Multiplication of Polynomials on Hermitian Symmetric spaces and Littlewood–Richardson Coefficients

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Abstract. Let K be a complex reductive algebraic group and V a representation of K. Let S denote the ring of polynomials on V. Assume that the action of K on S is multiplicity-free. If λ denotes the isomorphism class of an irreducible representation of K, let $\rho_\lambda \colon K \to GL(V_\lambda)$ denote the corresponding irreducible representation and S_λ the λ-isotypic component of S. Write $S_\lambda \cdot S_\mu$ for the subspace of S spanned by products of S_λ and S_μ. If V_ν occurs as an irreducible constituent of $V_\lambda \otimes V_\mu$, is it true that $S_\nu \subseteq S_\lambda \cdot S_\mu$? In this paper, the authors investigate this question for representations arising in the context of Hermitian symmetric pairs. It is shown that the answer is yes in some cases and, using an earlier result of Ruitenburg, that in the remaining classical cases, the answer is yes provided that a conjecture of Stanley on the multiplication of Jack polynomials is true. It is also shown how the conjecture connects multiplication in the ring S to the usual Littlewood–Richardson rule.

1 Introduction

1.1 A Question for Multiplicity Free Actions

Let K be a connected complex reductive group and let X be a complex affine algebraic variety with a K-action. We will assume that X is a multiplicity-free space, i.e., every finite dimensional irreducible representation of K appears with multiplicity at most one in the algebra $S = \mathbb{C}[X]$ of regular functions on X. Thus $S \simeq \bigoplus_{\lambda \in \Lambda} V_\lambda$, where Λ is a subset of $\widehat{K}$, the set of isomorphism classes of finite dimensional irreducible K-representations, and V_λ is a representative of the class $\lambda \in \widehat{K}$. If $\lambda \in \Lambda$, let $S_\lambda \subseteq S$ denote the λ-isotypic component of S, *i.e.*, S_λ is a K-stable subspace of S such that $S_\lambda \simeq V_\lambda$. If $\lambda, \mu \in \Lambda$, let $S_\lambda \cdot S_\mu$ be the subspace of S spanned by the products of elements in S_λ and S_μ. The subspace $S_\lambda \cdot S_\mu$ is K-stable and hence is a sum of certain S_ν's. We are interested in describing the set of all $\nu \in \Lambda$ such that $S_\nu \subseteq S_\lambda \cdot S_\mu$. Clearly, a necessary condition for $S_\nu \subseteq S_\lambda \cdot S_\mu$ is that $V_\nu \hookrightarrow V_\lambda \otimes V_\mu$.

Question Is it true that $S_\nu \subseteq S_\lambda \cdot S_\mu$ if and only if $V_\nu \hookrightarrow V_\lambda \otimes V_\mu$ for $\nu, \lambda, \mu \in \Lambda$?

It is easy to find examples for which the answer is negative. Consider the natural action of $K = SL_2(\mathbb{C})$ on $X = \mathbb{C}^2$, which is a multiplicity-free action. More precisely,

if d is a non-negative integer, then the space S_d of homogenous polynomials on X of degree d is an irreducible representation of K. In particular, Λ can be identified with the set of non-negative integers. Clearly, $S_1 \cdot S_1 = S_2$ and $V_1 \otimes V_1 \simeq V_0 \oplus V_2$. This counterexample can be fixed by replacing $K = SL_2(\mathbb{C})$ with $SL_2(\mathbb{C}) \times \mathbb{C}^\times$, where the multiplicative group $\mathbb{C}^\times$ acts by multiplication on $X = \mathbb{C}^2$. Then again $S_1 \cdot S_1 = S_2$, but now $V_1 \otimes V_1 \simeq \tilde{V}_0 \oplus V_2$ with $\tilde{V}_0 \not\simeq V_0$, since $\mathbb{C}^\times$ acts on $\tilde{V}_0$ with non-trivial weight. This example suggests that a natural class of multiplicity-free actions for which to study the question above is given by the class of irreducible linear multiplicity-free actions with the property that the image of K in $GL(X)$ contains the center $\mathbb{C}^\times$ of $GL(X)$. (A complete list of all such multiplicity-free actions can be found in [15].) In this article we consider the nice subclass of multiplicity-free actions that arise in the context of Hermitian symmetric spaces.

1.2 Two Conjectures

Let $G_\mathbb{R}/K_\mathbb{R}$ be an irreducible Hermitian symmetric space of non-compact type, and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}^+ \oplus \mathfrak{p}^-$ be the usual decomposition of $\mathfrak{g} = \mathrm{Lie}(G_\mathbb{R}) \otimes_\mathbb{R} \mathbb{C}$ (see Section 3 for more details). The complexification of the compact group $K_\mathbb{R}$ is a complex reductive group K with one-dimensional center $\mathbb{C}^\times$, and the action of K on $X = \mathfrak{p}^+$ is multiplicity free. Table 1 shows a complete list of the multiplicity-free actions that arise in this way. The last column of the table shows the rank of the Hermitian symmetric space $G_\mathbb{R}/K_\mathbb{R}$ which is (by definition) the real rank of $G_\mathbb{R}$.

$G_\mathbb{R}$	K	$X = \mathfrak{p}^+$	r
$SU(p,q)$	$S(GL_p(\mathbb{C}) \times GL_q(\mathbb{C}))$	$\mathbb{C}^p \otimes (\mathbb{C}^q)^*$	$\min\{p,q\}$
$Sp(n,\mathbb{R})$	$GL_n(\mathbb{C})$	$S^2(\mathbb{C}^n)$	n
$SO^*(2n)$	$GL_n(\mathbb{C})$	$\wedge^2(\mathbb{C}^n)$	$\lfloor n/2 \rfloor$
$SO_0(n,2)$	$SO_n(\mathbb{C}) \times \mathbb{C}^\times$	$\mathbb{C}^n$	2
$E\,\mathrm{III}$	$\mathrm{Spin}_{10}(\mathbb{C}) \cdot \mathbb{C}^\times$	$\mathbb{C}^{16}$ (spin)	2
$E\,\mathrm{VII}$	$E_6(\mathbb{C}) \cdot \mathbb{C}^\times$	$\mathbb{C}^{27}$ (min)	3

Table 1: Multiplicity-free actions associated with Hermitian symmetric spaces.

We will study two conjectures.

Conjecture A *For the multiplicity-free action associated with the Hermitian symmetric space $G_\mathbb{R}/K_\mathbb{R}$, i.e., for the K-action on $X = \mathfrak{p}^+$, $S_\nu \subseteq S_\lambda \cdot S_\mu$ if and only if $V_\nu \hookrightarrow V_\lambda \otimes V_\mu$, where $\lambda, \mu, \nu \in \Lambda$.*

The second conjecture is closely related to the first and connects the problem of describing the set $\{\nu \in \Lambda \mid S_\nu \subseteq S_\lambda \cdot S_\mu\}$ to the classical Littlewood–Richardson rule. To state the conjecture, we need some more notation. Schmid proved in [31]

that if $G_\mathbb{R}/K_\mathbb{R}$ has rank r, then the non-zero K-isotypic components of $S = \mathbb{C}[\mathfrak{p}^+]$ are naturally parametrized by partitions of length at most r, *i.e.*, the set

$$\Lambda = \{(\lambda_1, \ldots, \lambda_r) \in \mathbb{Z}^r \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 0\}.$$

This set Λ also parametrizes the irreducible (polynomial) representations of $GL_r(\mathbb{C})$ in the usual way. For $\lambda \in \Lambda$, let F_λ be the irreducible $GL_r(\mathbb{C})$-representation of lowest weight $-\lambda$. For $\lambda, \mu, \nu \in \Lambda$, we denote by $c_{\lambda\mu}^\nu$ the multiplicity of F_ν in $F_\lambda \otimes F_\mu$. The numbers $c_{\lambda\mu}^\nu$ are known as the Littlewood–Richardson coefficients for $GL_r(\mathbb{C})$.

Conjecture B   *Let r be the rank of the Hermitian symmetric space $G_\mathbb{R}/K_\mathbb{R}$ and identify Λ with the set of partitions of length at most r. Then for the K-action on $X = \mathfrak{p}^+$, $S_\nu \subseteq S_\lambda \cdot S_\mu$ if and only if $c_{\lambda\mu}^\nu \neq 0$, where $\lambda, \mu, \nu \in \Lambda$ and $c_{\lambda\mu}^\nu$ are Littlewood–Richardson coefficients for $GL_r(\mathbb{C})$.*

For the multiplicity-free actions that correspond to Hermitian symmetric spaces of tube type (see Section 3.3), this conjecture was made by Ruitenburg [28], who observed that it holds for $X = \mathbb{C}^p \otimes (\mathbb{C}^p)^*$.

The validity of Conjecture B is connected to the multiplication of Jack polynomials $P_\lambda^{(\alpha)}$, a well-known family of symmetric functions. Here λ is a partition and α is a real parameter. If $\alpha = 1$, then $P_\lambda^{(\alpha)} = s_\lambda$ is the Schur symmetric function. There is an expansion

$$P_\lambda^{(\alpha)} P_\mu^{(\alpha)} = \sum_\nu f_{\lambda\mu}^\nu(\alpha) P_\nu^{(\alpha)},$$

where the $f_{\lambda\mu}^\nu(\alpha)$ are rational functions in the parameter α. Note that since $P_\lambda^{(1)} = s_\lambda$, the value $f_{\lambda\mu}^\nu(1)$ is the Littlewood–Richardson coefficient $c_{\lambda\mu}^\nu$. In [32], Stanley conjectured that the $f_{\lambda\mu}^\nu(\alpha)$ have a certain positivity property, which we can formulate as follows. Let $c_\lambda(\alpha)$ and $c_\lambda'(\alpha)$ be the non-zero polynomials in α with non-negative integer coefficients as defined in [26, Chapter V, §10]. Then Stanley's conjecture is that $c_\lambda(\alpha)c_\mu(\alpha)c_\nu'(\alpha)f_{\lambda\mu}^\nu(\alpha)$ are polynomials in α with non-negative integer coefficients. Except for the non-negativity, this conjecture is true. As noted by Macdonald [26, Ex. VI.10.1 d], this follows from the integrality of Macdonald polynomials, proved independently by several authors [7, 8, 16, 18, 20, 23, 29]. The main result of this paper, which we will prove in Section 5, can then be phrased as follows.

Theorem 1.1   *Suppose that Stanley's conjecture is true. Then Conjecture B is true for all Hermitian symmetric spaces.*

We also prove (without assuming Stanley's conjecture) that Conjecture B is true if $G_\mathbb{R} = SU(p, q)$ or if the rank of the Hermitian symmetric space is at most 2 (see Sections 2 and 3).

What about the validity of Conjecture A? In Section 6, we show the equivalence of Conjecture A and Conjecture B for all Hermitian symmetric spaces of classical type. More precisely, for $\lambda, \mu, \nu \in \Lambda$ we express the multiplicity of V_ν in $V_\lambda \otimes V_\mu$ in terms of Littlewood–Richardson coefficients (see Table 4) and show that $V_\nu \hookrightarrow V_\lambda \otimes V_\mu$ if and only if $c_{\lambda\mu}^\nu \neq 0$. In the classical cases corresponding to $G_\mathbb{R} = Sp(n, \mathbb{R})$ and

$SO^*(2n)$, this last equivalence follows from Klyachko's saturation conjecture, which is now a theorem of Knutson and Tao [21] and also of Derksen and Weyman [3].

There is some overlap between the topics of this paper (especially Section 5) and earlier work of Ruitenburg [28]. For the reader's convenience, we have tried to give a relatively self-contained account.

2 A Classical Example

2.1 The Regular Representation of a Reductive Group

Let G be a complex reductive algebraic group. By reductive we mean linearly reductive, *i.e.*, all representations are completely reducible. (In particular, we do not assume here that G is connected.) Let $R = \mathbb{C}[G]$ denote the ring of regular functions on G. Recall that R carries a natural action of $G \times G$, the regular representation ρ_{reg}, given by

$$(\rho_{reg}(g_1, g_2)f)(x) = f(g_1^{-1}xg_2)$$

for $f \in R$ and $g_1, g_2, x \in G$. By the algebraic Peter–Weyl theorem, R decomposes, as a $G \times G$-representation, into a direct sum $R = \bigoplus_{\lambda \in \widehat{G}} R_\lambda$, where $R_\lambda \simeq V_\lambda \boxtimes V_\lambda^*$.

The following proposition is observed in [1, §3.2].

Proposition 2.1 *Let G be a complex reductive algebraic group, and let $R = \mathbb{C}[G]$ be the ring of regular functions on G. Then $R_\nu \subseteq R_\lambda \cdot R_\mu$ if and only if $V_\nu \hookrightarrow V_\lambda \otimes V_\mu$ for all $\lambda, \mu, \nu \in \widehat{G}$.*

Proof If $R_\nu \subseteq R_\lambda \cdot R_\nu$, then the $G \times G$-representation $V_\nu \boxtimes V_\nu^*$ appears in $(V_\lambda \boxtimes V_\lambda^*) \otimes (V_\mu \boxtimes V_\mu^*)$, which implies that the G-representation V_ν appears in $V_\lambda \otimes V_\mu$. We now prove the converse. For $\lambda \in \widehat{G}$ let $\chi_\lambda \in R$ denote the character of V_λ, *i.e.*, $\chi_\lambda(g) = \operatorname{tr} \rho_\lambda(g)$. Recall that since R_λ is spanned by the matrix coefficients of ρ_λ (with respect to any basis of V_λ), we have that $\chi_\lambda \in R_\lambda$. Moreover, in R, we have $\chi_\lambda \cdot \chi_\mu = \sum c_{\lambda\mu}^\nu \chi_\nu$, where $c_{\lambda\mu}^\nu$ is the multiplicity of V_ν in $V_\lambda \otimes V_\mu$. Let $\pi_\lambda \colon R \to R_\lambda$ denote the projection. Since $R_\lambda \cdot R_\mu$ is a $G \times G$-stable subspace of R, and since $G \times G$ is reductive, we have $R_\nu \subseteq R_\lambda \cdot R_\mu$ if and only if $\pi_\nu(R_\lambda \cdot R_\mu)$ is non-zero. If V_ν occurs in $V_\lambda \otimes V_\mu$ then $c_{\lambda\mu}^\nu \neq 0$. Hence $\pi_\nu(\chi_\lambda \cdot \chi_\mu) \neq 0$, so $\pi_\nu(R_\lambda \cdot R_\mu) \neq 0$, completing the proof. ∎

Remark 2.2 Note that the decomposition $R = \sum R_\lambda$ is a decomposition into isotypic components for the $G \times G$-action as well as the G-action induced by left multiplication on G (the latter action being not multiplicity free). We recall that the irreducible representations of $G \times G$ are parametrized by pairs $(\lambda, \mu) \in \widehat{G} \times \widehat{G}$. With this notation we then have $R_{(\lambda,\mu)} = 0$ if $\mu \neq \lambda^*$ and $R_{(\lambda,\lambda^*)} = R_\lambda$, where λ^* denotes the isomorphism class of the G-representation dual to V_λ.

2.2 Proof of the Conjectures for $SU(p, p)$

We use the notation of Subsection 2.1 with $G = GL_p(\mathbb{C})$. (To simplify the notation in the following, we will write GL_p for $GL_p(\mathbb{C})$.) The set $\widehat{GL_p}$ is parametrized by integer

sequences $\lambda = (\lambda_1, \ldots, \lambda_p)$ with $\lambda_1 \geq \cdots \geq \lambda_p$. More precisely, if $\lambda = (\lambda_1, \ldots, \lambda_p)$ let F_λ be the irreducible GL_p-representation with lowest weight

$$-\lambda_1 \varepsilon_1 - \lambda_2 \varepsilon_2 - \cdots - \lambda_p \varepsilon_p.$$

By abuse of notation we will identify elements of $\widehat{GL_p}$ with such sequences. Let M_p denote the space of complex $p \times p$-matrices and let $S = \mathbb{C}[M_p]$. The open embedding $GL_p \hookrightarrow M_p$ induces a $GL_p \times GL_p$-equivariant embedding of coordinate rings $S \hookrightarrow R = S[\det^{-1}]$. This embedding identifies S as the subring

$$S = \mathbb{C}[M_p] \simeq \sum_{\lambda \in \Lambda} R_\lambda \subseteq R = \mathbb{C}[GL_p] \,,$$

where $\Lambda = \{(\lambda_1, \ldots, \lambda_p) \in \mathbb{Z}^p \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p \geq 0\}$. Using notation as in the remark at the end of Subsection 2.1, the only non-vanishing isotypic components for the $GL_p \times GL_p$-action on S are of the form $S_{(\lambda, \lambda^*)} = R_\lambda$ for $\lambda \in \widehat{GL_p}$ with $\lambda_p \geq 0$. It follows that Conjecture A and Conjecture B for the $K = GL_p \times GL_p$-action on $X = M_p$ are an immediate consequence of the results in Subsection 2.1.

2.3 Proof of the Conjectures for $SU(p, q)$

Next we consider the action of $K = GL_p \times GL_q$ on $X = M_{p,q}$, where $M_{p,q}$ denotes the space of $p \times q$-matrices. Without loss of generality, we may assume $p \leq q$. Let $M_{p,q} \twoheadrightarrow M_p$ be the natural projection given by "forgetting the last $(q - p)$-columns". This projection is GL_p-equivariant with respect to the actions given by left multiplication and thus we obtain a GL_p-equivariant embedding $\mathbb{C}[M_p] \hookrightarrow \mathbb{C}[M_{p,q}]$. If x_{ij} are the canonical coordinate functions on $M_{p,q}$ then $\mathbb{C}[M_{p,q}] = \mathbb{C}[x_{ij} \mid 1 \leq i \leq p, 1 \leq j \leq q]$ and $\mathbb{C}[M_p]$ is identified (via the embedding given above) with the subring $\mathbb{C}[x_{ij} \mid 1 \leq i, j \leq p]$. Let $N_p^- \subseteq GL_p$ be the group of lower triangular unipotent $p \times p$-matrices and $N_q \subseteq GL_q$ the group of upper triangular unipotent $q \times q$-matrices. Let $\mathbb{C}[M_{p,q}]^{N_p^- \times N_q}$ be the ring of $N_p^- \times N_q$-invariants in $\mathbb{C}[M_{p,q}]$. It is a theorem of classical invariant theory due to Weyl (see, *e.g.*, [6]) that $\mathbb{C}[M_{p,q}]^{N_p^- \times N_q} = \mathbb{C}[u_1, \ldots, u_p]$, where $u_k(x)$ is the k-th principal minor of $x \in M_{p,q}$. Note that u_k is in fact in $\mathbb{C}[M_p]$ and hence

$$\mathbb{C}[M_p]^{N_p^- \times N_p} = \mathbb{C}[M_{p,q}]^{N_p^- \times N_q}.$$

Furthermore, the polynomials $u_1, \ldots, u_r$ are algebraically independent and for each $\lambda \in \Lambda = \{(\lambda_1, \ldots, \lambda_p) \in \mathbb{Z}^p \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p \geq 0\}$, the polynomial $u_\lambda = u_1^{\lambda_1} u_2^{\lambda_2 - \lambda_3} \cdots u_r^{\lambda_r}$ is a weight vector of weight $(-\lambda_1 \varepsilon_1 - \lambda_2 \varepsilon_2 - \cdots - \lambda_p \varepsilon_p, \lambda_1 \varepsilon_1 + \lambda_2 \varepsilon_2 + \cdots + \lambda_p \varepsilon_p)$ for the group $GL_p \times GL_q$. Thus, as a $GL_p \times GL_q$-representation,

$$\mathbb{C}[M_{p,q}] \simeq \bigoplus_{\lambda \in \Lambda} F_\lambda^{(p)} \boxtimes \left(F_\lambda^{(q)}\right)^*.$$

Here we use superscripts p and q to emphasize that $F_\lambda^{(p)}$ is a representation of GL_p and $F_\lambda^{(q)}$ is representation of GL_q.

Let $S = \mathbb{C}[M_p]$ and $T = \mathbb{C}[M_{p,q}]$. If $\lambda \in \Lambda$, let S_λ and T_λ denote the λ-isotypic components of S and T, respectively, *i.e.*, $S_\lambda \simeq F_\lambda^{(p)} \boxtimes (F_\lambda^{(p)})^*$ as a $GL_p \times GL_p$-representation and $T_\lambda \simeq F_\lambda^{(p)} \boxtimes (F_\lambda^{(p)})^*$ as a $GL_p \times GL_q$-representation. We note that $S_\lambda \hookrightarrow T_\lambda$ via the embedding $S \hookrightarrow T$. This follows from the observation that S_λ and T_λ are also the λ-isotypic components for the GL_p-action on S and T, respectively.

Suppose now that the $GL_p \times GL_q$-representation $F_\nu^{(p)} \boxtimes (F_\nu^{(q)})^*$ appears in $(F_\lambda^{(p)} \boxtimes (F_\lambda^{(q)})^*) \otimes (F_\mu^{(p)} \boxtimes (F_\mu^{(q)})^*)$. Then the GL_p-representation $F_\nu^{(p)}$ appears in $F_\lambda^{(p)} \otimes F_\mu^{(p)}$. By the previous subsection, it follows that $S_\nu \subseteq S_\lambda \cdot S_\mu$. In particular, $u_\nu \in S_\lambda \cdot S_\mu$. Since $u_\nu \in T_\nu$ (via the embedding $S \hookrightarrow T$) this implies that $T_\nu \subseteq T_\lambda \cdot T_\mu$. It follows that Conjecture A and Conjecture B are also true for the $GL_p \times GL_q$-action on $X = M_{p,q}$ for $q > p$.

Remark 2.3 The argument in the previous section is a special case of a "reduction to tube type" argument introduced by Wallach in [34]. We will discuss this kind of argument for general Hermitian symmetric pairs in more detail in Section 4.

3 Polynomials on Hermitian Symmetric Spaces

3.1 Hermitian Symmetric Spaces

We recall some well-known results and constructions related to Hermitian symmetric spaces which we will use in the following sections. Let Ω be an irreducible Hermitian symmetric space of non-compact type. Distinguishing a point $o \in \Omega$, we have $\Omega \simeq G_\mathbb{R}/K_\mathbb{R}$, where $G_\mathbb{R}$ is the connected group of biholomorphic transformations of Ω and $K_\mathbb{R}$ is the stabilizer group of o. The group $G_\mathbb{R}$ is a simple non-compact Lie group and $K_\mathbb{R}$ is a maximal compact subgroup of $G_\mathbb{R}$. (These groups coincide with the groups of Table 1, up to local isomorphism.) Let $\mathfrak{g}$ and $\mathfrak{k}$ denote the complexified Lie algebras of $G_\mathbb{R}$ and $K_\mathbb{R}$, respectively, and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition. There is an element z in the center of $\mathfrak{k}$ such that $\mathfrak{k} = \mathbb{C}z \oplus [\mathfrak{k}, \mathfrak{k}]$ with $\mathrm{ad}(z)$ having the eigenvalues 0 and ± 1 on $\mathfrak{g}$. Define $\mathfrak{p}_\pm = \{x \in \mathfrak{g} \mid [z, x] = \pm x\}$. Then $\mathfrak{p} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$, $[\mathfrak{p}^\pm, \mathfrak{p}^\pm] = 0$, $[\mathfrak{p}^\pm, \mathfrak{p}^\mp] = \mathfrak{k}$ and $[\mathfrak{k}, \mathfrak{p}^\pm] = \mathfrak{p}^\pm$. Harish-Chandra contructed a canonical $K_\mathbb{R}$-equivariant open embedding $\Omega \hookrightarrow \mathfrak{p}^+$ that sends o to the origin in $\mathfrak{p}^+$; via this embedding we will view Ω as an open subset of $\mathfrak{p}^+$.

3.2 Strongly Orthogonal Roots

Let $\mathfrak{h}_\mathbb{R}$ be a Cartan subalgebra of $\mathfrak{k}_\mathbb{R}$. The complexification $\mathfrak{h}$ of $\mathfrak{h}_\mathbb{R}$ is a Cartan subalgebra of $\mathfrak{k}$ and of $\mathfrak{g}$. Let $\Delta \subseteq \mathfrak{h}^*$ be the root system of $(\mathfrak{g}, \mathfrak{h})$. For $\alpha \in \Delta$, let $\mathfrak{g}_\alpha$ denote the root subspace of $\mathfrak{g}$ corresponding to α. Define the set of compact roots as $\Delta_c = \{\alpha \in \Delta \mid \mathfrak{g}_\alpha \subseteq \mathfrak{k}\}$ and the set of non-compact roots as $\Delta_n = \{\alpha \in \Delta \mid \mathfrak{g}_\alpha \subseteq \mathfrak{p}\}$. We may choose a system of positive roots Δ^+ for Δ such that $\mathfrak{p}^+$ is the sum of the root subspaces for the roots in $\Delta_n^+ = \Delta_n \cap \Delta^+$. The set $\Delta_c^+ = \Delta_c \cap \Delta^+$ is a system of positive roots for Δ_c. Following Harish-Chandra we define a maximal set $\{\gamma_1, \ldots, \gamma_r\}$ of strongly orthogonal roots in Δ_n^+ inductively

as follows. Let γ_1 be the largest root in Δ_n^+ (with respect to the usual ordering on Δ induced by the choice of Δ^+); for $1 < i \leq r$, let γ_i be the largest root in Δ_n^+ that is orthogonal to $\gamma_1, \ldots, \gamma_{i-1}$. Table 2 shows the γ_i's explicitly for all cases. For

$\mathfrak{g}$	$\mathfrak{k}$	r	$\{\gamma_1, \ldots, \gamma_r\}$
$\mathfrak{sl}_{p+q}(\mathbb{C})$	$\mathfrak{s}(\mathfrak{gl}_p(\mathbb{C}) \oplus \mathfrak{gl}_q(\mathbb{C}))$	$\min\{p,q\}$	$\{e_1 - e_{p+q}, e_2 - e_{p+q-1}, \ldots\}$
$\mathfrak{sp}_{2n}(\mathbb{C})$	$\mathfrak{gl}_n(\mathbb{C})$	n	$\{2e_1, \ldots, 2e_n\}$
$\mathfrak{so}_{2n}(\mathbb{C})$	$\mathfrak{gl}_n(\mathbb{C})$	$\lfloor n/2 \rfloor$	$\{e_1 + e_2, e_3 + e_4, \ldots\}$
$\mathfrak{so}_{n+2}(\mathbb{C})$	$\mathfrak{so}_n(\mathbb{C}) \oplus \mathbb{C}$	2	$\{e_1 + e_2, e_1 - e_2\}$
$\mathfrak{e}_6(\mathbb{C})$	$\mathfrak{so}_{10}(\mathbb{C}) \oplus \mathbb{C}$	2	$\left\{ {}^{1\;2\;3\;2\;1}_{\quad 2}, {}^{1\;1\;1\;1\;1}_{\quad 0} \right\}$
$\mathfrak{e}_7(\mathbb{C})$	$\mathfrak{e}_6(\mathbb{C}) \oplus \mathbb{C}$	3	$\left\{ {}^{2\;3\;4\;3\;2\;1}_{\quad 2}, {}^{0\;1\;2\;2\;2\;1}_{\quad 1}, \alpha_7 \right\}$

Table 2: Strongly orthogonal roots.

$1 \leq i \leq r$, define $\pi_i := \sum_{j=1}^{i} \gamma_j$; the π_i are Δ_c^+-integral and dominant weights. Let $\Lambda = \{(\lambda_1, \ldots, \lambda_r) \in \mathbb{Z}^r \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 0\}$ be the set of partitions of length at most r. If $\lambda = (\lambda_1, \ldots, \lambda_r) \in \Lambda$, then $\sum_{i=1}^{r} \lambda_i \gamma_i = \sum_{i=1}^{r} (\lambda_i - \lambda_{i+1})\pi_i$ is a Δ_c^+-integral and dominant weight. (Here it is understood that $\lambda_{r+1} = 0$.) In [31], Schmid gave an explicit decomposition of $S = \mathbb{C}[\mathfrak{p}^+]$ as a K-representation as follows. If $\lambda \in \Lambda$, let V_λ be the irreducible $\mathfrak{k}$-representation with lowest weight $-\sum_{i=1}^{r} \lambda_i \gamma_i$. Then $S \simeq \bigoplus_{\lambda \in \Lambda} V_\lambda$. Furthermore, S_λ is contained in the space of homogeneous polynomials of degree $|\lambda|$, where $|\lambda| = \sum_{i=1}^{r} \lambda_i$.

3.3 Tube Domains and a Result of Schmid

For the convenience of the reader and to provide a context for the material in Sections 4 and 5, we include here some more details from Schmid [31] and Koranyi and Wolf [22]. For each $\alpha \in \Delta_n^+$ let $\{h_\alpha, e_\alpha, e_{-\alpha}\}$ be an $\mathfrak{sl}_2$-triple such that $e_\alpha \in \mathfrak{g}_\alpha$, $e_{-\alpha} \in \mathfrak{g}_{-\alpha}$, $h_\alpha \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ and $e_{-\alpha} = \sigma(e_\alpha)$, where $\sigma \colon \mathfrak{g} \to \mathfrak{g}$ denotes complex conjugation with respect to $\mathfrak{g}_\mathbb{R} = \mathrm{Lie}(G_\mathbb{R})$. Let $e_+ = e_{\gamma_1} + \cdots + e_{\gamma_r}$. Then e_+ is on the Shilov boundary of Ω in $\mathfrak{p}^+$. In fact, the Shilov boundary is the $K_\mathbb{R}$-orbit through e_+, and hence isomorphic to $K_\mathbb{R}/M_\mathbb{R}$, where $M_\mathbb{R} = \mathrm{Stab}_{K_\mathbb{R}}(e_+) = \{k \in K_\mathbb{R} \mid ke_+ = e_+\}$. Let $M = \mathrm{Stab}_K(e_+) = \{k \in K \mid ke_+ = e_+\}$ and let $\mathfrak{m}$ be the Lie algebra of M. Define an automorphism $\tau \colon \mathfrak{g} \to \mathfrak{g}$ by $\tau = \mathrm{Ad}\exp(i\pi(e_+ + \overline{e}_+)/2)$. Then $\tau^4 = 1$, $\tau^2(\mathfrak{k}) = \mathfrak{k}$, $\mathfrak{m} = \{x \in \mathfrak{k} \mid \tau(x) = x\}$ and the following are equivalent (see [22, Proposition 4.4 and Lemma 4.8]):

(i) Ω is a tube domain;
(ii) $\tau^2 = 1$;
(iii) $\tau(\mathfrak{k}) = \mathfrak{k}$;
(iv) $\dim \mathfrak{k} - \dim \mathfrak{m} = \dim \mathfrak{p}^+$.

Suppose from now on that Ω is a tube domain. Then by (iv), the K-orbit through e_+ is open and dense in $\mathfrak{p}^+$ and the corresponding open embedding $K/M \hookrightarrow \mathfrak{p}^+$ gives a K-equivariant inclusion of coordinate rings $S = \mathbb{C}[\mathfrak{p}^+] \hookrightarrow \mathbb{C}[K/M]$. By the algebraic Peter-Weyl theorem, $\mathbb{C}[K/M] \simeq \bigoplus_{\lambda \in \widehat{K}} V_\lambda \otimes (V_\lambda^*)^M$. We may identify $\widehat{K}$ with the set of Δ_c^+-dominant integral weights in $\mathfrak{h}^*$. If $\lambda \in \widehat{K}$, let V_λ be the irreducible K-representation with lowest weight $-\lambda$. Then $V_\lambda^M \neq 0$ (or equivalently $(V_\lambda^*)^M \neq 0$) if and only if λ is in the lattice generated by the γ_i, *i.e.*, if and only if λ is of the form $\lambda = \sum \lambda_i \gamma_i$ with $\lambda_i \in \mathbb{Z}$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r$ (see [31, Lemma 1]). Furthermore, if $V_\lambda^M \neq 0$, then $\dim V_\lambda^M = 1$. Schmid then showed (see [31, Behauptung c]) that via the embedding $S = \mathbb{C}[\mathfrak{p}^+] \hookrightarrow \mathbb{C}[K/M]$,

$$ S = \mathbb{C}[\mathfrak{p}^+] \simeq \bigoplus_{\lambda \in \Lambda} V_\lambda \otimes (V_\lambda^*)^M \hookrightarrow \mathbb{C}[K/M], $$

where

$$ \Lambda = \{\lambda \in \widehat{K} \mid \lambda = \sum \lambda_i \gamma_i,\ \lambda_i \in \mathbb{Z},\ \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 0\}. $$

The connection with root systems of type A is as follows. Since we assume that Ω is of tube type, the pair $(\mathfrak{k}, \mathfrak{m})$ is a symmetric pair with involution τ. Let $\mathfrak{k}_{\mathbb{R}} = \mathfrak{m}_{\mathbb{R}} \oplus \mathfrak{s}_{\mathbb{R}}$ be the corresponding decomposition, *i.e.*,

$$ \mathfrak{m} = \{x \in \mathfrak{k} \mid \tau(x) = x\} \quad \text{and} \quad \mathfrak{s} = \{x \in \mathfrak{k} \mid \tau(x) = -x\}. $$

This complexifies to $\mathfrak{k} = \mathfrak{m} \oplus \mathfrak{s}$. Define $\mathfrak{h}_{\mathbb{R}}^+ = \mathfrak{h}_{\mathbb{R}} \cap \mathfrak{m}_{\mathbb{R}}$ and $\mathfrak{h}_{\mathbb{R}}^- = \mathfrak{h}_{\mathbb{R}} \cap \mathfrak{s}_{\mathbb{R}}$. Then $\mathfrak{h} = \mathfrak{h}^+ \oplus \mathfrak{h}^-$ and $\mathfrak{h}^-$ is a maximal abelian subspace of $\mathfrak{s}$. By a result of Moore [25] (see also the remarks in the next section), if $\alpha \in \Delta_c^+$, then $\alpha|_{\mathfrak{h}^-} = 0$ or $\alpha|_{\mathfrak{h}^-} = \frac{1}{2}(\gamma_i - \gamma_j)$ with $i < j$. Thus the restricted root system Σ is of type A_{r-1}. Schmid's result above (which is a version of the Cartan–Helgason theorem; see also Section 5) implies that the K-representations occurring in $\mathbb{C}[K/M]$ are exactly the V_λ, where $\lambda = \sum_{i=1}^r \lambda_i \gamma_i$ with $\lambda_i \in \mathbb{Z}$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r$. Table 3 shows the symmetric pairs $(\mathfrak{k}, \mathfrak{m})$ associated to the Hermitian symmetric spaces $\Omega = G_{\mathbb{R}}/K_{\mathbb{R}}$ of tube type. The last column shows the root multiplicity m of the restricted roots. This root multiplicity will play an important role later in Section 5.

Remark 3.1   Note that in our context here, K is not simply connected and M is in general not connected. Furthermore, K is not semisimple: it has a one-dimensional center. In Section 5, to avoid technical difficulties, we will work with symmetric pairs (K, M), where K is a connected and simply connected semisimple complex algebraic group. In Subsection 5.8, we will then return to the context of this section.

4   Reduction to Tube Type

4.1   A Construction Due to Wallach

We retain the notation from the previous section. In particular, let $\{\gamma_1, \ldots, \gamma_r\}$ be Harish-Chandra's strongly orthogonal roots with the convention that γ_1 is the *largest*

$G_{\mathbb{R}}$	$\mathfrak{k}$	$\mathfrak{m}$	m
$SU(p,p)$	$\mathfrak{s}(\mathfrak{gl}_p(\mathbb{C}) \oplus \mathfrak{gl}_p(\mathbb{C}))$	$\mathfrak{sl}_p(\mathbb{C})$	2
$Sp(n,\mathbb{R})$	$\mathfrak{gl}_n(\mathbb{C})$	$\mathfrak{so}_n(\mathbb{C})$	1
$SO^*(4p)$	$\mathfrak{gl}_{2p}(\mathbb{C})$	$\mathfrak{sp}_{2p}(\mathbb{C})$	4
$SO_0(n,2), n \geq 3$	$\mathfrak{so}_n(\mathbb{C}) \oplus \mathbb{C}$	$\mathfrak{so}_{n-1}(\mathbb{C})$	$n-2$
$E\,\mathrm{VII}$	$\mathfrak{e}_6(\mathbb{C}) \oplus \mathbb{C}$	$\mathfrak{f}_4(\mathbb{C})$	8

Table 3: The symmetric pairs $(\mathfrak{k},\mathfrak{m})$ associated to $\Omega = G_{\mathbb{R}}/K_{\mathbb{R}}$ of tube type.

root in Δ_n^+ and $\gamma_1 > \gamma_2 > \cdots > \gamma_r$. We point out to the reader that this is different from the convention in much of the literature (*e.g.*, [25, 31, 34]), where γ_1 is taken to be the smallest root in Δ_n^+. If $(\mathfrak{g}, \mathfrak{k})$ is of tube type (and only then), our γ_i corresponds to the other's γ_{r-i}. Let $\mathfrak{h}^-$ be the subspace of $\mathfrak{h}$ spanned by the coroots of $\gamma_1, \ldots, \gamma_r$ and let $\mathfrak{h}^+ = \{h \in \mathfrak{h} \mid \gamma_i(h) = 0, \ 1 \leq i \leq r\}$. If $(\mathfrak{g}, \mathfrak{h})$ is of tube type, the spaces $\mathfrak{h}^-$ and $\mathfrak{h}^+$ agree with the ones of 3.3. The following results are due to Moore [25]. If $\alpha \in \Delta_c^+$, then $\alpha|_{\mathfrak{h}^-} = \frac{1}{2}(\gamma_i - \gamma_j)$ with $i < j$, $\alpha|_{\mathfrak{h}^-} = \frac{1}{2}\gamma_i$, or $\alpha|_{\mathfrak{h}^-} = 0$. If $\alpha \in \Delta_n^+$, then $\alpha|_{\mathfrak{h}^-} = \frac{1}{2}(\gamma_i + \gamma_j)$ with $i \leq j$ or $\alpha|_{\mathfrak{h}^-} = \frac{1}{2}\gamma_i$. The Hermitian symmetric pair $(\mathfrak{g}, \mathfrak{k})$ is of tube type if and only if for every $\alpha \in \Delta$, $\alpha|_{\mathfrak{h}^-} = \pm\frac{1}{2}(\gamma_i \pm \gamma_j)$, where $1 \leq i \leq j \leq r$. The following construction is due to Wallach [34]. Let $\Delta_0 = \{\alpha \in \Delta \mid \alpha|_{\mathfrak{h}^-} = \pm\frac{1}{2}(\gamma_i \pm \gamma_j)\}$. Then Δ_0 is a root subsystem of Δ. Define $\tilde{\mathfrak{g}}_0 = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta_0} \mathfrak{g}_\alpha$, $\tilde{\mathfrak{k}}_0 = \mathfrak{k} \cap \tilde{\mathfrak{g}}_0$, and $\mathfrak{p}_0^\pm = \mathfrak{p}^\pm \cap \tilde{\mathfrak{g}}_0$. Then $\tilde{\mathfrak{g}}_0 = \tilde{\mathfrak{k}}_0 \oplus \mathfrak{p}_0^+ \oplus \mathfrak{p}_0^-$, $[\tilde{\mathfrak{k}}_0, \mathfrak{p}_0^\pm] = \mathfrak{p}_0^\pm$, $[\mathfrak{p}_0^\pm, \mathfrak{p}_0^\pm] = 0$ and $[\mathfrak{p}_0^\pm, \mathfrak{p}_0^\mp] \subseteq \tilde{\mathfrak{k}}_0$. Define $\mathfrak{k}_0 = [\mathfrak{p}_0^+, \mathfrak{p}_0^-]$ and $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0^+ \oplus \mathfrak{p}_0^-$. By [34, Lemma 2.2], $\mathfrak{g}_0$ is a simple Lie subalgebra of $\mathfrak{g}$ and $(\mathfrak{g}_0, \mathfrak{k}_0)$ is an irreducible Hermitian symmetric pair of tube type of rank r. Furthermore, $\mathfrak{h}^- \subseteq \mathfrak{k}_0$ and $\mathfrak{h}_0 = (\mathfrak{h}^+ \cap \mathfrak{k}_0) \oplus \mathfrak{h}^-$ is a Cartan subalgebra of $\mathfrak{k}_0$ (and of $\mathfrak{g}_0$). By slight abuse of notation, the set $\{\gamma_1, \ldots, \gamma_r\}$, via restriction to $\mathfrak{h}_0$, is also a maximal set of strongly orthogonal roots for $\mathfrak{p}_0^+$.

4.2 Reduction to Tube Type

Let $\mathfrak{n}^- = \bigoplus_{\alpha \in \Delta_c^+} \mathfrak{g}_{-\alpha}$ and let $\mathbb{C}[\mathfrak{p}^+]^{\mathfrak{n}^-}$, *i.e.*, the space of lowest vectors of the $\mathfrak{k}$-module. By Schmid's result, the lowest weights in $\mathbb{C}[\mathfrak{p}^+]$ are of the form $-\sum_{i=1}^r \lambda_i \gamma_i$. For $1 \leq k \leq r$, let u_k be a lowest weight vector in $\mathbb{C}[\mathfrak{p}^+]$ of weight $-\sum_{i=1}^k \gamma_i$. Then $\mathbb{C}[\mathfrak{p}^+]^{\mathfrak{n}^-} = \mathbb{C}[u_1, \ldots, u_r]$ and the functions $u_1, \ldots, u_r$ are algebraically independent. Recall that via the Killing form $(\mathfrak{p}^+)^* \simeq \mathfrak{p}^-$, and hence we can identify $\mathbb{C}[\mathfrak{p}^+]$ with the symmetric algebra $S(\mathfrak{p}^-)$. Similarly, we can identify $\mathbb{C}[\mathfrak{p}_0^+]$ with $S(\mathfrak{p}_0^-)$. Thus $\mathbb{C}[\mathfrak{p}_0^+]$ may be viewed as a subring of $\mathbb{C}[\mathfrak{p}^+]$. Wallach showed [34, Lemma 3.3] that $u_k \in \mathbb{C}[\mathfrak{p}_0^+]$ for $1 \leq k \leq r$. Using Schmid's result again, this time for the pair $(\mathfrak{g}_0, \mathfrak{k}_0)$, we have

$$\mathbb{C}[\mathfrak{p}_0^+]^{\mathfrak{k}_0 \cap \mathfrak{n}^-} = \mathbb{C}[\mathfrak{p}_0^+]^{\tilde{\mathfrak{k}}_0 \cap \mathfrak{n}^-} = \mathbb{C}[\mathfrak{p}^+]^{\mathfrak{n}^-}.$$

Let $S = \mathbb{C}[\mathfrak{p}_0^+]$ and $T = \mathbb{C}[\mathfrak{p}^+]$. If $\lambda = \sum_{i=1}^r \lambda_i \gamma_i$, then $S_\lambda = S \cap T_\lambda$. Furthermore, S_λ is an isotypic component of S as a $\tilde{\mathfrak{k}}_0$-module as well as a $\breve{\mathfrak{k}}_0$-module.

Proposition 4.1 *Let $(\mathfrak{g}, \mathfrak{k})$ be an irreducible Hermitian symmetric pair of rank r and let $(\mathfrak{g}_0, \mathfrak{k}_0)$ be the associated Hermitian symmetric pair of tube type of the same rank. Then Conjecture B is true for $(\mathfrak{g}, \mathfrak{k})$ if and only if it is true for $(\mathfrak{g}_0, \mathfrak{k}_0)$.*
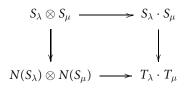
Proof We will show that $S_\nu \subseteq S_\lambda \cdot S_\mu$ if and only if $T_\nu \subseteq T_\lambda \cdot T_\mu$. By the remarks above, $S_\nu \subseteq S_\lambda \cdot S_\mu$ trivially implies that $T_\nu \subseteq T_\lambda \cdot T_\mu$. To prove the converse we will use an induction argument sketched by Enright and Wallach in [5]. Define $\mathfrak{q} = \tilde{\mathfrak{k}}_0 \oplus \mathfrak{u}^+$, where $\mathfrak{u}^+$ is the sum of all root spaces $\mathfrak{g}_\alpha$ with $\alpha \in \Delta_c^+$ such that $\alpha|_{\mathfrak{h}^-} = \frac{1}{2}\gamma_i$ for some $1 \leq i \leq r$. Then $\mathfrak{q}$ is a parabolic subalgebra of $\mathfrak{k}$ with Levi factor $\tilde{\mathfrak{k}}_0$ and abelian nilradical $\mathfrak{u}^+$ (see proof of Lemma 1 in [5]). Let $\mathfrak{q}^- = \tilde{\mathfrak{k}}_0 \oplus \mathfrak{u}^-$ be the opposite parabolic. If E is an irreducible finite dimensional $\tilde{\mathfrak{k}}_0$-module, let $N(E)$ denote the $\mathfrak{k}$-module obtained by inducing from $\mathfrak{q}^-$, *i.e.*, $N(E) = U(\mathfrak{k}) \otimes_{U(\mathfrak{q}^-)} E$.

Claim If ν is of the form $\nu = \lambda + \mu + \sum_{i=1}^r a_i \gamma_i$ with $\sum_{i=1}^r a_i = 0$, then the inclusion $S_\lambda \otimes S_\mu \hookrightarrow N(S_\lambda) \otimes N(S_\lambda)$ induces an equality of $-\nu$-weight spaces

$$[S_\lambda \otimes S_\mu]_{-\nu} = [N(S_\lambda) \otimes N(S_\mu)]_{-\nu}$$

Proof of Claim As an $\mathfrak{h}$-module, $N(E) \simeq U(\mathfrak{u}^+) \otimes E$. The weights in $U(\mathfrak{u}^+)$ restricted to $\mathfrak{h}^-$ are of the form $\sum_{i=1}^r \frac{1}{2}n_i \gamma_i$, where the n_i are non-negative integers. The restriction of the weights in $S_\lambda \otimes S_\mu$ to $\mathfrak{h}^-$ are all of the form $-\lambda - \mu + \sum_{i \leq j} \frac{1}{2}m_{ij}(\gamma_i - \gamma_j)$, where the m_{ij} are non-negative integers.

To show that $T_\nu \subseteq T_\lambda \cdot T_\mu$ implies $S_\nu \subseteq S_\lambda \cdot S_\mu$, we take $E = S_\lambda$ and $F = S_\mu$. By the universal property of generalized Verma modules, there are canonical quotient maps $N(S_\lambda) \to T_\lambda$ and $N(S_\mu) \to T_\mu$ such that the following diagram commutes:

$$
\begin{array}{ccc}
S_\lambda \otimes S_\mu & \longrightarrow & S_\lambda \cdot S_\mu \\
\downarrow & & \downarrow \\
N(S_\lambda) \otimes N(S_\mu) & \longrightarrow & T_\lambda \cdot T_\mu
\end{array}
$$

If $\nu \in \Lambda$ such that $T_\nu \subseteq T_\lambda \cdot T_\mu$ then $|\nu| = |\lambda| + |\mu|$ and hence ν satisfies the hypothesis of the claim. If we restrict the commutative diagram above to $-\nu$-weight spaces, the vertical arrow on the left is an isomorphism. It follows that the inclusion $[S_\lambda \cdot S_\mu]_{-\nu} \hookrightarrow [T_\lambda \cdot T_\mu]_{-\nu}$ is surjective and hence an isomorphism. Since $S^{\tilde{\mathfrak{k}}_0 \cap \mathfrak{n}^-} = T^{\mathfrak{n}^-}$, we conclude that $S_\lambda \cdot S_\mu$ contains a lowest weight vector of weight $-\nu$ and hence $S_\nu \subseteq S_\lambda \cdot S_\mu$. ∎

4.3 A Pieri Rule for Hermitian Symmetric Pairs

Enright and Wallach [5] proved the following Pieri rule for multiplication of functions in $S = \mathbb{C}[\mathfrak{p}^+]$ by an induction on the rank of the Hermitian symmetric pair.

Theorem 4.2 (**Enright-Wallach [5]**)   *With notation as above, for every $\lambda \in \Lambda$ and $k \in \mathbb{N}$, $S_\lambda \cdot S_{k\gamma_1} = \sum_\nu S_\nu$, where the sum is over all $\mu \in \Lambda$ with $|\nu| = |\lambda| + k$ and $\nu_1 \geq \lambda_1 \geq \nu_2 \geq \cdots \geq \lambda_{r-1} \geq \nu_r \geq \lambda_r$.*

Corollary 4.3   *Let $(\mathfrak{g}, \mathfrak{k})$ be an irreducible Hermitian symmetric pair of rank ≤ 2. Then Conjecture B is true for $(\mathfrak{g}, \mathfrak{k})$.*

Proof By Proposition 4.1 we may assume that $(\mathfrak{g}, \mathfrak{k})$ is of tube type. In this case (see Schlichtkrull [30]), the $\mathfrak{k}$-module with lowest weight $-\gamma_1 - \cdots - \gamma_r$ is one-dimensional, *i.e.*, $\dim S_{\gamma_1 + \ldots + \gamma_r} = 1$. Since S is a domain, this implies that for every $\lambda \in \Lambda$,

$$S_\lambda \cdot S_{\gamma_1 + \ldots + \gamma_r} = S_{\lambda + \gamma_1 + \ldots + \gamma_r}.$$

Suppose now that $r = 2$. Then for $\mu \in \Lambda$,

$$S_{(\mu_1, \mu_2)} = S_{(\mu_1 - \mu_2, 0)} \cdot S_{(\mu_2, \mu_2)}.$$

By using the Pieri rule of the theorem above, we can then compute the decomposition of $S_\lambda \cdot S_\mu$ for any $\lambda, \mu \in \Lambda$. If $\lambda, \mu, \nu \in \Lambda$ are such that $|\lambda| + |\mu| = |\nu|$, then one finds that $S_\nu \subseteq S_\lambda \cdot S_\mu$ if and only if $c_{\lambda\mu}^\nu \neq 0$. Explicitly, if $a = \lambda_1 - \lambda_2$, $b = \mu_1 - \mu_2$ and $c = \nu_1 - \nu_2$, then

$$c_{\lambda\mu}^\nu = \begin{cases} 1 & \text{if } c = a + b - 2k \text{ for some non-negative integer } k, \\ 0 & \text{otherwise.} \end{cases} \qquad \blacksquare$$

Remark 4.4   In [4], the theorem above was proved for $k = 1$. This special case is enough to determine all the K-invariant ideals of $S = \mathbb{C}[\mathfrak{p}^+]$. The authors of [4] were not aware at the time of writing that Ruitenburg in [28] proved a Pieri rule and determined all the K-invariant ideals of $S = \mathbb{C}[\mathfrak{p}^+]$ in a uniform manner for all cases corresponding to Hermitian symmetric spaces of tube type. Ruitenburg did not work in the context of Hermitian symmetric spaces, but instead used the structure of Riemannian symmetric spaces and spherical functions. In the next section we will use the same approach. Finally, note that much earlier Stanley [32] proved a Pieri rule for the multiplication of Jack polynomials, from which the above Pieri rule could be deduced using the ideas of the next section.

5 Spherical Functions and Jack Polynomials

5.1 Preliminaries

We begin with some preliminaries about symmetric spaces and introduce some notation which we will use throughout the section. Let K be a connected and simply connected complex semisimple algebraic group. Let τ be an involution of K and let $M = K^\tau$; the group M is connected (see [24, p. 171]). There exists a compact real form $(K_\mathbb{R}, M_\mathbb{R})$ of the pair (K, M); here $K_\mathbb{R}$ is a maximal compact subgroup of K which is preserved by τ, and $M_\mathbb{R} = (K_\mathbb{R})^\tau$ is a maximal compact subgroup of M. This can be seen as follows. There exists a real form $\mathfrak{k}_0$ of $\mathfrak{k}$ such that the involution $d\tau$

of $\mathfrak{k}$ is the complexification of a Cartan involution of $\mathfrak{k}_0$ (see [11, Lemma III.4.1]). Let $\mathfrak{k}_0 = \mathfrak{m}_{\mathbb{R}} + \mathfrak{s}_0$ denote the corresponding Cartan decomposition; the complexification $\mathfrak{k} = \mathfrak{m} + \mathfrak{s}$ is the decomposition of $\mathfrak{k}$ into $+1$ and -1 eigenspaces for τ. Write $\mathfrak{s}_{\mathbb{R}} = i\mathfrak{s}_0$. We can define a Cartan involution $d\tau'$ of $\mathfrak{k}$ by requiring that $d\tau'$ act as multiplication by 1 on $\mathfrak{k}_{\mathbb{R}} := \mathfrak{m}_{\mathbb{R}} + \mathfrak{s}_{\mathbb{R}}$, and as multiplication by -1 on $i\mathfrak{k}_0 + i\mathfrak{s}_{\mathbb{R}}$. Since K is simply connected, there is a corresponding involution τ' of K, and the fixed point set $K_{\mathbb{R}} = K^{\tau'}$ is a maximal compact subgroup of K with Lie algebra $\mathfrak{k}_{\mathbb{R}}$ (see [10, p. 252]). In particular, $K_{\mathbb{R}}$ is connected and simply connected. Moreover, by consideration of the Lie algebra one can see that $K_{\mathbb{R}}$ is preserved by τ. Since $K_{\mathbb{R}}$ is simply connected, the fixed point set $M_{\mathbb{R}} := (K_{\mathbb{R}})^{\tau'}$ is connected (again by [24]). Finally, $d\tau'$ restricts to a Cartan involution of $\mathfrak{m}$, and $M_{\mathbb{R}} = M^{\tau'}$, so $M_{\mathbb{R}}$ is a maximal compact subgroup of M. The space K/M is a symmetric space, and it is the complexification (in the sense of differential geometry) of the Riemannian symmetric space $K_{\mathbb{R}}/M_{\mathbb{R}}$.

5.2 Restricted Roots and Characters

Let $\mathfrak{a}_{\mathbb{R}}$ be a maximal abelian subspace of $\mathfrak{s}_{\mathbb{R}}$, let $\mathfrak{h}_{\mathbb{R}}$ be any maximal abelian subspace of $\mathfrak{k}_{\mathbb{R}}$ containing $\mathfrak{a}_{\mathbb{R}}$, and let $\mathfrak{h}$ denote the complex span of $\mathfrak{t}_{\mathbb{R}}$ in $\mathfrak{k}$. Then $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{k}$ [10, p. 259]. Let $H \supseteq A$ denote the algebraic tori in K whose Lie algebras are $\mathfrak{h}$ and $\mathfrak{a}$, respectively, and $H_{\mathbb{R}} \supseteq A_{\mathbb{R}}$ the compact tori of $K_{\mathbb{R}}$ whose Lie algebras are $\mathfrak{h}_{\mathbb{R}}$ and $\mathfrak{a}_{\mathbb{R}}$. The torus A is called a maximal split (or anisotropic) torus. Let $X^*(A)$ denote the group of characters of A, viewed as a subset of $\mathfrak{a}^*$; similarly we have $X^*(H) \subseteq \mathfrak{h}^*$. The group algebra over $\mathbb{C}$ of $X^*(A)$ can be identified with the coordinate ring $\mathbb{C}[A]$; write $e^\lambda \in \mathbb{C}[A]$ for the element of $\mathbb{C}[A]$ corresponding to the character $\lambda \in X^*(A)$.

Let $\Sigma \subseteq X^*(A)$ denote the set of restricted roots; that is, the elements of Σ are the non-zero weights for the A-action on $\mathfrak{k}$. Then Σ is a root system in the real subspace of $\mathfrak{a}^*$ it spans (see [27, §4]). Choose a positive system of roots Φ^+ for $(\mathfrak{k}, \mathfrak{h})$ and let Σ^+ denote the corresponding set of positive restricted roots. This choice of positive system induces an ordering on $\mathfrak{a}^*$ as usual, by the rule $\mu \le \lambda$ iff $\lambda - \mu$ is a non-negative linear combination of positive restricted roots. Let $X^*(H)^+ \subseteq \mathfrak{h}^*$ and $X^*(A)^+ \subseteq \mathfrak{a}^*$ denote the sets of dominant weights corresponding to the choice of positive system. We write V_λ for the irreducible representation of K with lowest weight $-\lambda \in X^*(H)^+$. Note that the decomposition $\mathfrak{h} = (\mathfrak{h} \cap \mathfrak{m}) \oplus \mathfrak{a}$ allows us to view $\mathfrak{a}^*$ as a summand in $\mathfrak{h}^*$. If $\lambda \in 2X^*(A)$, then $\lambda \in X^*(H)$; this follows since $A \cap M$ consists of the elements $a \in A$ with $a^2 = 1$.

5.3 A Result of Ruitenburg

Let $\mathbb{C}[K]$ denote the coordinate ring of K. By the algebraic Peter–Weyl theorem, as a K-representation,

$$\mathbb{C}[K/M] = \bigoplus_{\lambda \in \widehat{K}} V_\lambda \otimes (V_\lambda^*)^M.$$

The group M has the property that for any $\lambda \in \widehat{K}$, the dimension of the space V_λ^M is either 1 or 0 (see [11, Ch. IV §3]). The Cartan–Helgason theorem (see [11, Ch. V §2],

or [33]) states that the irreducible K-representations occurring in $\mathbb{C}[K/M]$ (that is, the K-representations with a non-zero M-fixed vector) are exactly the V_λ, where $\lambda \in 2X^*(A)^+$. For each such weight λ choose a left M-invariant function φ_λ in the V_λ-isotypic component of $\mathbb{C}[K/M]$; this choice is unique up to scaling (we will choose a scaling in the discussion after Proposition 5.3). We may view φ_λ as a function on K, bi-invariant under M; these are called spherical functions. The spherical functions φ_λ, as λ runs through $2X^*(A)^+$, form a basis of the ring $\mathbb{C}[K/M]^M$ as a vector space. Thus, we can expand the product of two spherical functions as follows: $\varphi_\lambda \varphi_\mu = \sum a_{\lambda\mu}^\nu \varphi_\nu$, where the $a_{\lambda\mu}^\nu$ are constants. Let $R = \mathbb{C}[K/M]$ and let R_λ denote the V_λ-isotypic component of R. In [28], Ruitenburg proved the following result.

Theorem 5.1 (Ruitenburg [28, Theorem 3.1]) *For fixed $\lambda, \mu, \nu \in 2X^*(A)^+$, $R_\nu \subseteq R_\lambda \cdot R_\mu$ if and only if $a_{\lambda\mu}^\nu \neq 0$.*

To prove this result, Ruitenburg used the compact real form described above. We will later use Ruitenburg' result to prove our main theorem from the introduction.

Remark 5.2 If $\chi_\lambda \in \mathbb{C}[K]$ denotes the character of the representation $\rho_\lambda \colon K \to GL(V_\lambda)$, then up to scaling, φ_λ is the projection of χ_λ onto the space of M-invariants (with respect to the left action of M on $\mathbb{C}[K]$). This follows from [11, Theorem 4.2, Ch. IV].
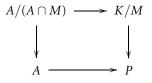
5.4 A Result of Richardson and the Functions P_λ

The negative of the Killing form induces a positive definite inner product on $\mathfrak{a}_\mathbb{R}$; let $s_\alpha \in GL(\mathfrak{a}_\mathbb{R})$ denote the reflection in the hyperplane $\alpha = 0$ in $\mathfrak{a}_\mathbb{R}$. The Weyl group of the pair (K, M), the "little Weyl group", is

$$W = N_M(A)/Z_M(A) \simeq N_{M_\mathbb{R}}(A_\mathbb{R})/Z_{M_\mathbb{R}}(A_\mathbb{R}).$$

(Here, if $G \supseteq H$ are groups, $N_G(H)$ and $Z_G(H)$ denote the normalizer and centralizer of H in G, respectively). The group W acts on $\mathfrak{a}_\mathbb{R}$ and can be identified with its image in $GL(\mathfrak{a}_\mathbb{R})$, which is generated by the reflections s_α. (See [27] and [10, Ch. VII] for proofs of these facts.)

The natural map $A/(A \cap M) \to K/M$ induces a map of coordinate rings $\mathbb{C}[K/M] \to \mathbb{C}[A/(A \cap M)]$. The induced map $\mathbb{C}[K/M]^M \to \mathbb{C}[A/(A \cap M)]^W$ is an isomorphism. This is a reformulation of a theorem of Richardson [27]. Indeed, let P be the subset of K consisting of elements of the form $k\tau(k)^{-1}$. There is a commutative diagram

$$
\begin{array}{ccc}
A/(A \cap M) & \longrightarrow & K/M \\
\downarrow & & \downarrow \\
A & \longrightarrow & P
\end{array}
$$

Here the horizontal maps are the inclusions. The vertical maps are isomorphisms; the left vertical map takes $a(A \cap M)$ to a^2, and the right vertical map takes kM to

$k\tau(k)^{-1}$. Note that M acts by left translation on K/M and by conjugation on P, and the right vertical map is M-equivariant. Richardson's result states that the induced map $\mathbb{C}[P]^M \to \mathbb{C}[A]^W$ is an isomorphism, from which the version stated above follows. Composing the isomorphisms $\mathbb{C}[K/M]^M \to \mathbb{C}[A/(A \cap M)]^W$ and $\mathbb{C}[A/(A \cap M)]^W \to \mathbb{C}[A]^W$ yields an isomorphism $\mathbb{C}[K/M]^M \to \mathbb{C}[A]^W$. For $\lambda \in 2X^*(A)^+$, let Q_λ denote the image of the spherical function φ_λ. Finally, we define the function P_λ in $\mathbb{C}[A]^W$ by $P_\lambda(a) = Q_\lambda(a^{-1})$ for $a \in A$. Thus, by definition, we have $P_\lambda(a^2) = \varphi_\lambda(a^{-1})$. Note that in light of the remark at the end of the previous subsection, $P_\lambda = Q_{\lambda^*}$ (up to scaling), where $\lambda^* \in 2X^*(A)^+$ is such that $V_{\lambda^*} \simeq (V_\lambda)^*$. Since the spherical functions φ_λ form a basis for $\mathbb{C}[K/M]^M$, the preceding discussion implies that the functions P_λ form a basis for $\mathbb{C}[A]^W$. Furthermore, the structure constants for the multiplication of the functions P_λ and the spherical functions φ_λ are the same, *i.e.*, $P_\lambda P_\mu = \sum a_{\lambda\mu}^\nu P_\nu$, where the $a_{\lambda\mu}^\nu$ are the same constants as at the end of the previous subsection.

5.5 A Characterization of the Functions P_λ

In the cases most of interest in this paper, the functions P_λ will turn out to be specializations of Jack polynomials. Before we show this in the next subsection, we give an alternative characterization of the functions P_λ. For $\lambda \in 2X^*(A)^+$, define $m_\lambda \in \mathbb{C}[A]^W$ by $m_\lambda = (1/|W_\lambda|) \sum_{w \in W} e^{w\lambda/2}$, where $|W_\lambda|$ is the order of the stabilizer of λ in W. Define a function δ on the compact torus $A_\mathbb{R}$ by $\delta(a) := \prod_{\alpha \in \Sigma^+} |1 - e^\alpha(a)|^{\text{mult}(\alpha)}$, where $\text{mult}(\alpha)$ is the multiplicity of the restricted root α. Finally, define an inner product on $\mathbb{C}[A]$ by the rule

$$\langle f, g \rangle_A = \int_{A_\mathbb{R}} f(a)\overline{g(a)}\delta(a)\, da.$$

The following result is known, but because a complete proof seems hard to find in the literature, we provide the argument. As a side remark, it is perhaps interesting to note that most of the ideas in the proof already appear in a paper of Harish-Chandra from 1958 [9].

Proposition 5.3 *The functions P_λ satisfy the following properties:*

(a) *There exist constants $c_{\lambda\mu}$ such that*

$$P_\lambda = \sum_{\mu/2 \leq \lambda/2} c_{\lambda\mu} m_\mu \quad with \quad c_{\lambda\lambda} \neq 0.$$

(b) *For all $\lambda \neq \mu$, $\langle P_\lambda, P_\mu \rangle_A = 0$.*

Proof We prove the equivalent of the proposition for the functions Q_λ. For $\lambda \in 2X^*(A)^+$, define $n_\lambda \in \mathbb{C}[A]^W$ by $n_\lambda = (1/|W_\lambda|) \sum_{w \in W} e^{-w\lambda/2}$. The fact that $Q_\lambda = \sum c_{\lambda\mu} n_\mu$ with $c_{\lambda\lambda} \neq 0$ is proved in [9, p. 274]. Harish-Chandra [9, p. 275] also gives a Freudenthal-type recursion formula for φ_λ which implies that $Q_\lambda = \sum c_{\lambda\mu} n_\mu$, where the sum is over $\mu \in 2X^*(A)^+$ such that $\mu/2 \in X^*(A)$ and $\mu/2 \leq \lambda/2$. (It is easy to see that the sum is over $\mu \leq \lambda$, but the ordering is defined using integral linear

combinations of positive roots, so the fact that we can divide this inequality by 2 does not seem obvious without using Harish-Chandra's formula. Arguments analogous to those in Humphreys [13, Lemma B, §13.3] show that the leading term of Harish-Chandra's formula is non-zero, so the formula determines φ_λ up to scaling.) This implies (a). As for (b), we have already observed that the Q_λ form a basis of $\mathbb{C}[A]^W$. If $\lambda \neq \mu$, then

$$\int_{K_\mathbb{R}} \varphi_\lambda(k)\overline{\varphi_\mu(k)}\,dk = 0,$$

by the usual orthogonality relation for matrix coefficients of different representations. The function $\varphi_\lambda\overline{\varphi_\mu}$ is bi-invariant under the group $M_\mathbb{R}$, so using the integration formula of Theorem 5.10 of [11, Ch. I §5], we see that the above integral is a non-zero constant times

$$\int_{A_\mathbb{R}/(A_\mathbb{R} \cap M_\mathbb{R})} \varphi_\lambda(a)\overline{\varphi_\mu(a)}\delta(a^2)da.$$

Via the isomorphism $A_\mathbb{R}/(A_\mathbb{R} \cap M_\mathbb{R}) \to A_\mathbb{R}$ that takes $a(A_\mathbb{R} \cap M_\mathbb{R})$ to a^2, this leads to the desired orthogonality. ∎

Henceforth, we will normalize the φ_λ so that the coefficient $c_{\lambda\lambda}$ equals 1.

5.6 Jack Polynomials

Much of the following discussion generalizes to arbitrary root systems. But for the application to our main results, we only need the case where the root system Σ is of type A_{r-1}, so from now on, to simplify the exposition, we will assume this. The torus A is isomorphic to a maximal torus $T_{SL_{r-1}}$ via an isomorphism respecting the roots and the character lattices. The reason is that our assumption that K is simply connected implies that the integrality conditions characterizing the character lattice $X^*(A)$ are the same as the integrality conditions characterizing the character lattice $X^*(T_{SL_{r-1}})$ (see [10, Ch. VII §8]). Therefore, we can write the positive roots as $\varepsilon_i - \varepsilon_j$, $1 \leq i < j \leq r$, where ε_i are elements of $X^*(A)$ such that $\sum_{i=1}^r \varepsilon_i = 0$. Let $\gamma_i := 2\varepsilon_i$. Then the positive roots are $\frac{1}{2}(\gamma_i - \gamma_j)$, $1 \leq i < j \leq r$, as in Subsection 3.3. The coordinate ring $\mathbb{C}[A]$ can be identified with the ring $\mathbb{C}[x_1^{\pm 1}, \ldots, x_r^{\pm 1}]/(\prod x_i - 1)$ in such a way that $e^{\varepsilon_i} = e^{\gamma_i/2}$ corresponds to x_i. In particular, we have a projection homomorphism $\pi\colon \mathbb{C}[x_1, \ldots, x_r] \to \mathbb{C}[A]$. If we let the symmetric group S_r act on $\mathbb{C}[x_1, \ldots, x_r]$ as usual, then π induces a homomorphism of rings of invariants $\pi\colon \mathbb{C}[x_1, \ldots, x_r]^{S_r} \to \mathbb{C}[A]^W$.

Let $\Lambda = \{(\lambda_1, \ldots, \lambda_r) \in \mathbb{Z}^r \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 0\}$. If $\lambda \in \Lambda$, then $\sum_{i=1}^r \lambda_i\gamma_i$ is an element of $2X^*(A)^+$; thus we have a natural map $\Lambda \to 2X^*(A)^+$, which is surjective. By abuse of notation, if $\lambda \in \Lambda$, we will denote its image in $2X^*(A)^+$ also by λ. With this convention, the image of the monomial $x^\lambda := x_1^{\lambda_1} \cdots x_r^{\lambda_r}$ under the mapping π is the element $e^{\lambda/2}$. It follows that for $\lambda \in \Lambda$, the function $m_\lambda \in \mathbb{C}[A]$ that was defined in Subsection 5.3 (just before the proposition) is the image of the monomial symmetric function corresponding to the partition λ.

Corresponding to any partition $\lambda \in \Lambda$ there is an element $P_\lambda^{(\alpha)} \in \mathbb{C}(\alpha)[x_1, \ldots, x_r]$, called a Jack polynomial (or Jack symmetric function). Here α is a parameter, not to be confused with a root (unfortunately it is customary to denote this parameter by α). These polynomials are defined in [26, Section 10]; here we recall the formulation of [20]. If α is such that $1/\alpha$ is a non-negative integer, then $\Delta^{1/\alpha}(x) := \prod_{i \neq j}(1 - x_i/x_j)^{1/\alpha}$ is an element of the Laurent polynomial ring $\mathbb{C}[x_1^{\pm 1}, \ldots, x_r^{\pm 1}]$. Define an inner product on $\mathbb{C}[x_1, \ldots, x_r]$ by the rule $\langle f, g \rangle_\alpha = [f(x)g(x^{-1})\Delta^{1/\alpha}(x)]_0$ where the subscript 0 denotes taking the constant term of a Laurent polynomial. This inner product can be defined in an alternative way using integration; the alternative definition makes sense for all nonnegative real α; see [26]. Then the Jack polynomials are characterized by the fact that the coefficient in $P_\lambda^{(\alpha)}$ of the monomial symmetric function corresponding to λ is 1, and by the fact that if $\lambda \neq \mu$, then $\langle P_\lambda^{(\alpha)}, P_\mu^{(\alpha)} \rangle_\alpha = 0$ for all $\lambda, \mu \in \Lambda$ and α with $1/\alpha \in \mathbb{N}$ (see [20]). The functions $P_\lambda^{(1)}$ are the Schur functions s_λ. A version of the following proposition appears in [28].

Proposition 5.4 *Let $\lambda \in \Lambda$ and identify λ with its image in $2X^*(A)^+$. Then under the map $\pi \colon \mathbb{C}[x_1, \ldots, x_r]^{S_r} \to \mathbb{C}[A]^W$, the Jack polynomial $P_\lambda^{(2/m)}$ is mapped to the function P_λ.*

Proof Let f and g be homogeneous polynomials in $\mathbb{C}[x_1, \ldots, x_r]$. If f and g have the same degree, then $\langle f, g \rangle_{2/m} = \langle \pi(f), \pi(g) \rangle_A$; if f and g have different degrees, then $\langle f, g \rangle_{2/m} = 0$. It follows that the images of the Jack polynomials $P_\lambda^{(2/m)}$ satisfy the properties of Proposition 5.3. Since these properties characterize the P_λ, the result follows.   ∎

5.7 Stanley's Conjecture on the Multiplication of Jack Polynomials

One can expand the product of Jack polynomials as a sum of Jack polynomials

$$P_\lambda^{(\alpha)} P_\mu^{(\alpha)} = \sum f_{\lambda\mu}^\nu(\alpha) P_\nu^{(\alpha)},$$

where the $f_{\lambda\mu}^\nu(\alpha)$ are rational functions of α. If $\alpha = 1$, then the $f_{\lambda\mu}^\nu(\alpha) = c_{\lambda\mu}^\nu$ are Littlewood–Richardson coefficients for the group $GL_r(\mathbb{C})$; if $\alpha = 2/m$, then the $f_{\lambda\mu}^\nu(\alpha) = a_{\lambda\mu}^\nu$ are the structure coefficients for the multiplication of the spherical functions. Stanley conjectured that the $f_{\lambda\mu}^\nu(\alpha)$ have a certain positivity property, which we can formulate as follows. (Stanley's formulation used another inner product, which we do not want to define here, but the equivalence of the formulations follows from [26, Ch. VI, Section 10].) For a partition λ, let $c_\lambda(\alpha)$ and $c_\lambda'(\alpha)$ be the polynomials in α with non-negative integer coefficients as defined in [26, Ch. VI, Section 10, (10.21)]. It is known that $c_\lambda(\alpha)c_\mu(\alpha)c_\nu'(\alpha)f_{\lambda\mu}^\nu(\alpha)$ is a polynomial with integer coefficients. This follows from another conjecture of Stanley and Macdonald (see [26, Ch. VI, Section 10, (10.26?) and (10.33)]), which was proved by Knop and Sahi in [20], and also (in part) by Lapointe and Vinet [23].

Conjecture (Stanley [32, Conjecture 8.3]) *For fixed partitions $\lambda, \mu, \nu \in \Lambda$, the polynomial $c_\lambda(\alpha)c_\mu(\alpha)c_\nu'(\alpha)f_{\lambda\mu}^\nu(\alpha)$ has non-negative integer coefficients.*

We will only need the following consequence of Stanley's conjecture.

Corollary 5.5 *Assume that Stanley's conjecture holds. Then for fixed partitions $\lambda, \mu, \nu \in \Lambda$, the following are equivalent:*

(i) $f^\nu_{\lambda\mu}(\alpha)$ *is non-zero for all positive real values of α.*
(ii) *The Littlewood–Richardson coefficient $c^\nu_{\lambda\mu}$ is non-zero.*
(iii) $a^\nu_{\lambda\mu}$ *is non-zero.*

Proof Clearly, (i) implies (ii) and (iii). Conversely, suppose (ii) holds. We have $f^\nu_{\lambda\mu}(1) = c^\nu_{\lambda\mu}$. If this is non-zero, then some coefficient of the polynomial $c_\lambda(\alpha)c_\mu(\alpha)c'_\nu(\alpha)f^\nu_{\lambda\mu}(\alpha)$ must be non-zero. Stanley's conjecture implies that this polynomial is non-zero for all positive real values of α, proving (i). The proof that (iii) implies (i) is similar. ∎

5.8   Main Theorem

We now return to the notation of Section 3. By Proposition 4.1, Conjecture B is true if and only if it is true for pairs of tube type, so we assume the pair $(\mathfrak{g}, \mathfrak{k})$ is of tube type. Recall that Λ denotes the set of partitions of length at most r, identified with a subset of $\widehat{K}$ by sending λ to V_λ, the irreducible K-representation of lowest weight $-\sum \lambda_i \gamma_i$. We can decompose $S = \mathbb{C}[\mathfrak{p}^+]$ into K-isotypic components: $S \simeq \bigoplus_{\lambda \in \Lambda} S_\lambda$, where S_λ is isomorphic to V_λ. Note that in this case K is not semisimple; it has a one-dimensional center. However, we can replace K by K', the simply connected cover of the derived group $[K, K]$, and M by the fixed point set $M' \subseteq K'$ of the corresponding involution. We have a surjective group homomorphism $K' \times \mathbb{C}^\times \to K$ and hence we may view representations of K as representations of $K' \times \mathbb{C}^\times$. Thus, we can view the above decomposition as a $K' \times \mathbb{C}^\times$-module decomposition of S.

Since M has finite intersection with the center of K, the Lie algebras of M and M' coincide. Decomposing into eigenspaces of the involution τ gives $\mathfrak{k}' = \mathfrak{m} \oplus \mathfrak{s}'$ and $\mathfrak{k} = \mathfrak{m} \oplus \mathfrak{s}$, and $\mathfrak{s} = \mathfrak{s}' \oplus \mathfrak{z}$ where $\mathfrak{z}$ is the center of $\mathfrak{k}$. In fact, we have $\mathfrak{k}'_\mathbb{R} = \mathfrak{m}_\mathbb{R} \oplus \mathfrak{s}'_\mathbb{R}$, and we can take our maximal abelian subspace of $\mathfrak{s}'_\mathbb{R}$ to be $\mathfrak{a}'_\mathbb{R} := \mathfrak{h}^-_\mathbb{R} \cap \mathfrak{s}'_\mathbb{R}$. We obtain a corresponding algebraic torus A' of K'. By Moore's result [25] the restricted roots of $(\mathfrak{k}', \mathfrak{m})$ are of the form $\frac{1}{2}(\gamma_i - \gamma_j)$. The integrality conditions of [10, Ch. VII §8] imply that $\varepsilon_j := \frac{1}{2}\gamma_j \in X^*(A')$. Therefore, the analysis of Subsections 5.6 and 5.7 applies to $\mathbb{C}[K'/M']$.

Theorem 5.6 *Suppose that Stanley's conjecture is true. Then for $\lambda, \mu, \nu \in \Lambda$, $S_\nu \subseteq S_\lambda \cdot S_\mu$ if and only if the Littlewood–Richardson coefficient $c^\nu_{\lambda\mu}$ is non-zero.*

Proof By Proposition 4.1, we may assume $(\mathfrak{g}, \mathfrak{k})$ is of tube type. In this case, recall from 3.3 that we have an inclusion $S = \bigoplus_{\lambda \in \Lambda} S_\lambda \hookrightarrow \bigoplus_{\lambda \in \widehat{K}} R_\lambda$, where $R = \mathbb{C}[K/M]$ is a multiplicity-free K-representation. Replacing R by $R' = \mathbb{C}[K'/M' \times \mathbb{C}^\times] = \mathbb{C}[K'/M'] \otimes \mathbb{C}[t, t^{-1}]$, we can view S as a subring of R'; in particular, if $\lambda \in \Lambda$, then $S_\lambda = R'_\lambda$. As above, for each $\lambda \in \widehat{K}$ we have an M-invariant element φ_λ of R'_λ. If λ and μ are in Λ, then, since S is a subring of R', we have $\varphi_\lambda \varphi_\mu = \sum a^\nu_{\lambda\mu} \varphi_\nu$. By Ruitenburg's result, we have $S_\nu \subseteq S_\lambda \cdot S_\mu$ if and only if $a^\nu_{\lambda\mu} \neq 0$. By our discussion in the previous subsection, $a^\nu_{\lambda\mu} = f^\nu_{\lambda\mu}(2/m)$. Therefore, assuming Stanley's conjecture, $a^\nu_{\lambda\mu} \neq 0$ if and only if $c^\nu_{\lambda\mu} \neq 0$. The theorem follows. ∎

6 Multiplicities and Littlewood–Richardson Coefficients

6.1 Equivalence of Conjectures A and B

Let $(\mathfrak{g}, \mathfrak{k})$ be an irreducible Hermitian symmetric pair of rank r and let $\{\gamma_1, \ldots, \gamma_r\}$ be the set of strongly orthogonal roots in $\mathfrak{p}^+$ as defined in Section 3. Let

$$\Lambda = \{(\lambda_1, \ldots, \lambda_r) \in \mathbb{Z}^r | \lambda_1 \geq \cdots \geq \lambda_r \geq 0\}.$$

In this section, if $\lambda \in \Lambda$, let $\tilde{\lambda}$ denote the Δ_c^+-dominant integral weight given by $\tilde{\lambda} = \lambda_1 \gamma_1 + \cdots + \lambda_r \gamma_r$. Then define

$$V_\lambda = \text{irreducible } \mathfrak{k}\text{-module with lowest weight } -\tilde{\lambda},$$

$$F_\lambda^{(r)} = \text{irreducible } \mathfrak{gl}_r\text{-module with lowest weight } -\lambda.$$

For $\lambda, \mu, \nu \in \Lambda$, let $[V_\lambda \otimes V_\mu, V_\nu]$ denote the multiplicity of V_ν in $V_\lambda \otimes V_\mu$.

Proposition 6.1   *Let $(\mathfrak{g}, \mathfrak{k})$ be an irreducible Hermitian symmetric pair of classical type of rank r and let $\lambda, \mu, \nu \in \Lambda$ such that $|\nu| = |\lambda| + |\mu|$. Then $[V_\lambda \otimes V_\mu, V_\nu] \neq 0$ if and only if $c_{\lambda\mu}^\nu \neq 0$.*

As an immediate consequence we obtain the following:

Corollary 6.2   *Let $(\mathfrak{g}, \mathfrak{k})$ be an irreducible Hermitian symmetric pair of classical type of rank r. Then Conjecture A is true for $(\mathfrak{g}, \mathfrak{k})$ if and only if Conjecture B is true for $(\mathfrak{g}, \mathfrak{k})$.*

6.2 Littlewood–Richardson Coefficients

Before we prove Proposition 6.1 we recall some known facts about Littlewood–Richardson coefficients that are needed in the proof. We first need a little bit more notation about partitions. We use the same notation as in Macdonald's book [26]. A partition is a sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r, \ldots)$ of non-negative integers such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq \cdots$ and $\lambda_i = 0$ for $i \gg 1$. The integer $\ell(\lambda) = \min\{i | \lambda_i \neq 0\}$ is the length of λ and $|\lambda| = \sum_i \lambda_i$ is the size of λ. The partition conjugate to λ is the partition λ' given by $(\lambda')_i = \#\{\lambda_j \mid \lambda_j \geq i\}$. Let λ, μ, ν be partitions of length at most r. By the Littlewood–Richardson rule it follows that for any $n \geq r$,

$$[F_\lambda^{(n)} \otimes F_\mu^{(n)}, F_\nu^{(n)}] = [F_\lambda^{(r)} \otimes F_\mu^{(r)}, F_\nu^{(r)}] = c_{\lambda\mu}^\nu.$$

Another interpretation of Littlewood–Richardson coefficients is in terms of Schur functions. For partition λ and μ, we have $s_\lambda s_\mu = \sum_\nu c_{\lambda\mu}^\nu s_\nu$. A simple, but very useful necessary condition for $c_{\lambda\mu}^\nu \neq 0$ is that $|\lambda| + |\mu| = |\nu|$. We will also need that $c_{\lambda\mu}^\nu = c_{\lambda'\mu'}^{\nu'}$. This follows since there is an involution ω on the ring of symmetric functions such that $\omega(s_\lambda) = s_{\lambda'}$. Finally, we will need that

$$c_{\lambda\mu}^\nu \neq 0 \quad \text{if and only if} \quad c_{2\lambda,2\mu}^{2\nu} \neq 0.$$

This is a special case of Klyachko's saturation conjecture [17], which is now a theorem of Knutson and Tao [21] and also of Derksen and Weyman [3].

6.3 Proof of Proposition 6.1

We will prove the proposition case by case.

Case (i) $\mathfrak{g}_\mathbb{R} = \mathfrak{su}(p,q)$. It is slightly more convenient to work with $\mathfrak{g}_\mathbb{R} = \mathfrak{u}(p,q)$ instead of $\mathfrak{su}(p,q)$. In this case, $(\mathfrak{g},\mathfrak{k}) = (\mathfrak{gl}_n, \mathfrak{gl}_p \times \mathfrak{gl}_q)$ and $r = p$. Here we assume that $p \leq q$. From Table 2, if $\lambda = (\lambda_1, \ldots, \lambda_p)$, then

$$\tilde{\lambda} = (\lambda_1, \ldots, \lambda_p, 0, \ldots, 0, -\lambda_p, \ldots, -\lambda_1).$$

Hence $V_\lambda \simeq F_\lambda^{(p)} \boxtimes \left(F_\lambda^{(q)}\right)^*$ and it follows that $[V_\lambda \otimes V_\mu, V_\nu] = (c_{\lambda\mu}^\nu)^2$. The equivalence is now obvious.

Case (ii) $\mathfrak{g}_\mathbb{R} = \mathfrak{sp}(n,\mathbb{R})$. In this case, $(\mathfrak{g},\mathfrak{k}) = (\mathfrak{sp}_{2n}, \mathfrak{gl}_n)$ and $r = n$. From Table 2, if $\lambda = (\lambda_1, \ldots, \lambda_n)$ then $\tilde{\lambda} = (2\lambda_1, \ldots, 2\lambda_n) = 2\lambda$. Hence $V_\lambda \simeq F_{2\lambda}^{(n)}$ and it follows that $[V_\lambda \otimes V_\mu, V_\nu] = c_{2\lambda, 2\mu}^{2\nu}$. The equivalence is now an immediate consequence of the saturation conjecture.

Case (iii) $\mathfrak{g}_\mathbb{R} = \mathfrak{so}^*(2n)$. In this case, $(\mathfrak{g},\mathfrak{k}) = (\mathfrak{so}_{2n}, \mathfrak{gl}_n)$ and $r = \lfloor n/2 \rfloor$. From Table 2, if $\lambda = (\lambda_1, \ldots, \lambda_r)$ then $\tilde{\lambda} = (\lambda_1, \lambda_1, \lambda_2, \lambda_2, \ldots) = (2\lambda')'$. Hence $V_\lambda \simeq F_{(2\lambda')'}^{(n)}$, and it follows that $[V_\lambda \otimes V_\mu, V_\nu] = c_{(2\lambda')',(2\mu')'}^{(2\nu')'} = c_{2\lambda',2\mu'}^{2\nu'}$. The equivalence is now a consequence of the saturation conjecture and the fact that $c_{\lambda'\mu'}^{\nu'} = c_{\lambda\mu}^\nu$.

Case (iv) $\mathfrak{g}_\mathbb{R} = \mathfrak{so}(n,2)$. In this case, $(\mathfrak{g},\mathfrak{k}) = (\mathfrak{so}_{n+2}, \mathfrak{so}_n \oplus \mathbb{C})$ and $r = 2$. We may assume that $n \geq 4$ since $\mathfrak{so}(3,2) \simeq \mathfrak{sp}(2,\mathbb{R})$. Let $\{\varepsilon_1, \ldots, \varepsilon_{\lfloor n/2 \rfloor}\}$ be the canonical basis for the standard Cartan subalgebra of $\mathfrak{so}_n$. Extend this basis to the standard basis (with non-standard labeling) $\{\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{\lfloor n/2 \rfloor}\}$ of $\mathfrak{so}_{n+2}$. With this convention, $\gamma_1 = \varepsilon_0 + \varepsilon_1$ and $\gamma_2 = \varepsilon_0 - \varepsilon_1$. So, if $\lambda = (\lambda_1, \lambda_2)$ then $\tilde{\lambda} = (\lambda_1 + \lambda_2)\varepsilon_0 + (\lambda_1 - \lambda_2)\varepsilon_1$. Thus, as a $\mathfrak{k} = \mathfrak{so}_n \oplus \mathbb{C}$-module, $V_\lambda \simeq E_{(\lambda_1 - \lambda_2)\varepsilon_1}^{(n)} \boxtimes \mathbb{C}_{-\lambda_1 - \lambda_2}$, where $E_{(\lambda_1 - \lambda_2)\varepsilon_1}^{(n)}$ is the irreducible $\mathfrak{so}_n$-module with lowest weight $-(\lambda_1 - \lambda_2)\varepsilon_1$. In the following we will show that $[V_\lambda \otimes V_\mu, V_\nu] = c_{\lambda\mu}^\nu$.

Lemma 6.3 *Suppose that $n \geq 4$. If a and b are non-negative integers then*

$$E_{a\varepsilon_1}^{(n)} \otimes E_{b\varepsilon_1}^{(n)} \simeq \bigoplus_{k=0}^{b} \bigoplus_{l=0}^{b-k} E_{(a+b-2k-l)\varepsilon_1 + l\varepsilon_2}^{(n)}.$$

Proof The assumption $n \geq 4$ makes sure that we are in the "stable range". We can then use the branching rule [12, Theorem 2.1.2] to decompose the tensor product $E_{a\varepsilon_1}^{(n)} \otimes E_{b\varepsilon_1}^{(n)}$. The lemma then follows by the usual Pieri rule for Littlewood–Richardson coefficients. ∎

We now return to the proof of Proposition 6.1. Let $\lambda, \mu, \nu \in \Lambda$ such that $|\lambda| + |\mu| = |\nu|$. We note that $F_\lambda^{(2)} \simeq F_{(\lambda_1 - \lambda_2, 0)}^{(2)} \otimes F_{(\lambda_2, \lambda_2)}^{(2)}$ and $F_{(\lambda_2, \lambda_2)}^{(2)}$ is one-dimensional, which implies that we can calculate $c_{\lambda\mu}^\nu$ by using the Pieri rule. If we put $a = \lambda_1 - \lambda_2$, $b = \mu_1 - \mu_2$, and $c = \nu_1 - \nu_2$, then

$$c_{\lambda\mu}^\nu = \begin{cases} 1 & \text{if } c = a + b - 2k \text{ for some non-negative integer } k, \\ 0 & \text{otherwise.} \end{cases}$$

By Lemma 6.3, $[E_{a\varepsilon_1}^{(n)} \otimes E_{b\varepsilon_1}^{(n)}, E_{c\varepsilon_1}^{(n)}] = c_{\lambda\mu}^\nu$ and hence $[V_\lambda \otimes V_\mu, V_\nu] = c_{\lambda\mu}^\nu$. This completes the proof of Proposition 6.1. ∎

6.4   A Conjecture for the Exceptional Groups

We conjecture that Proposition 6.1 also holds in the two exceptional cases.

Case (i) $\mathfrak{g}_\mathbb{R} = E\,\mathrm{III}$. In this case, $(\mathfrak{g}, \mathfrak{k}) = (\mathfrak{e}_6, \mathfrak{so}_{10} \oplus \mathbb{C})$ and $r = 2$. Let $\omega_1, \ldots, \omega_6$ be the fundamental weights of $\mathfrak{e}_6$ (Bourbaki ordering). By Table 2, $\gamma_1 = \omega_2$, $\gamma_2 = \omega_1 - \omega_2 + \omega_6$. Thus if $\lambda = (\lambda_1, \lambda_2)$, then $\tilde\lambda = \lambda_2\omega_1 + (\lambda_1 - \lambda_2)\omega_2 + \lambda_2\omega_6$. Let $\varpi_1, \ldots, \varpi_5$ be the fundamental weights of $\mathfrak{so}_{10}$ (Bourbaki ordering). If $\lambda = (\lambda_1, \lambda_2) \in \Lambda$, then as a $\mathfrak{k} = \mathfrak{so}_{10} \oplus \mathbb{C}$-module, $V_\lambda \simeq E_{(\lambda_1-\lambda_2)\varpi_1+\lambda_2\varpi_5} \boxtimes \mathbb{C}_{-\lambda_1-\lambda_2}$, where $E_{(\lambda_1-\lambda_2)\varpi_1+\lambda_2\varpi_5}$ denotes the irreducible $\mathfrak{so}_{10}$-module of lowest weight $-(\lambda_1 - \lambda_2)\varpi_1 - \lambda_2\varpi_5$. We conjecture that if $\lambda, \mu, \nu \in \Lambda$ with $|\lambda| + |\mu| = |\nu|$, then $[V_\lambda \otimes V_\mu, V_\nu] = c_{\lambda\mu}^\nu$.

Case (ii) $\mathfrak{g}_\mathbb{R} = E\,\mathrm{VII}$. In this case, $(\mathfrak{g}, \mathfrak{k}) = (\mathfrak{e}_7, \mathfrak{e}_6 \oplus \mathbb{C})$ and $r = 3$. Let $\omega_1, \ldots, \omega_7$ be the fundamental weights of $\mathfrak{e}_7$ (Bourbaki ordering). By Table 2, $\gamma_1 = \omega_1$, $\gamma_2 = -\omega_1 + \omega_6$ and $\gamma_3 = -\omega_6 + 2\omega_7$. If $\lambda = (\lambda_1, \lambda_2, \lambda_3)$, then $\tilde\lambda = (\lambda_1 - \lambda_2)\omega_1 + (\lambda_2 - \lambda_3)\omega_6 + 2\lambda_3\omega_7$. Let $\varpi_1, \ldots, \varpi_6$ be the fundamental weights of $\mathfrak{e}_6$ (Bourbaki ordering). If $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \Lambda$, then as a $\mathfrak{k} = \mathfrak{e}_6 \oplus \mathbb{C}$-module, $V_\lambda \simeq W_{(\lambda_1-\lambda_2)\varpi_1+(\lambda_2-\lambda_3)\varpi_6} \boxtimes \mathbb{C}_{-\lambda_1-\lambda_2-\lambda_3}$, where $W_{(\lambda_1-\lambda_2)\varpi_1+(\lambda_2-\lambda_3)\varpi_6}$ denotes the irreducible $\mathfrak{e}_6$-module with lowest weight $-(\lambda_1 - \lambda_2)\varpi_1 - (\lambda_2 - \lambda_3)\varpi_6$. We conjecture that if $\lambda, \mu, \nu \in \Lambda$ with $|\lambda| + |\mu| = |\nu|$, then

$$[V_\lambda \otimes V_\mu, V_\nu] = \frac{c_{\lambda\mu}^\nu(c_{\lambda\mu}^\nu + 1)}{2}.$$

We arrived at this conjecture by computing examples using the computer algebra package LiE [2].

K	$X = \mathfrak{p}^+$	$[V_\lambda \otimes V_\mu, V_\nu]$
$GL_p(\mathbb{C}) \times GL_q(\mathbb{C})$	$\mathbb{C}^p \otimes (\mathbb{C}^q)^*$	$(c_{\lambda\mu}^\nu)^2$
$GL_n(\mathbb{C})$	$S^2(\mathbb{C}^n)$	$c_{2\lambda,2\mu}^{2\nu}$
$GL_n(\mathbb{C})$	$\wedge^2(\mathbb{C}^n)$	$c_{(2\lambda')', (2\mu')'}^{(2\nu')'} = c_{2\lambda,2\mu}^{2\nu}$
$SO_n(\mathbb{C}) \times \mathbb{C}^\times$	$\mathbb{C}^n$	$c_{\lambda\mu}^\nu$
$\mathrm{Spin}_{10}(\mathbb{C}) \times \mathbb{C}^\times$	$\mathbb{C}^{16}$ (spin)	$c_{\lambda\mu}^\nu$ (?)
$E_6(\mathbb{C}) \times \mathbb{C}^\times$	$\mathbb{C}^{27}$ (min)	$\frac{c_{\lambda\mu}^\nu(c_{\lambda\mu}^\nu+1)}{2}$ (?)

Table 4: Multiplicities $[V_\lambda \otimes V_\mu, V_\nu]$ in terms of Littlewood–Richardson coefficients.

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