denotes the quadratic twist of E by f in the above-mentioned sense and $f \in K(C)$ is a function such that $v_{P_i}(f) \equiv 1 \pmod{2}$ and $v_Q(f) \equiv 0 \pmod{2}$ for all $Q \notin \{P_i\}$.

If $K = \overline{K}$, then the existence of a twist of π by (P_1, \dots, P_{2n}) follows directly from the fact that $\text{Pic}^0(C)$ is 2-divisible. Moreover, if we fix $2n$ points P_1, \dots, P_{2n} , then there exist precisely $2^{2g(C)}$ twists by $(P_i)_{i=1}^{2n}$.

If P is one of the $2n$ distinguished points, then the fiber of P changes in the following way (see [16, V.4]).

$$I_\nu \leftrightarrow I_\nu^*(\nu \geq 0), \quad \text{II} \leftrightarrow \text{IV}^*, \quad \text{III} \leftrightarrow \text{III}^*, \quad \text{IV} \leftrightarrow \text{II}^* .$$

Let $\pi: X \rightarrow C$ be a Jacobian elliptic surface, $P_1, \dots, P_{2n} \in C$ points. Let $\tilde{\pi}: \tilde{X} \rightarrow C$ be a twist by the P_i . Then there exist a double cover $\varphi: C_1 \rightarrow C$ ramified at the P_i , such that the minimal non-singular models of $X \times_C C_1 \rightarrow C_1$ and $\tilde{X} \times_C C_1 \rightarrow C_1$ are isomorphic as fiber surfaces. Denote this model by $\pi_1: X_1 \rightarrow C_1$.

Recall from [21, Exercise 10.16] that

$$(1) \quad \text{rank}(MW(\pi_1)) = \text{rank}(MW(\pi)) + \text{rank}(MW(\tilde{\pi})).$$

Moreover, the singular fibers change as follows:

Fiber of π at P_i	I_ν or I_ν^*	II or IV^*	III or III^*	IV or II^*
Fiber of π_1 at $\varphi^{-1}(P_i)$	$I_{2\nu}$	IV	I_0^*	IV^*

3 Construction

Let K be an algebraically closed field of characteristic different from 2 and 3.

Consider the following construction.

Construction 3.1 Let $\pi: X \rightarrow \mathbf{P}^1$ be a Jacobian elliptic surface whose singular fibers are three fibers of type I_1 and one fiber of type III^* . Let $f \in K(t)$ be a function of degree two, such that the fibers of π over the critical values of f are non-singular.

Let $\alpha, \beta \in \mathbf{P}^1$ be the two distinct points such that $f(\alpha) = f(\beta)$ is the point whose fiber is of type III^* . Let g be a degree four cyclic covering, with only ramification over α, β . Let $\varphi: Y \rightarrow \mathbf{P}^1$ be the non-singular relatively minimal model of the fiber product $X \times_{\mathbf{P}^1} \mathbf{P}^1$ with respect to π and $f \circ g: \mathbf{P}^1 \rightarrow \mathbf{P}^1$.

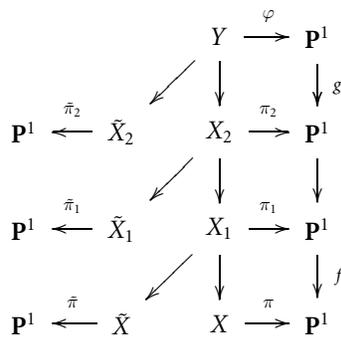


Table 2: Overview of all maps used in this section ($g = g_2 \circ g_2'$).

Proposition 3.2 The Mordell–Weil rank of φ (of Construction 3.1) is at least 15, and is precisely 15 if and only if the rank of the twist of π by the two critical values of f is 0.

Proof The assumptions imply that X is a rational surface and hence using Theorem 2.5 we have that $\text{rank} MW(\pi) = 1$. Let $\pi_1: X_1 \rightarrow \mathbf{P}^1$ be the fiber product $X \times_{\mathbf{P}^1} \mathbf{P}^1$ with respect to $f: \mathbf{P}^1 \rightarrow \mathbf{P}^1$ and π . Let $\tilde{\pi}: \tilde{X} \rightarrow \mathbf{P}^1$ be the twist of π by the two critical values of f . Then by (1) and Theorem 2.5

$$\text{rank}(MW(\pi_1)) = \text{rank}(MW(\pi)) + \text{rank}(MW(\tilde{\pi})) = 1 + \text{rank}(MW(\tilde{\pi})).$$

Note that π_1 has two fibers of type III* and six fibers of type I₁. Let P_1 and P_2 be the points with a fiber of type III*.

Let $g_2: \mathbf{P}^1 \rightarrow \mathbf{P}^1$ be the degree two function with critical values P_1 and P_2 . Define $\pi_2: X_2 \rightarrow \mathbf{P}^1$ to be the non-singular relatively minimal model of the fiber product $X \times_{\mathbf{P}^1} \mathbf{P}^1$ with respect to π_1 and g_2 .

Let $\tilde{\pi}_1$ be the twist of π_1 by P_1 and P_2 . Then $\tilde{\pi}_1$ has two fibers of type III and six fibers of type I₁, hence the corresponding surface is rational and by Theorem 2.5 $\tilde{\pi}_1$ has Mordell–Weil rank 6. It follows that $\text{rank}(MW(\pi_2)) = 7 + \text{rank}(MW(\tilde{\pi}_1))$. Furthermore, π_2 has two fibers of type I₀*, and 12 fibers of type I₁.

Let $\tilde{\pi}_2$ be the twist of π_2 by the two points with fiber of type I₀*. Then $\tilde{\pi}_2$ has 12 fibers of type I₁ and the corresponding surface is rational with Mordell–Weil rank eight. So

$$\text{rank}(MW(\varphi)) = \text{rank}(MW(\pi_2)) + \text{rank}(MW(\tilde{\pi}_2)) = 15 + \text{rank}(MW(\tilde{\pi})). \quad \blacksquare$$

Remark 3.3 If we suppose that $\text{rank}(MW(\tilde{\pi})) = 0$, then it is relatively easy to find explicit generators for $MW(\varphi)$. In that case the pull-backs of the generators of $MW(\pi)$, $MW(\tilde{\pi}_1)$, $MW(\tilde{\pi}_2)$ generate a subgroup of $MW(\varphi)$ of index 2^m , for some $m \geq 0$. Since all these three surfaces are rational, we can take a specific Weierstrass model for these surfaces such that all Mordell–Weil groups are generated by polynomials of degree at most 2. (See [18].)

Remark 3.4 In the case $K = \mathbf{C}$, there exists another proof. Since Y and \tilde{X} are both K3 surfaces and there exists a finite map between them, the Picard numbers of both surfaces coincide (see [7, Corollary 1.2]). From an easy exercise using Kodaira’s classification of singular fibers it follows that the configuration of singular fibers of φ is the one mentioned in the theorem. By Kodaira’s classification of singular fibers and the Shioda–Tate formula 2.4, we conclude

$$2 + 15 + \text{rank}(MW(\tilde{\pi})) = \rho(X) = \rho(Y) = 2 + \text{rank}(MW(\varphi)).$$

Proposition 3.2 enables us to prove the first theorem.

Proof of Theorem 1.1 Let $c \in K^*$ be such that $c^2 \neq -1$. Then the rational elliptic surface E'_c associated to the Weierstrass equation $y^2 = x^3 + t^3(t - c)x + t^5$ has a fiber of type III* and three fibers of type I₁. One easily shows that if $E'_c \cong E'_{c'}$, then $c'^2 = c^2$. (If $E'_c \cong E'_{c'}$, then there exists an automorphism $h: \mathbf{P}^1 \rightarrow \mathbf{P}^1$ fixing 0 and ∞ , and a constant $\lambda \in K$, verifying $h(t)^3(h(t) - c) = \lambda^4 t^3(t - c')$ and $h(t)^5 = \lambda^6 t^5$. This implies that $\lambda^2 = 1$ and $c' = \lambda c$.)

Let $a \neq b$ and

$$f_{a,b}(s) = \frac{4abs}{(a - b)s^2 + 2(a + b)s + a - b}.$$

The critical values of $f_{a,b}$ are a and b , and $f^{-1}(0) = \{0, \infty\}$. Hence by Proposition 3.2 the elliptic surface associated to the Weierstrass equation

$$y^2 = x^3 + f_{a,b}(s^4)^3(f_{a,b}(s^4) - c)x + f_{a,b}(s^4)^5$$

satisfies the properties stated in the theorem. After a coordinate change which clears denominators, we obtain the equation of $E_{a,b,c}$.

This family contains a three-dimensional sub-family of non-isomorphic elliptic surfaces, because it is a finite base change of a three-dimensional family of non-isomorphic elliptic surfaces.

Assume now that $K = \mathbf{C}$. Let \mathcal{M} be the moduli space of Jacobian elliptic $K3$ surfaces (cf. [15]). Let $U \subset \mathcal{M}$ be the set of elliptic surfaces with non-constant j -invariant. Let NL_{18} be the locus in \mathcal{M} corresponding to elliptic $K3$ with Picard number at least 18.

Suppose that a generic twist of E'_c would have positive Mordell–Weil rank. Then the constructed family $E_{a,b,c}$ would map to a 3-dimensional component C of NL_{18} , moreover the general member of the family $E_{a,b,c}$ has non-constant j -invariant, hence $\dim C \cap U = 3$. From [9, Theorem 1.1] it follows that $\dim NL_{18} \cap U \leq 2$, a contradiction. From Proposition 3.2 it follows that the generic member of $E_{a,b,c}$ has Mordell–Weil rank precisely 15. ■

4 A Method for Bounding the Picard Number

In the previous sections we showed the existence of a family of $K3$ surfaces such that the general member has Mordell–Weil rank 15. In this section we give an explicit example. In general it is hard to determine the Mordell–Weil rank of a non-rational elliptic surface. In the case of elliptic $K3$ surfaces, one might be able to compute the Mordell–Weil rank using the Tate conjectures (which are proven for elliptic $K3$ surfaces over finite fields.)

In this section K is supposed to be a number field. Recall the following facts.

Suppose $\pi: Y \rightarrow \mathbf{P}^1$ is an elliptic surface defined over a number field K . Fix a model for π over \mathcal{O}_K , the ring of integers of K . Let \mathfrak{p} be a prime of \mathcal{O}_K . Assume that Y has good reduction at \mathfrak{p} . Let $q: = \#\mathcal{O}_K/\mathfrak{p}\mathcal{O}_K$. Let \bar{Y} be the reduction of Y modulo \mathfrak{p} . Then the reduction map $r_{\mathfrak{p}}: NS(Y_{\bar{\mathbf{Q}}}) \rightarrow NS(\bar{Y}_{\bar{\mathbf{F}}_q})$ is injective (see [11, Proposition 6.2]). It turns out that one can determine the rank of $NS(\bar{Y}_{\bar{\mathbf{F}}_q})$ for varieties \bar{Y} for which the Tate conjectures hold. In any case, the method mentioned below gives us an *upper* bound for $\text{rank } NS(\bar{Y}_{\bar{\mathbf{F}}_q})$.

Definition 4.1 Suppose $(\Lambda, \langle \cdot, \cdot \rangle)$ is a lattice. Let G be a Gram matrix of Λ with respect to $\langle \cdot, \cdot \rangle$. By definition, the *discriminant* of $(\Lambda, \langle \cdot, \cdot \rangle)$ is the determinant of G , which we denote by $\Delta(\Lambda, \langle \cdot, \cdot \rangle)$ or $\Delta(\Lambda)$, if no confusion arises.

It is well known that the Néron–Severi group (modulo torsion) together with the intersection pairing forms a lattice. This implies the following proposition.

Proposition 4.2 *Let Y/K be a smooth projective surface. Suppose \mathfrak{p} is a prime of good reduction. Let $q = \#\mathcal{O}_K/\mathfrak{p}\mathcal{O}_K$. Suppose that the reduction map*

$$r_{\mathfrak{p}}: NS(Y_{\overline{\mathbf{Q}}}) \otimes \mathbf{Q} \rightarrow NS(\overline{Y}_{\overline{\mathbf{F}}_q}) \otimes \mathbf{Q}$$

is an isomorphism. Then the determinant of the Gram matrices of the intersection pairings on $NS(Y_{\overline{\mathbf{Q}}})$ and $NS(\overline{Y}_{\overline{\mathbf{F}}_q})$ differ by a square.

Proof Since \mathfrak{p} is a prime of good reduction, we have that $NS(Y_{\overline{\mathbf{Q}}})$ is a sublattice of $NS(\overline{Y}_{\overline{\mathbf{F}}_q})$ (see [11, Proposition 6.2]). Our assumptions imply that both lattices have the same rank. A standard result in lattice theory gives

$$\Delta(NS(Y_{\overline{\mathbf{Q}}})) = [NS(\overline{Y}_{\overline{\mathbf{F}}_q}) : NS(Y_{\overline{\mathbf{Q}}})]^2 \Delta(NS(\overline{Y}_{\overline{\mathbf{F}}_q}))$$

where $\Delta(\Lambda)$ denotes the determinant of the Gram matrix of the lattice Λ . This yields the proposition. \blacksquare

One can construct examples of surfaces such that for every prime \mathfrak{p} of good reduction the image of $r_{\mathfrak{p}}$ is *not* of finite index in $NS(\overline{Y}_{\overline{\mathbf{F}}_q})$. (See [6, 19, 25] for examples such that $\rho(Y) = 1$. An easy example is the Kummer surface Y of $E \times E$, for an elliptic curve E/K without potential complex multiplication, then $\rho(Y) = 19$. By Proposition 4.4 all good reductions have an even Picard number).

Proposition 4.2 turns out to be useful in showing that $r_{\mathfrak{p}}$ is not surjective. In Section 5 we give such an example.

Conjecture 4.3 (Tate Conjecture) *Let Y/\mathbf{F}_q be a smooth surface. Let F_q^* be the automorphism of $H_{\text{ét}}^2(Y, \mathbf{Q}_\ell)$ induced by the Frobenius automorphism of \mathbf{F}_q on Y . Let $Q(t)$ be $\det(I - tF_q^* | H_{\text{ét}}^2(Y, \mathbf{Q}_\ell))$. Then $\rho(Y)$ equals the number of reciprocal zeroes of Q of the form $q\zeta$, with ζ a root of unity.*

This conjecture is known to be true for several classes of varieties, see for example [24].

The following proposition shows that considering only one prime might not be sufficient to obtain a sharp bound on the Picard number, namely that over finite fields the number of possible Picard numbers seems smaller than over fields of characteristic zero.

Proposition 4.4 *Let Y/\mathbf{F}_q be a smooth projective surface for which Conjecture 4.3 holds, e.g., K3 surfaces. Then $\rho(Y) - \dim_{\mathbf{Q}_\ell} H_{\text{ét}}^2(Y, \mathbf{Q}_\ell)$ is even.*

Proof After replacing \mathbf{F}_q by a finite extension, if necessary, we may assume that we have a set of \mathbf{F}_q -rational divisors generating $NS(\overline{Y}_{\overline{\mathbf{F}}_q})$. This implies that the characteristic polynomial of Frobenius on $H_{\text{ét}}^2(Y, \mathbf{Q}_\ell)$ is of the form $g(t)(t - 1/q)^{\rho(Y)}$, where

$g \in \mathbf{Q}[t]$ is a polynomial such that all its reciprocal zeroes have absolute value q [5, Theorem 1.6]. Since the Tate Conjecture 4.3 holds for Y , it follows that $g(\pm q) \neq 0$, hence g has no zeroes on the real line. In particular, g has even degree.

From Conjecture 4.3 it follows that

$$\dim_{\mathbf{Q}_\ell} H_{\text{ét}}^2(Y, \mathbf{Q}_\ell) - \rho(Y) = \deg g(t),$$

which yields the proposition. ■

Remark 4.5 In this remark we try to indicate why there might exist many surfaces such that r_p is not surjective for any prime of good reduction. Using the period map one can show that every integer r , such that $1 \leq r \leq 20$ occurs as the Picard number of an algebraic $K3$ surface Y over \mathbf{C} . Since $\dim H_{\text{ét}}^2(Y, \mathbf{Q}_\ell) = 22$, it follows from Proposition 4.4 that $\rho(\overline{Y}_{\mathbf{F}_q})$ is even, hence one might expect many examples of $K3$ surfaces defined over number fields such that for every prime of good reduction the image of the reduction map r_p is not of finite index in $NS(\overline{Y}_{\mathbf{F}_q})$.

Suppose we can show that for two different primes p_1, p_2 of good reduction the rank of the Néron–Severi lattices is the same, but the discriminants of the Néron–Severi lattices differ by a non-square. Then we conclude by Proposition 4.2 that the rank of $NS(Y_{\overline{\mathbf{Q}}})$ is at least one lower than the rank of $NS(\overline{Y}_{\mathbf{F}_{q_1}})$. This method was suggested to the author by Ronald van Luijk, see also [12].

The above remarks are only useful, if for a given surface Y/\mathbf{F}_q one can efficiently compute $\rho(Y)$ and the determinant of the Néron–Severi lattice of Y . In general this is not the case, but for surfaces for which Conjecture 4.3 holds, this can be done. Milne [13] proved that if Y is a surface for which Conjecture 4.3 holds, then the following conjecture also holds.

Conjecture 4.6 (Artin–Tate Conjecture) Let Y/\mathbf{F}_q be a smooth surface. Let F_q be the Frobenius automorphism of \mathbf{F}_q acting on Y . Let $Q(t) := \det(1 - tF_q^* | H_{\text{ét}}^2(Y, \mathbf{Q}_\ell))$. Then

$$(*) \quad \lim_{s \rightarrow 1} \frac{Q(q^{-s})}{(1 - q^{1-s})^{\rho'(Y)}} = \frac{(-1)^{\rho'(Y)-1} \# \text{Br}(Y) \Delta(NS(Y_{\mathbf{F}_q}))}{q^{\alpha(Y)} (\#NS(Y_{\mathbf{F}_q})_{\text{tor}})^2},$$

with $\alpha(Y) = \chi(Y, \mathcal{O}_Y) - 1 + \dim \text{Pic}^0(Y)$ and $\text{Br}(Y)$ is the Brauer group of Y . With $NS(Y_{\mathbf{F}_q})$ we indicate the subgroup of $NS(Y_{\overline{\mathbf{F}_q}})$ generated by \mathbf{F}_q -rational divisors and with $\rho'(Y) = \text{rank } NS(Y_{\mathbf{F}_q})$.

If Y/\mathbf{F}_q is an elliptic $K3$ surface, such that $\gcd(q, 6) = 1$, then both the Tate and the Artin–Tate conjecture are known to be true (see [17]). We are actually interested in the discriminant of $NS(Y_{\overline{\mathbf{F}_q}})$. This forces us to apply Conjecture 4.6 over a field extension such that $\rho'(Y) = \rho(Y)$. (For an overview of cases for which the Tate and the Artin–Tate conjecture holds, see [24].)

The Tate conjecture reduces our problem to finding the characteristic polynomial of Frobenius on $H_{\text{ét}}^2(Y, \mathbf{Q}_\ell)$. This can be done by using the Lefschetz fixed point

formula, *i.e.*, one calculates the trace of the Frobenius automorphism (and several of its powers) on the cohomology by counting points on the surface. From knowing these traces, one deduces the characteristic polynomial. (This is explained in detail in [11, Section 7]).

We discuss some of the other quantities that have to be computed. Since we are only interested in knowing $\Delta(NS(Y_{\overline{\mathbf{F}}_q}))$ up to squares, we might just disregard all quantities in (*) that are a square.

Proposition 4.7 *Suppose q is a prime power, with $\gcd(q, 6) = 1$. Let $\pi: Y \rightarrow \mathbf{P}^1$ be an elliptic K3 surface, defined over \mathbf{F}_q . Assume that q is a square and $\rho(Y) = \rho'(Y)$. Then*

$$\Delta(NS(Y_{\overline{\mathbf{F}}_q})) \equiv -\lim_{s \rightarrow 1} \frac{Q(q^{-s})}{(1 - q^{1-s})^{\rho(Y)}} \pmod{\mathbf{Q}^{*2}}.$$

Proof It is known that for an elliptic surface $\pi: Y \rightarrow \mathbf{P}^1$, the Brauer group $\text{Br}(Y)$ is isomorphic to $\text{III}(Y/\mathbf{P}^1)$ [3, Chapter 5], where $\text{III}(Y/\mathbf{P}^1)$ is the Tate–Shafarevich group of Y/\mathbf{P}^1 . It is classically known that the number of elements of $\text{III}(Y/\mathbf{P}^1)$ is a square [14, Remark 6.11], although this remark is not completely correct, it is correct in the case of an elliptic curve over a function field), so we may disregard $\#\text{Br}(Y)$. The parity of $\rho'(Y)$ follows from

$$\rho(Y) = \rho'(Y) \equiv \dim H_{\text{ét}}^2(Y, \mathbf{Q}_\ell) \equiv 0 \pmod{2},$$

using Proposition 4.4. Since $\gcd(q, 6) = 1$, we know that Conjecture 4.6 holds. Combining these facts yields the proposition. ■

5 Proof of Theorem 1.2

We apply the above mentioned strategy in the following example.

Consider the following elliptic K3 surface X associated with

$$y^2 = x^3 - (2t - 1)^3(4t - 1)^2x + t(2t - 1)^3(4t - 1)^3.$$

This surface has two fibers of type I_0^* (at $t = 1/4$ and $t = 1/2$), a fiber of type III^* at $t = \infty$ and three fibers of type I_1 . One can easily show that $p = 17$ and $p = 19$ are primes of good reduction, and all singular fibers are over points in $\mathbf{P}^1(\mathbf{F}_p)$. The components of the singular fibers, the zero-section and the class of a smooth fiber generate a rank 17 sublattice of $NS(X)$. One can easily show that these generators considered over \mathbf{F}_{17} (resp., \mathbf{F}_{19}) are rational over \mathbf{F}_{17^6} (resp., \mathbf{F}_{19^6}); this is a straightforward application of Tate’s algorithm [23]. A more precise application of Tate’s algorithm yields an explicit degree 17 factor T_p of the characteristic polynomial Q_p of the Frobenius of \mathbf{F}_p acting on $H_{\text{ét}}^2(\overline{X}, \mathbf{Q}_\ell)$. In particular, one obtains that all roots α_p of T_p satisfy $\alpha_p^6 = p^{-6}$, if $p \in \{17, 19\}$. Since $\dim H_{\text{ét}}^2(\overline{X}, \mathbf{Q}_\ell)$ equals 22, we have to find a complimentary degree 5 factor \tilde{G}_p of Q_p . From the fact that all reciprocal roots of Q_p have absolute value p , at least one of the roots of \tilde{G}_p is $\pm 1/p$. By counting

points over \mathbf{F}_{17} and \mathbf{F}_{17^2} and using Poincaré duality (cf. [11, §7]), we obtain that $G_{17} = (17x - 1)\tilde{G}_{17}$, with \tilde{G}_{17} given by

$$1 + 17x + 136x^2 + 4913x^3 + 83521x^4$$

and similarly for $p = 19$, we obtain that $G_{19} = (19x + 1)\tilde{G}_{19}$, with \tilde{G}_{19} given by

$$1 - 9x - 228x^2 - 3249x^3 + 130321x^4.$$

One easily shows that both polynomials have no reciprocal root of the form $p\zeta$, with ζ a root of unity. This implies that $\rho(\overline{X}_{\mathbf{F}_p}) = 18$, for $p = 17, 19$.

Let H_p the polynomial obtained by taking all roots of G_p to the power six. Then $(p^6X - 1)^{18}H_p$ is the characteristic polynomial of the Frobenius of \mathbf{F}_{p^6} acting on $H_{\text{ét}}^2(\overline{X}_{\mathbf{F}_{p^6}}, \mathbf{Q}_\ell)$ for $p = 17, 19$.

Proposition 4.7 implies that

$$\Delta(NS(X_{\mathbf{F}_{17^6}})) \equiv 5 \cdot 19 \cdot 101516605992547 \cdot 11 \cdot 875005421 \pmod{\mathbf{Q}^{*2}}$$

and

$$\begin{aligned} \Delta(NS(X_{\mathbf{F}_{19^6}})) &\equiv 809308043 \cdot 95814202607062823339 \cdot \\ &\quad \cdot 2297 \cdot 774901 \cdot 7 \cdot 13 \cdot 419 \cdot 16620229 \pmod{\mathbf{Q}^{*2}}. \end{aligned}$$

The strategy explained in the previous section now implies that $\rho(X) \leq 17$. From Proposition 3.2 we know that $\rho(X) \geq 17$, proving that $\text{rank } NS(X) = 17$, and $\text{rank } MW(\pi) = 0$. Applying Proposition 3.2 again gives that

$$y^2 = x^3 + 2(t^8 + 2t^4 + 1)x - 4t^2(t^8 - 6t^4 + 1)$$

has Mordell–Weil rank 15.

Acknowledgements The author would like to thank Ronald van Luijk, Jasper Scholten and Jaap Top for many valuable discussions on this topic. The author would like to thank the referee for suggesting many improvements. The result of Section 3 is taken from the author’s Ph.D. thesis [8, Ch. 4].

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