THE PERFECT AND EXTREME SENARY FORMS

E. S. BARNES

This paper is devoted partly to a summary of some results which will be published in detail elsewhere,[1] partly to a description of a new extreme senary form, ϕ_6, and an n-dimensional generalization of it.

The main result is the determination of all classes of extreme senary forms. There are just six classes, represented by:

$$\begin{aligned}
\phi_0 &= \sum_1^6 x_i^2 + \sum_{i<j} x_i x_j,\\
\phi_1 &= \phi_0 - x_1x_2,\\
\phi_2 &= \phi_0 - x_1x_2 - x_1x_3,\\
\phi_3 &= \phi_0 - \tfrac{1}{2}(x_1x_2 + x_3x_4 + x_5x_6),\\
\phi_4 &= \phi_0 - \tfrac{1}{2}(x_1x_2 + x_3x_4 + x_3x_5 + x_3x_6 + x_4x_5 + x_4x_6 + x_5x_6),\\
\phi_6 &= \phi_0 - \tfrac{1}{2}(2x_1x_2 + x_1x_3 + x_1x_6 + x_2x_5 + x_4x_6 + 2x_5x_6).
\end{aligned}$$

Each of these forms has minimum $M = 1$ for integral $\mathbf{x} \neq \mathbf{0}$. They are listed in the following table, together with the symbol used by Coxeter **(3)**, the order of the group $\mathfrak{g}$ of automorphs, the number s of pairs of minimal vectors, and the value of $2^6D/M^6$ in decreasing order of this quantity.

Form	Coxeter's symbol	Order of $\mathfrak{g}$	s	$2^6D/M^6$
ϕ_0	A_6	2.7!	21	7
ϕ_3		96	21	$13.3^3/2^6$
ϕ_6		672	21	$7^3/2^6$
ϕ_1	$B_6 \sim D_6$	2^6.6!	30	4
ϕ_4	E_6^3	144.6!	27	$3^5/2^6$
ϕ_2	E_6	144.6!	36	3

Thus there is just one class of absolutely extreme forms, represented by ϕ_2; and, in agreement with Blichfeldt **(2)**,

$$\gamma_6^{\,6} = \max_f \frac{M^6}{D} = \frac{1}{D(\phi_2)} = \frac{64}{3}.$$

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[1] *The complete enumeration of extreme senary forms*, Phil. Trans. Roy. Soc. London, *A 249* (1957), 461-506.

The method used to establish these forms is the algorithm for perfect forms given by Voronoi **(6)**. Hence all classes of perfect senary forms are now determined; these are represented by the extreme forms $\phi_0, \ldots, \phi_4, \phi_6$ and

$$\phi_5 = \phi_0 - \tfrac{1}{2}(x_1x_2 + x_3x_4 + x_3x_5 + x_4x_5).$$

The forms ϕ_0, ϕ_1, ϕ_2 and ϕ_4 were found by Coxeter **(3)**, who gives a full description of their properties. Previously, both Voronoi **(6)** and Hofreiter **(4)** had established ϕ_0, ϕ_1 and ϕ_2 (this being Voronoi's notation for them). As is pointed out by Coxeter **(3)**, Hofreiter's list contains a fourth form, F_4, which is not extreme, but which is presumably an arithmetical slip for[2] ϕ_3.

The form ϕ_3 was recently discovered independently by Kneser **(5)** and the author **(1)**, and the reader is referred to these articles for a full description of its properties.

The remaining extreme form ϕ_6 and the perfect non-extreme form ϕ_5 are new. Apart from its existence as the simplest known form of its type, ϕ_5 has little structural interest; we merely note that it has

$$M = 1, \quad D = 3^4/2^{10}, \quad s = 22,$$

and that its group $\mathfrak{g}$ is of order 288.

On applying to $\phi_6(\mathbf{x})$ the transformation

$$\mathbf{x} = \frac{1}{7}\begin{pmatrix} 3 & 6 & 2 & 5 & 1 & 4 \\ 5 & 3 & 1 & 6 & 4 & 2 \\ 4 & 1 & 5 & 2 & -1 & 3 \\ -4 & -1 & 2 & -2 & 1 & -3 \\ -1 & -2 & -3 & -4 & 2 & 1 \\ -6 & -5 & -4 & -3 & -2 & -1 \end{pmatrix}\mathbf{y},$$

we obtain

$$(2) \qquad 4\phi_6(\mathbf{x}) = 2\sum_1^6 y_i^2 + 2\sum_{i<j} y_iy_j = 2\phi_0(\mathbf{y});$$

or, defining y_7 by the relation

$$(3) \qquad \sum_1^7 y_i = 0,$$

$$(4) \qquad 4\phi_6(\mathbf{x}) = \sum_1^7 y_i^2.$$

Since $2\phi_0$ has determinant 7, and the transformation (1) has determinant 1/7, it follows that $D(\phi_6) = 7^3/2^{12}$. It is also easily verified from (1) that $\mathbf{x}$ is integral if and only if $\mathbf{y}$ is integral and satisfies

$$(5) \qquad \sum_1^7 iy_i \equiv 0 \pmod 7.$$

[2]As Professor Coxeter has remarked to me, the correct value of $2^6D/M^6$ for ϕ_3 is obtained by replacing 53 by 52 in F_4.

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Thus ϕ_6 now appears as the case $n = 6$ of the form f_n defined, in terms of its lattice, by

(6) $$f_n = \sum_1^m y_i^2,$$

where the y_i take integral values subject to

(7) $$\sum_1^m y_i = 0,$$

(8) $$\sum_1^m iy_i \equiv 0 \pmod m$$

(where we have set $m = n + 1$ for convenience). This form is extreme for all $n \geqslant 6$, but the proof of this for odd n is rather involved. In what follows we shall restrict ourselves to even n (odd m).

Let $\mathbf{e}_1, \ldots, \mathbf{e}_m$ denote the unit vectors in m-space, with coordinates $y_1, \ldots, y_m$. Now f_n takes only even values subject to (7). It takes the value 2 only when $\mathbf{y} = \mathbf{e}_i - \mathbf{e}_j$, and clearly none of these sets satisfies (8). It takes the value 4 when

(9) $$\mathbf{y} = \mathbf{e}_a + \mathbf{e}_b - \mathbf{e}_c - \mathbf{e}_d$$

with

(10) $$a + b \equiv c + d \pmod m.$$

Thus $M(f_n) = 4$. Since the lattice defined by (7), (8) has determinant $m^{\frac{1}{2}}.\, m$, we have $D(f_n) = m^3$; so

$$(2/M)^n D = (n+1)^3/2^n.$$

The number of pairs of minimal vectors is[3]

$$s = \tfrac{1}{8}\, n(n+1)(n-2) \qquad (n \text{ even}).$$

For, for any a, b, the number of distinct unordered pairs c, d satisfying (10) is $\frac{1}{2}(m-1)$, of which one pair is a, b. Since a, b may be chosen in $\frac{1}{2}m(m-1)$ ways, this gives $2s = \frac{1}{2}m(m-1)\frac{1}{2}(m-3)$, as asserted.

We now show that f_n is eutactic. Regarding y_m as the redundant variable in (6), we have

$$f_n = \left(\sum_1^n y_i\right)^2 + \sum_1^n y_i^2,$$

with adjoint

$$F_n = n\sum_1^n y_i^2 - 2\sum_{i<j} y_i y_j.$$

The linear forms associated with the minimal vectors (9) are then

(11) $$\lambda_k = y_a + y_b - y_c - y_d \qquad (k = 1, \ldots, s),$$

[3]For odd n, $s = \frac{1}{8}(n-1)^2(n+1)$.

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with $y_m = 0$. We now see that

$$\tfrac{1}{2}(m-3)F_n = \sum_1^s \lambda_k^2. \tag{12}$$

For, by symmetry, each y_i occurs in $4s/m = \frac{1}{2}(m-1)(m-3)$ forms λ_k, giving a coefficient $\frac{1}{2}(m-1)(m-3)$ of y_i^2 in $\sum\lambda_k^2$. A pair y_i, y_j occurs with like signs in $\frac{1}{2}(m-3)$ forms (as above); and with opposite signs in $m-3$ forms, this being the number of solutions of $d - b \equiv i - j$ with d, b distinct from i and j. This gives a coefficient $m - 3 - 2(m-3) = -(m-3)$ of $y_i y_j$ in $\sum\lambda_k^2$. Collecting these results and setting $y_m = 0$, we obtain (12).

To prove that f_n is perfect, we begin by defining the vectors

$$\mathbf{f}_i = \mathbf{e}_i + \mathbf{e}_{i+1} - \mathbf{e}_{i+2} - \mathbf{e}_{i-1} \qquad (i = 1, \ldots, m),$$

where all suffixes and congruences are to be interpreted modulo m. Then

$$\sum_1^m \mathbf{f}_i = 0, \tag{13}$$

but any $m - 1$ of the $\mathbf{f}_i$ are independent. Hence we may take a new coordinate system $z_1, \ldots, z_n$ defined by

$$\mathbf{z} = \sum_1^n z_i \mathbf{f}_i .$$

We have now to show that if all minimal vectors $\mathbf{z}$ of f_n satisfy a quadratic relation

$$p(\mathbf{z}) = \sum_1^n p_{ij} z_i z_j = 0, \tag{14}$$

then all $p_{ij} = 0$. Using (12), we shall express some of the minimal vectors in terms of $\mathbf{f}_1, \ldots, \mathbf{f}_m$; if $\mathbf{f}_m$ occurs in the expression, we must of course replace it by

$$-\sum_1^n \mathbf{f}_i$$

before using (14).

(i) Suppose first that $m \geqslant 9$ (and odd). We use in (14) the minimal vectors

$$\mathbf{f}_i + \mathbf{f}_{i+1} + \ldots + \mathbf{f}_{i+r} = \mathbf{e}_{i+r} + \mathbf{e}_{i+1} - \mathbf{e}_{i+r+2} - \mathbf{e}_{i-1} \qquad (r \not\equiv \pm 1, -3) \tag{15}$$

$$\mathbf{f}_i + \mathbf{f}_{i+2} + \ldots + \mathbf{f}_{i+2r} = \mathbf{e}_i + \mathbf{e}_{i+2r+1} - \mathbf{e}_{i+2r+2} - \mathbf{e}_{i-1} \qquad (2r \not\equiv -1, -2, -3), \tag{16}$$

$$-\mathbf{f}_{i-3} + \mathbf{f}_i + \mathbf{f}_{i+1} + \ldots + \mathbf{f}_{i+m-6} = \mathbf{e}_{i-6} + \mathbf{e}_{i+1} - \mathbf{e}_{i-3} - \mathbf{e}_{i-2}, \tag{17}$$

the given conditions ensuring that the suffixes on the right are distinct. Values of i giving a suffix $0 (\mathrm{mod}\ m)$ on the left will not be considered.

We obtain at once from (15), with $r = 0$,

$$p(\mathbf{f}_i) = p_{ii} = 0. \tag{18}$$

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Using (15) with indices $r, r-1$, and subtracting, we obtain

$$2p_{i,i+r} + 2p_{i+1,\,i+r} + \ldots + 2p_{i+r-1,\,i+r} + p_{i+r,\,i+r} = 0;$$

hence, from (18),

(19) $$p_{i,i+r} + p_{i+1,\,i+r} + \ldots + p_{i+r-1,\,i+r} = 0 \qquad (r = 3, \ldots, m-4).$$

Replacing i, r by $i+1, r-1$ and subtracting from (19) gives

(20) $$p_{i,i+r} = 0 \qquad (r = 4, \ldots, m-4).$$

From (16) with $r = 1$, and $p_{ii} = 0$,

(21) $$p_{i,i+2} = 0, \quad p_{1,m-1} = 0;$$

from (15) with $r = 2$ and (21),

(22) $$p_{i,i+1} + p_{i+1,\,i+2} = 0.$$

From (19) with $r = 3$ and (21)

(23) $$p_{i,i+3} + p_{i+2,\,i+3} = 0.$$

We now use (17), and (15) with $r = m - 6$; subtracting the results gives

$$p_{i-3,i} + p_{i-3,\,i+1} + \ldots + p_{i-3,\,i+m-6} = 0.$$

This holds certainly for $i = 4$ and $i = 1$ (no suffix then being zero), whence

$$p_{14} + p_{15} + \ldots + p_{1,m-2} = 0,$$
$$p_{1,m-2} + p_{2,m-2} + \ldots + p_{m-5,\,m-2} = 0.$$

By (20), these reduce to

$$p_{14} + p_{1,m-2} = p_{1,m-2} + p_{m-5,\,m-2} = 0,$$

whence

$$p_{14} = p_{m-5,\,m-2}.$$

By (23) with $i = 1$ and $i = m - 5$, this gives

(24) $$p_{34} = p_{m-3,\,m-2}.$$

But (22) gives

$$p_{34} = -p_{45} = p_{56} = \ldots = -p_{m-3,\,m-2},$$

and so, with (24),

$$p_{34} = p_{m-3,\,m-2} = 0.$$

Now (22) and (23) give at once

$$p_{i,i+1} = 0, \quad p_{i,i+3} = 0.$$

This, with (18), (20) and (21), establishes that all $p_{ij} = 0$ except possibly $p_{1,m-2}$ and $p_{2,m-1}$. That these are also zero now follows immediately by using the minimal vectors (from (15) with $r = 1$)

$$-(\mathbf{f}_m + \mathbf{f}_2) = \mathbf{f}_1 + \mathbf{f}_3 + \mathbf{f}_4 + \ldots + \mathbf{f}_{m-1},$$
$$-(\mathbf{f}_{m-2} + \mathbf{f}_m) = \mathbf{f}_1 + \mathbf{f}_2 + \ldots + \mathbf{f}_{m-3} + \mathbf{f}_{m-1}.$$

Thus all $p_{ij} = 0$, and so f_n is perfect.

(ii) When $m = 7$, (17) is no longer a minimal vector and the above analysis fails. All 21 vectors are now

$$\mathbf{f}_i, \mathbf{f}_i + \mathbf{f}_{i+2}, \mathbf{f}_i + \mathbf{f}_{i+1} + \mathbf{f}_{i+2} \qquad (i = 1, \ldots, 7), \tag{25}$$

and a simple calculation establishes the perfection of $f_n = \phi_6$.

It is easy to verify that $\mathbf{f}_1, \ldots, \mathbf{f}_{m-1}$ give not only a basis of the space, but also a basis for the lattice of f_n. Hence we find the cyclically symmetrical representation

$$f_n(\mathbf{x}) = \sum_1^m (x_i + x_{i-1} - x_{i-2} - x_{i+1})^2 \qquad (n \text{ even}),$$

with $x_m = 0$, the suffixes being taken modulo m. A similar representation exists also for n odd.

We now consider the group $\mathfrak{g}$ of automorphs of ϕ_6; we denote by $\mathfrak{G}$ the contragadient group of automorphs of its adjoint F_6. Since $\mathfrak{g}$ must permute the minimal vectors (more precisely, the pairs $\pm\mathbf{m}$ of minimal vectors), $\mathfrak{G}$ permutes the associated linear forms. From (11), any permutation of the λ_k leaves F_6 invariant. It follows that $\mathfrak{G}$ is precisely the group of linear transformations permuting the associated linear forms.

From this result, we shall show that $\mathfrak{G}$ (and so $\mathfrak{g}$) has order 672, is transitive on the linear forms,[4] and is isomorphic to $\mathfrak{C}_2 \times PGL(2, 7)$. Corresponding to the representation (25), we may write the associated linear forms (11) as

$$z_i, z_i + z_{i+2}, z_i + z_{i+1} + z_{i+2} \qquad (i = 1, \ldots, 7),$$

where

$$z_i = y_i + y_{i+1} - y_{i+2} - y_{i-1},$$
$$\sum_1^m z_i = 0.$$

Obvious elements of $\mathfrak{G}$ are S: $y_i \to y_{i+1}$ and U: $y_i \to y_{3i}$ (where the suffixes are taken modulo 7). These show that $\mathfrak{G}$ is transitive on the linear forms. For $\mathrm{S}z_i = z_{i+1}$, so that the forms of each set (z_i), $(z_i + z_{i+2})$, $(z_i + z_{i+1} + z_{i+2})$ are permuted cyclically by S; while

$$\mathrm{U}z_7 = y_7 + y_3 - y_6 - y_4 = z_7 + z_2,$$
$$\mathrm{U}^2 z_7 = y_7 + y_2 - y_4 - y_5 = -(z_3 + z_4 + z_5).$$

Let now $\mathfrak{G}_7$ be the subgroup of $\mathfrak{G}$ which leaves z_7 invariant (or changes its sign). We shall say that two of the linear forms *combine* if their sum or difference is an associated linear form. This property being invariant under $\mathfrak{G}$, $\mathfrak{G}_7$ must permute the 8 forms combining with z_7, namely

$$z_1 + z_2 + z_3, \quad -z_5 - z_7, \quad z_6 + z_1, \quad -z_7 - z_2,$$
$$z_4 + z_5 + z_6, \quad z_5, \quad -z_6 - z_7 - z_1, \quad z_2. \tag{26}$$

[4]Contrary to what one might expect from (12), $\mathfrak{G}$ is no longer transitive for $n > 6$.

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Using the element T of $\mathfrak{G}_7$, of order 8, defined by

$$\mathrm{T}(z_1, \ldots, z_7) = (-z_7 - z_1 - z_2,$$
$$z_1 + z_2 + z_3,\ -z_3 - z_5,\ z_5 + z_6 + z_7,\ -z_6 - z_7 - z_1,\ z_1,\ z_7)$$

we see that $\mathfrak{G}_7$, considered as a permutation group on the set (26), is transitive on this set. The successive elements $\mathrm{T}^n z_2 (n = 1, \ldots, 8)$ are in fact the 8 elements (26), in the order written.

Let then $\mathfrak{G}_{7,2}$ be the subgroup of $\mathfrak{G}_7$ which leaves z_2 invariant. (We note that $\mathfrak{G}_{7,2}$ must leave the relative signs of z_7, z_2 unaltered, since $z_7 + z_2$ is a linear form, while $z_7 - z_2$ is not.) $\mathfrak{G}_{7,2}$ must permute the 2 forms $z_6 + z_7 + z_1$, $z_1 + z_2 + z_3$ which combine with both z_7 and z_2. A direct argument now shows that $\mathfrak{G}_{7,2}$ is of order 4 and consists of the elements $\pm \mathrm{I}$, $\pm \mathrm{V}$, where

$$\mathrm{V}(z_1, \ldots, z_7) = (-z_7 - z_1 - z_2,\ z_2,\ -z_6,\ -z_2 - z_4,\ -z_5 - z_7,\ -z_3,\ z_7).$$

It follows that $\mathfrak{G}$ has order 21.8.4 = 672, as asserted, and is generated by S, T, U, V and $-\mathrm{I}$. Since

$$\mathrm{V} = -\mathrm{S}^3\mathrm{T}^5\mathrm{S}, \quad \mathrm{U} = -\mathrm{T}^6\mathrm{S}^5\mathrm{T}^7,$$

we see that $\mathfrak{G} = \{\mathrm{S}, \mathrm{T}, -\mathrm{I}\} = \{-\mathrm{I}\} \times \{\mathrm{S}, \mathrm{T}\}$.

We may identify $\{\mathrm{S}, \mathrm{T}\}$ with $PGL(2, 7)$, consisting of the regular 2×2 matrices over $GF(7)$ (where multiples of the same matrix are identified), by means of the mapping

$$\mathrm{S} \leftrightarrow \sigma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$
$$\mathrm{T} \leftrightarrow \tau = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}.$$

For, as is well known, $PSL(2, 7)$ (consisting of those elements of $PGL(2, 7)$ whose determinant is a square) is a simple group of order 168 generated by σ and

$$\theta = \tau^4 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

and may be defined by the relations

$$(27) \qquad \sigma^7 = \theta^2 = (\sigma\theta)^3 = (\sigma^4\theta)^4 = 1.$$

Now τ has determinant 5, a non-square, so that σ, τ generate $PGL(2, 7)$; we may take the further defining relation as

$$(28) \qquad \tau^{-1}\sigma\tau = \sigma^4\theta\sigma^2\theta\sigma^5.$$

A simple calculation shows that the relations (27), (28) are satisfied if σ, τ, θ are replaced by S, T, T^4 respectively. We note that, under the given mapping, the elements of $PSL(2, 7)$ correspond to the elements of $\mathfrak{G}$ of determinant $+1$ (expressed as integral unimodular transformations on, say, $z_1, \ldots, z_6$).

We consider finally a conjecture made by Coxeter (**3**, p. 392), that every eutactic form with $s \geqslant \frac{1}{2}n(n+1)$ is perfect, and so extreme (the condition $s \geqslant \frac{1}{2}n(n+1)$ being of course *necessary* for perfection). We can use the form f_n considered above to show that the conjecture is unfortunately false.

Let $f(x_1, \ldots, x_n)$ be any eutactic form with $s \geqslant \frac{1}{2}n(2n+1)$; f_n is such a form for even $n \geqslant 10$ (in fact for all $n \geqslant 10$). Define

$$g(x_1, \ldots, x_{2n}) = f(x_1, \ldots, x_n) + f(x_{n+1}, \ldots, x_{2n}).$$

Then $M(g) = M(f)$, and is attained when and only when one f is zero and the other assumes its minimum. Hence the number of pairs of minimal vectors of g is $2s \geqslant \frac{1}{2}. 2n(2n+1)$. Since f is eutactic and the adjoint of g is a multiple of $F(x_1, \ldots, x_n) + F(x_{n+1}, \ldots, x_{2n})$, g is clearly eutactic. But g is not perfect, since it is disconnected: all minimal vectors satisfy, for example, the relation $x_1 x_{n+1} = 0$.

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University of Sydney