CAPTURE/ESCAPE BOUNDARY IN THE COLLINEAR RESTRICTED THREE-BODY PROBLEM

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ABSTRACT. We study the cantorian structure of the successive intersections of the invariant manifolds of infinity (parabolic orbits) with a certain surface of section. The first of these intersections is computed numerically. The structure of the set of orbits of capture or escape after n binary collisions is given.

1. FORMULATION OF THE PROBLEM. EQUATIONS OF MOTION.

Two bodies (called primaries) of point masses m_1 and m_2 are moving in an elliptic collision orbit under the influence of their mutual gravitational attraction and a third body of mass $m_3 \approx 0$ (attracted by the previous two but not influencing their motion) moves in the line defined by the two primaries. The collinear restricted three body problem is to describe the motion of this third body.

We select units of length, time and mass such that the length of the major axis of the collision elliptic orbit equals 2, the period is 2π and $m_1 = m$, $m_2 = 1 - m$ with $m \in (0,1)$. Units are taken in such a way that the gravitation constant equals 1. We take the center of masses at the origin.

Let $-x_1, x_2$ ($x_i \geqslant 0$) be the coordinates of m_1, m_2, respectively. Then the motion of the two primaries is given by

$$x_1 = (1-m)(1-\cos E) \quad , \qquad x_2 = m(1-\cos E) \quad , \tag{1}$$

with $t = E - \sin E$. The parameter E is the so called eccentric anomaly and the origin of time is taken at a collision between m_1 and m_2. We remark that t, E are defined modulus 2π and from now on this will be understood without explicit mention each time that t or E appears.

Let $x \geqslant 0$ be the coordinate of the third body. We assume $x \geqslant x_2$ (see Figure 1). The equation of motion of the third body is:

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$$d^2x/dt^2 = (1-m)/(x-x_2)^2 - m/(x+x_1)^2 \ , \tag{2}$$

with x_1, x_2 given by (1).

We have a singularity in (2) when $x=x_2$. Furthermore, if $x_2=0$ we obtain a triple collision because $x_1=0$, too. If $x_2>0$ we encounter a binary collision between m_2 and m_3. A change of variables will regularize such last collision as usual. In what follows we will not consider triple collision orbits.

Let $h_{123}=\dot{x}^2/2 - (1-m)/(x-x_2) - m/(x+x_1)$ be the energy per unit mass of the third body, and $h_{23}=(\dot{x}-\dot{x}_2)^2/2 - (1-m)/(x-x_2)$ the energy associated with the binary m_2,m_3. We relate the motion close to collision m_2,m_3 with the motion of the third body near infinity. As h_{123} does not have a finite limit when $x-x_2 \to 0$ and h_{23} is not suitable when $x\to\infty$, we define $h=(x^r-x_2^r)h_{123}/x^r + x_2^r h_{23}/x^r$ with $r \geqslant 1$. Using the behaviour of bodies near collision (Siegel-Moser, pp30), we see that h is well defined along solutions of (2) (if and only if $r\geqslant 1$) and that $h\to h_{23}$ if $x-x_2\to 0$, and $h\to h_{123}$ if $x\to\infty$.

We scale the time in order to regularize binary collisions m_1,m_2 and m_2,m_3 introducing the s variable (Stiefel-Scheifele, pp20)

$$dt = (x-x_2)(1-\cos E)ds \ .$$

Then (2) becomes

$$\frac{dx}{ds} = \left\{\frac{x_2^r\dot{x}_2}{x^r} + \left[\left(\frac{x_2^r\dot{x}_2}{x^r}\right)^2 + 2\left(\frac{x^r-x_2^r}{x^r}\ \frac{m}{x+x_1} + \frac{1-m}{x-x_2} + h - \frac{x_2^r\dot{x}_2^2}{2x^r}\right)\right]^{1/2}\right\}\frac{dt}{ds} \ ,$$

$$\frac{dh}{ds} = \frac{x^r-x_2^r}{x^r}\,\frac{dh_{123}}{ds} + \frac{x_2^r}{x^r}\,\frac{dh_{23}}{ds} + (h_{23}-h_{123})\,\frac{d}{ds}\left(\frac{x_2^r}{x^r}\right) , \tag{3}$$

$$\frac{dt}{ds} = (x-x_2)(1-\cos E) \ ,$$

where

$$dh_{123}/ds = (m\dot{x}_1/(x+x_1)^2 - (1-m)\dot{x}_2/(x-x_2)^2)\cdot dt/ds \ ,$$

$$dh_{23}/ds = (\dot{x}-\dot{x}_2)(-m/(x+x_1)^2 - \ddot{x}_2)\cdot dt/ds \ ,$$

$$d/ds(x_2/x)^r = rx_2^{r-1}(\dot{x}_2x - x_2\dot{x})/x^{r+1}\cdot dt/ds \ ,$$

$$h_{23}-h_{123} = -\dot{x}\dot{x}_2 + \dot{x}_2^2/2 + m/(x+x_1) \ ,$$

$$\dot{x}_1 = (1-m)\sin E/(1-\cos E) \ ,$$

$$\dot{x}_2 = m\sin E/(1-\cos E) \ ,$$

$$\ddot{x}_1 = -(1-m)/(1-\cos E)^2 ,$$

$$\ddot{x}_2 = -m/(1-\cos E)^2 .$$

We remark that $r \geqslant 1$ is enough to make dx/ds and $(x^r - x_2^r)x^{-r}dh_{123}/ds$ regular, but values $r \geqslant 2$ and $r \geqslant 3$ are necessary for $x_2^r x^{-r} dh_{23}/ds$ and $(h_{23} - h_{123})\, d/ds(x_2/x)^r$, respectively. From now on we take $r = 3$.

From (3) it follows that the motion of m_3 is determined by four initial conditions

$$s = s_0 , \quad (x - x_2)(s_0) = 0 , \quad h(s_0) = h_0 , \quad t(s_0) = t_0 .$$

Note that every orbit of m_3 has at least one collision m_2, m_3.

Due to the autonomous character of the equations the value of s_0 is irrelevant. Let us take $s_0 = 0$. Then $h(0) = h_0$, $t(0) = t_0$, or equivalently $E(0) = E_0$, are enough to determine the motion if we assume that bodies m_2, m_3 are at collision. Then we have:

LEMMA 1 (see Proposition 2.1 of Llibre-Simó,1980a). Orbits of the third body excluding triple collision are determined by points of a cylinder K excluding one generatrix (see Figure 2).

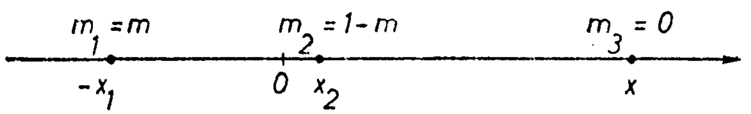

Figure 1.

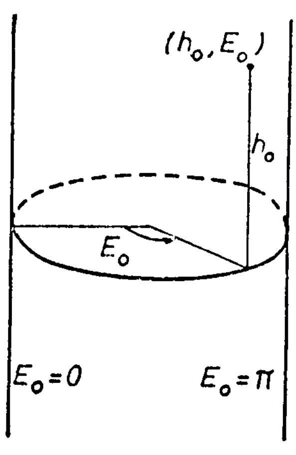

Figure 2.

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We remark that more than one point can give the same orbit. The generatrix $E_0 = 0$ which is related to triple collision is excluded.

We define a mapping in a subset D_1 of K, $f: D_1 \longrightarrow K$ in the following way: Given a point $(h_0, E_0) \in K$ and $s_0 = 0$ let us consider the orbit defined by (h_0, E_0). We have that $(x - x_2)(s_0) = 0$. Let s_1 be the next zero of $x - x_2$ for increasing values of s, if it exists. Then, if $h(s_1) = h_1$ and $E(s_1) = E_1$, we define $f(h_0, E_0) = (h_1, E_1)$. We call f the *Poincaré map* of the collinear restricted three body problem associated with K.

The set D_1 is associated with the orbits of m_3 which collide again with m_2. We call D_1 the set of *elliptic orbits* for increasing time. It is clear that D_1 is open. Now we study the complementary of D_1. If f is not defined on the point (h_0, E_0) as $\ddot{x}(t) < 0$ for all $t > t_0$, the function $\dot{x}(t)$ is monotonically decreasing. There exist $\lim_{t \to \infty} \dot{x}(t) = \dot{x}(+\infty) \geq 0$ and $\lim_{t \to \infty} x(t) = +\infty$. We say (h_0, E_0) is *hyperbolic (parabolic)* for $t \to +\infty$ if $\dot{x}(+\infty) > 0$ $(\dot{x}(+\infty) = 0)$. The set of hyperbolic (parabolic) initial conditions for $t \to +\infty$ in K will be called H_1 (P_1). The set H_1 is open. We call D_{-1} the subset of K where f^{-1} is defined, we have $D_{-1} = f(D_1)$. D_{-1} is called the set of elliptic orbits for decreasing time. We define H_{-1} (P_{-1}) as hyperbolic (parabolic) orbits for $t \to -\infty$ in a natural way.

We remark that equation (2) is invariant with respect to the symmetry $(x, \dot{x}, t) \to (x, -\dot{x}, -t)$. This symmetry on K is given by $S(h_0, E_0) = (h_0, 2\pi - E_0)$. Then we have $f^{-1} = S^{-1} \circ f \circ S$ and hence $D_{-1} = S(D_1)$, $P_{-1} = S(P_1)$, $H_{-1} = S(H_1)$.

2. THE INTEGRABLE CASE $m = 0$.

The system (2) for $m = 0$ reduces to a rectilinear two body problem. The energy $h = h_{123} = h_{23} = \dot{x}^2/2 - 1/x$ is constant for each orbit and the solutions such that for a given E_0 verify $(x - x_2)(E_0) = 0$, are associated with elliptic, parabolic, hyperbolic collision orbits according as the energy h is negative, zero or positive.

Furthermore the cylinder K is divided by the circle $h_0 = 0$ (parabolic orbits) in two regions: elliptic orbits $(h_0 < 0)$ and hyperbolic ones $(h_0 > 0)$.

Now the Poincaré map f is given by:

$$f(h_0, E_0) = (h_1, E_1) \quad , \tag{4}$$

with

$$h_1 = h_0 ,$$
$$E_1 = E_0 + E(2\pi(-2h_0)^{-3/2}) ,$$

where $E(t)$ is the solution of the equation $t = E - \sin E$. It is clear that f is defined only for values of $h_0 < 0$. Therefore, the domain of definition D_1 is a semi-cylinder and $D_1 = D_{-1} = f(D_1)$. Here D_1 is the circle $h_0 = 0$ of K and $P_1 = P_{-1}$, of course the semi-cylinder $h_0 > 0$ is $H_1 = H_{-1}$.

We look at D_1 as an annulus of radius $h_0 \in (-\infty, 0)$. Then the Poincaré map on D_1 is a twist map, which means that the concentric circles h_0 = constant < 0 are rotated by an angle $E(2\pi(-2h_0)^{-3/2})$ which tends increasing to ∞ as h_0 approaches 0. Thus the image of any radius is a curve spiralling infinitely about the origin as it approaches $h_0 = 0$.

3. THE NONINTEGRABLE CASE $m > 0$

In order to study P_1 and P_{-1} when $m > 0$ we need to consider the flow in the neighborhood of ∞. Following McGehee we introduce the variables

$$x = 2/q^2 , \quad \dot{x} = -p , \quad dt = (4/q^3)dw ,$$

with $0 < q < \infty$. Then (2) becomes

$$\begin{aligned} dq/dw &= p , \\ dp/dw &= [\,(1-m)/(1-q^2x_2/2)^2 + m/(1+q^2x_1/2)^2\,]\,q , \\ dt/dw &= 4/q^3 . \end{aligned} \tag{5}$$

In (McGehee) it was proved that in the infinity there is an hyperbolic periodic orbit whose invariant manifolds are P_1 (stable) and P_{-1} (unstable).

The symmetry of the problem in these new coordinates is given by $(q,p,t,w) \longrightarrow (q,-p,-t,-w)$. Then if the manifold P_{-1} has the expression $q = F(p,t)$, the P_1 is $q = F(-p,-t)$. In (Llibre-Simó, 1980a) it has been shown that $F(p,t)$ is 2π-periodic in t and is not analytic in t but in E. Moreover

$$F(p,t) = \sum_{n \geqslant 0} a_n(t)p^n , \tag{6}$$

with

$$\begin{aligned} a_0 &= a_2 = a_3 = a_4 = a_6 = 0 , \\ a_1 &= 1 , \\ a_5 &= -5m(1-m)/16 , \\ a_7 &= 35m(1-m)(1-2m)/128 , \\ a_8 &= -3m(1-m)(-5t/2 + \textstyle\int_0^t (1-\cos E)^3 dE) , \\ &\cdots\cdots \end{aligned} \tag{7}$$

Using the above results we have:

LEMMA 2 (see Proposition 3.2 of Llibre-Simó, 1980a). If $m > 0$ is sufficiently small the set P_1 of parabolic initial conditions for $t \to +\infty$ in K is a simple closed curve (if we add the generatrix $E_0 = 0$ to K) which divides K in two components D_1 and H_1.

The geometry of the curves P_1 and P_{-1} will be studied numerically in the next section.

LEMMA 3. If $m > 0$ is sufficiently small, let $\gamma = \{(h_0, E_0) : h_0 = h_0(s), E_0 = E_0(s), s \in [0,1]\}$ be an arc having in P_1 the point belonging to $s = 0$. We suppose that at this point γ and P_1 are not tangent. Then $f(\gamma) = \{(h_1, E_1) : h_1 = h_1(s), E_1 = E_1(s), s \in [0,1]\}$ approaches P_{-1} spiraling, i.e. $E_1(s) \to \infty$ when $s \to 0$.

The proof of this lemma is obtained taking into account that near the periodic orbit at infinity the equations of motion are approximately the same as in the Sitnikov problem (see Moser).

LEMMA 4 (see Theorem 4.1 of Llibre-Simó, 1980a). If $m > 0$ is small enough then $P_1 \neq P_{-1}$ and they intersect at a non-tangential homoclinic point p on $E_0 = \pi$.

Using this fact the Bernoulli shift can be embedded as a subsystem of the Poincaré map f in a neighborhood of the homoclinic point p (for more details see Llibre-Simó, 1980a).

4. NUMERICAL STUDY OF P_1 and P_{-1}

In this section we shall describe the geometry of the curve P_{-1} for values of $m \in (0,1)$, see Figures 3, 4 and 5.

Since $P_1 = S(P_{-1})$, $S(h_0, E_0) = (h_0, 2\pi - E_0)$ and the curve P_{-1} is increasing in E_1, we have that $P_1 \cap P_{-1}$ reduces to an unique point, the homoclinic point p of Lemma 4. Moreover, we compute the angle β of the intersection of P_1 and P_{-1} at p. These values together with the value of the energy at the intersection point are given in Table I.

In Figures 6 and 7 we have plotted the angle β in front of the mass ratio m. As it is clear from Figure 7 this angle varies linearly with m, if m is sufficiently small. This is due to the analytical dependence of the angle β with respect m. The coefficient of m in the development of β in power series of m was computed analytically in Llibre-Simó, 1980a. Its numerical value is $\tan\beta = 1.19\ldots m + O(m^2)$.

In short we have numerical evidence of the following:

CONJECTURE. For all value of $m \in (0,1)$ the intersection of P_1 with P_{-1} reduces to an unique transversal homoclinic point.

In order to compute P_{-1}, for the different values of m, we start

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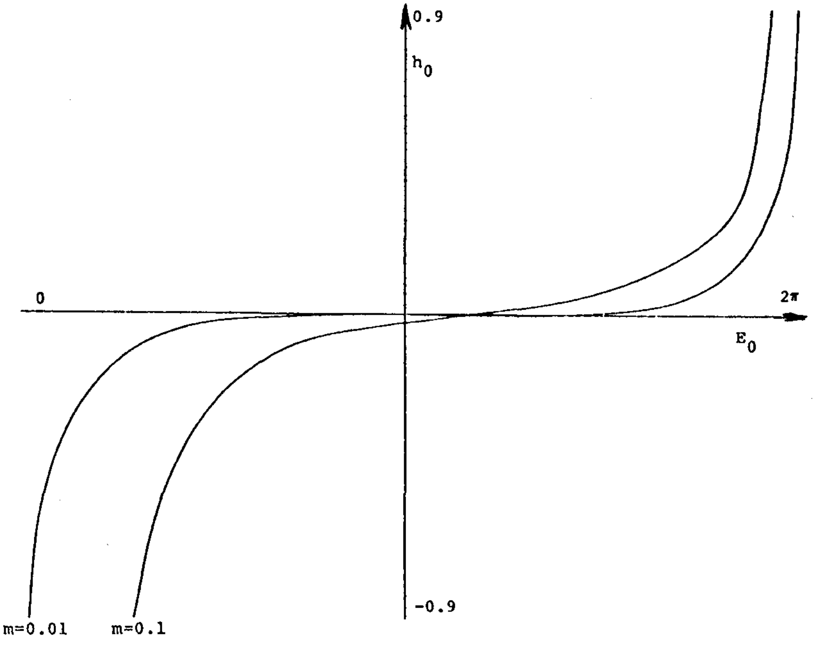

Figure 3.

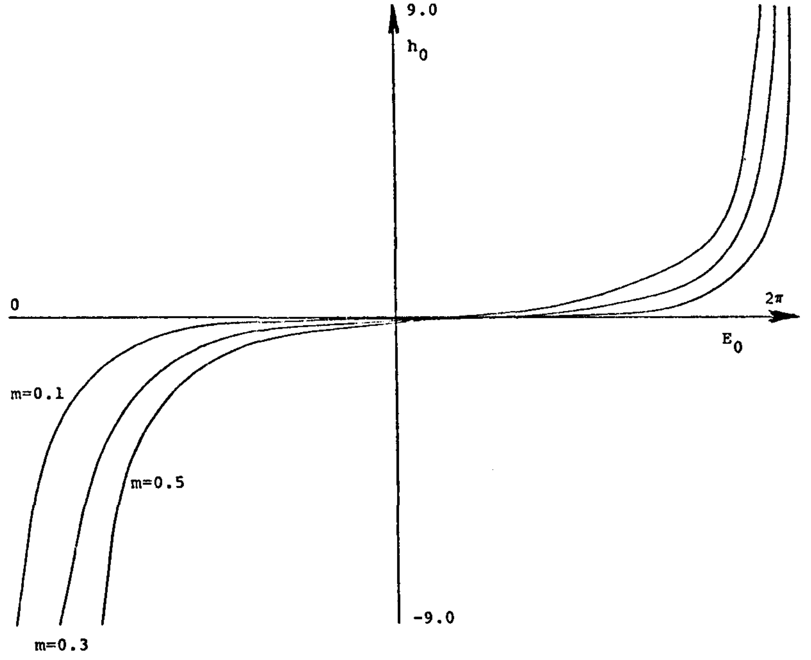

Figure 4.

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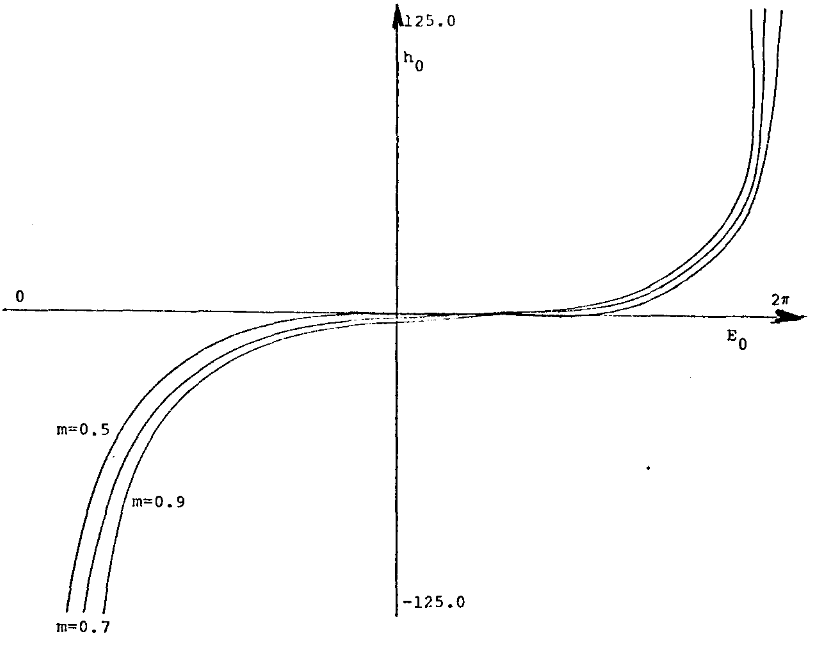

Figure 5.

m	E_0	$\beta/2$
.0001	$-2.9450 \cdot 10^{-5}$	$5.97 \cdot 10^{-5}$
.0002	$-5.8903 \cdot 10^{-5}$	$1.19 \cdot 10^{-4}$
.0003	$-8.8331 \cdot 10^{-5}$	$1.77 \cdot 10^{-4}$
.0004	$-1.1771 \cdot 10^{-4}$	$2.39 \cdot 10^{-4}$
.0005	$-1.4717 \cdot 10^{-4}$	$2.98 \cdot 10^{-4}$
.001	$-2.9500 \cdot 10^{-4}$	$5.67 \cdot 10^{-4}$
.005	$-1.4767 \cdot 10^{-3}$	$2.87 \cdot 10^{-3}$
.01	$-2.9570 \cdot 10^{-3}$	$5.67 \cdot 10^{-3}$
.05	$-1.5021 \cdot 10^{-2}$	$2.89 \cdot 10^{-2}$
.1	$-3.0661 \cdot 10^{-2}$	$5.64 \cdot 10^{-2}$
.2	$-6.3891 \cdot 10^{-2}$	0.1106
.3	$-9.9873 \cdot 10^{-2}$	0.1673
.4	-0.1388	0.2202
.5	-0.1812	0.2700
.6	-0.2276	0.3176
.7	-0.2787	0.3616
.8	-0.3364	0.4022
.9	-0.4040	0.4380

Table I.

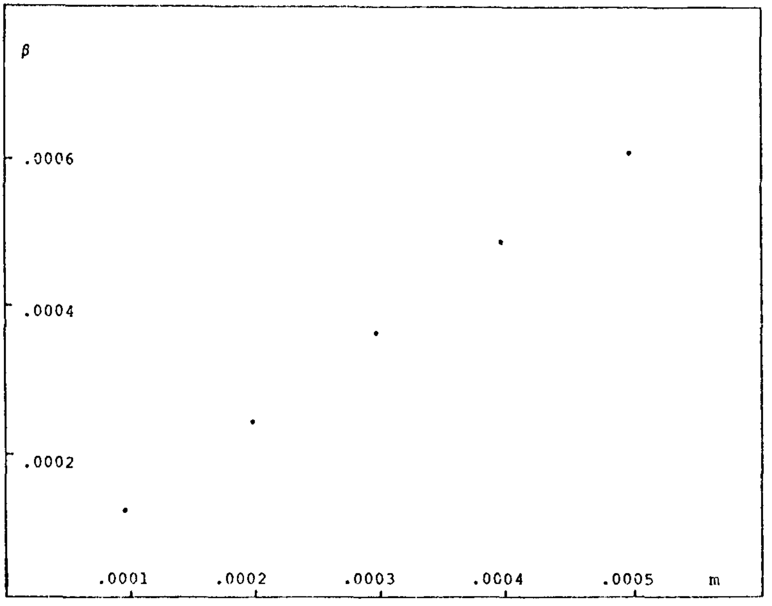

Figure 6.

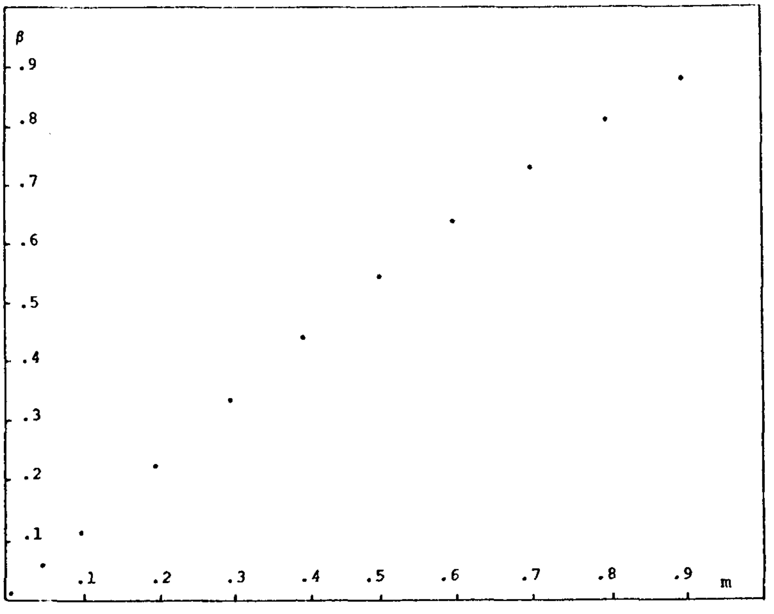

Figure 7.

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with initial conditions on the manifold of infinity given by (6) and (7) retaining terms up to the seventh order in p and taking a starting value of $p = 0.01$.

At the beginning, we integrate the differential system (5) until the singularity due to the binary collision m_2, m_3 makes regularization necessary. At that point we change the set of variables (q,p,t,w) to another one (u,h_{23},t,s) which is obtained by applying a Levi-Civita transformation to the system m_2, m_3. The differential equations in this new set of variables are

$$\frac{d^2u}{ds^2} = \frac{h_{23}}{2}u - \frac{mu^3}{2(u^2+1-\cos E)^2} + \frac{mu^3}{(1-\cos E)^2} ,$$

$$\frac{dh_{23}}{ds} = -2u\frac{du}{ds}\left[\frac{m}{(u^2+1-\cos E)^2} - \frac{m}{(1-\cos E)^2}\right] ,$$

$$\frac{dt}{ds} = u^2 .$$

The integration is done with a Runge-Kutta-Fehlberg, Rk78, routine with local error of 10^{-13}, 10^{-15}.

5. FINAL EVOLUTIONS

We call final evolutions of the collinear restricted three body problem the behavior of the infinitesimal body when time tends to $\pm\infty$. There are eight types of final evolutions. Parabolic (hyperbolic) evolutions, P or P_- (H or H_-) in which the infinitesimal body reaches infinity with zero (positive) velocity as time tends to $+\infty$ or $-\infty$, respectively. Lagrange evolutions L or L_- in which the orbit of the infinitesimal body remains bounded as time tends to $+\infty$ or $-\infty$, respectively. Finally, oscillatory evolutions OS are characterized by the fact $\limsup_{t\to+\infty} x(t) = +\infty$ and $\liminf_{t\to+\infty} x(t) < +\infty$. Similarly, we have oscillatory evolutions OS_-.

When we consider the two final evolutions associated to one orbit, there are sixteen possibilities. For example, $H_- \cap P$ means that as $t\to-\infty$ the evolution is of hyperbolic type and as $t\to+\infty$ it is parabolic.

In (Llibre-Simó, 1980a) it is shown that in a neighborhood of the homoclinic point p there are points with associated orbits of all the possible sixteen types of final evolutions.

We shall say that a point $q \in K$ gives rise to a parabolic orbit for f^n (f^{-n}), and we shall say that this orbit is of type P_n (P_{-n}) if it goes parabolically to infinity when $t\to+\infty$ $(-\infty)$ after $n-1$ binary collisions m_2, m_3. In a similar way we shall define hyperbolic orbits for f^n (f^{-n}) and we shall denote them by H_n (H_{-n}). We define the set

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C_n (C_{-n}) as those points $x \in K$ such that $f^n(x)$ $(f^{-n}(x))$ is a point of triple collision, i.e. a point of K on the generatrix E = 0.

As it follows from Section 2, all the final evolutions for $m = 0$ are given in Table II. In order to study all the kinds of final evolutions for $m > 0$ we consider the successive intersections of the invariant manifolds of the infinity, formed by the orbits of P_1 and P_{-1} with the surface of section $x - x_2 = 0$.

Let γ be an arc contained in D_1 and having an ending point on P_1, see Figure 8. By using Lemma 3, the domain of definition of f^2 restricted to γ is obtained taking out of γ infinite closed intervals I_i such that $f(I_i) \not\subset D_1$ (see Figures 8 and 9). It is clear that these intervals accumulate at P_1. According with our notation we shall call D_n (D_{-n}) the domain

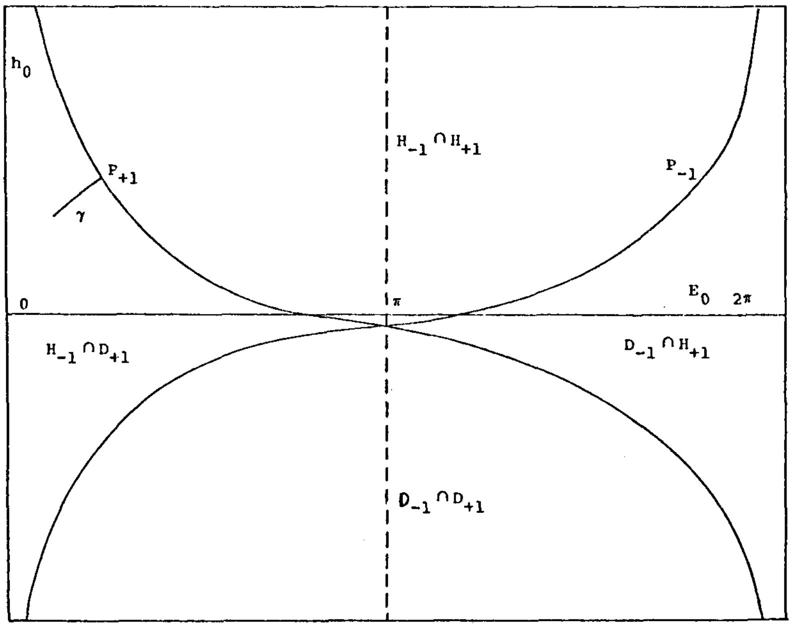

Figure 8.

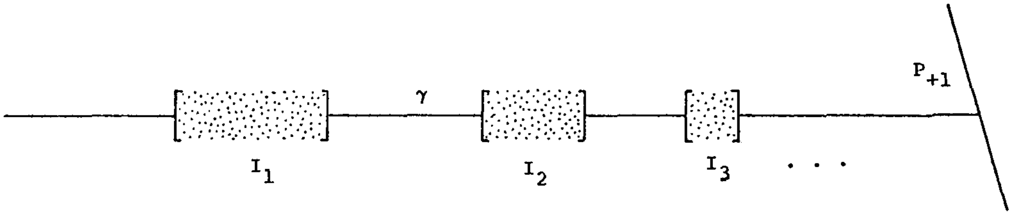

Figure 9.

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of definition of f^n (f^{-n}), recall that $D_{-n} = S(D_n)$. By continuity the domain D_2 is D_1 minus an infinite set of bands B_1^n, B_2,... which cut the arc γ at the closed intervals I_1, I_2, ..., respectively. These bands are the antiimage by f of $D_{-1} - D_1$.

Each one of these bands has two boundary curves. The points of the one that is closer to P_1 correspond to points of C_2 and the other one correspond to points of P_2. The interior of a band is formed by points of H_2. By using the symmetry S we obtain the sets D_{-2}, C_{-2}, P_{-2} and H_{-2}.

Again, to obtain the domain of definition D_3 of f^3 on γ, we must exclude not only

$$\bigcup_{i=1}^{\infty} I_i$$

but

$$\bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} I_{ij}$$

where each I_{ij} is a closed interval. The intervals I_{ij} as varying j accumulate to the interval I_i as it is shown in Figure 10. By continuity the domain D_3 is D_2 minus an infinite set of bands $B_{11}, B_{12}, \ldots, B_1, B_{21}, B_{22}, \ldots, B_2, B_{31}, B_{32}, \ldots$ which intersect the arc γ at the closed intervals $I_{11}, I_{12}, \ldots, I_1, I_{21}, I_{22}, \ldots, I_2, I_{31}, I_{32}, \ldots$ It is clear that $\bigcap_{n\geqslant 0} D_n$ is a cantorian set.

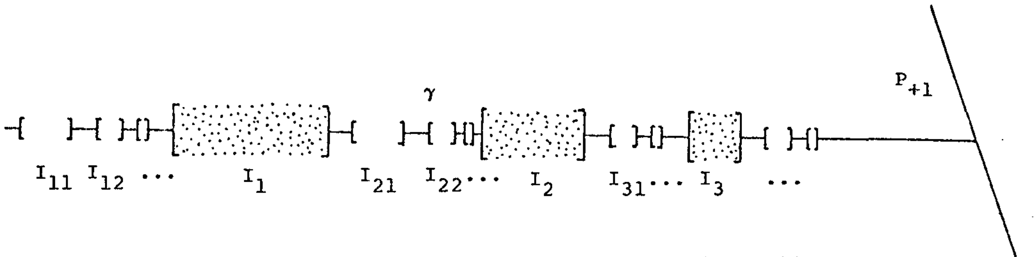

Figure 10.

In short, we have Table II for $m > 0$.

Remark. In (Serra) it has been proved: (1) that for $m = 0$ the set of initial conditions on K which give rise to triple collision orbits is dense in the set of bounded orbits, (2) for any $\delta > 0$ there exists a $m_0 > 0$ such that if the mass parameter $m \in [0, m_0]$, the set of bounded orbits which are not contained in the closure of the set of triple collision orbits has Lebesgue measure less than δ.

The set of orbits with final evolution of type	is homeomorphic to $m = 0$	is homeomorphic to $m \in (0,1)$
$P_{\pm 1}$	open interval	open interval
$P_{\pm n}$, $n \geq 2$	empty	countable union of disjoint open intervals
$H_{\pm 1}$	open 2-dimensional set	open 2-dimensional set
$H_{\pm n}$, $n \geq 2$	empty	countable union of disjoint open 2-dimensional sets
$P_{-1} \cap P_{+1}$	open interval	the point p
$P_{-n} \cap P_{+m}$, $n+m \geq 3$	empty	countable set of points
$P_{-n} \cap H_{+m}$ or $H_{-n} \cap P_{+m}$ with $n,m \geq 1$	empty	countable union of disjoint open intervals
$H_{-1} \cap H_{+1}$	open 2-dimensional set	open 2-dimensional set
$H_{-n} \cap H_{+m}$, $n+m \geq 3$	empty	countable union of disjoint open 2-dimensional sets
$C_{-1} \cap P_{+1}$ or $P_{-1} \cap C_{+1}$	one point	one point
$C_{-n} \cap P_{+m}$ or $P_{-n} \cap C_{+m}$ with $n+m \geq 3$	empty	countable set of points
$C_{-1} \cap H_{+1}$ or $H_{-1} \cap C_{+1}$	open interval	open interval
$C_{-n} \cap H_{+m}$ or $H_{-n} \cap C_{+m}$ with $n+m \geq 3$	empty	countable union of disjoint open intervals

Table II.

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