PART V

GRAVITATIONAL PROBLEM OF THREE OR MORE BODIES

https://doi.org/10.1017/S0252921100062370 Published online by Cambridge University Press

FAMILIES OF PERIODIC PLANETARY-TYPE ORBITS IN THE N-BODY PROBLEM AND THEIR APPLICATION TO THE SOLAR SYSTEM

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ABSTRACT. A new approach to the study of the Solar System and planetary systems in general is proposed, through the use of periodic planetary-type orbits of the general N-body problem. In such an orbit, one body (called Sun) has a large mass and the rest N-1 bodies (called planets) have small but not negligible masses and it can be proved that monoparametric families of periodic orbits of the N-body problem exist in a rotating frame of reference , all being of the planetary type

Two cases are studied in detail, N=3 and N=4. In N=3, apart from a general discussion, we present a detailed analysis of the Sun-Jupiter-Saturn system and a study is made on which configurations with the masses of these two planets, or a multiple of them, are stable or unstable. Also, part of a family is shown to represent the Jupiter family of comets. It was found that commensurabilities are not in general associated with instabilities. For N=4 we present three families of periodic orbits. The motion corresponding to a branch of one of the above families has many similarities with the actual motion of the three inner satellites of Jupiter.

It is shown that there exist many commensurable cases in the obtained periodic orbits and that the resonant orbits increase as the number of bodies increases. Based on these results, an attempt is made to explain the existence of commensurabilities in the Solar System.

Finally, it is mentioned that a periodic motion of the planetary type can be used as a reference orbit for accurate computations for the actual motions of the planets or satellites of the Solar System. In this way the small divisor difficulties existing in the classical approach will not appear.

1. INTRODUCTION

The purpose of this paper is to present a new approach to the study of the solar system and of planetary systems in general, based on the study

V. Szebehely (ed.), Dynamics of Planets and Satellites and Theories of Their Motion, 263-281.

https://doi.org/10.1017/S0252921100062370 Published online by Cambridge University Press

of families of periodic orbits of the general N-body problem (N>3). It can be proved (Hadjidemetriou 1975a, 1976b,c) that families of periodic orbits exist in the general planar N-body problem, in a rotating frame or reference, for fixed values of the masses of all the bodies. In particular, we shall consider here the case where only one body, say P_2, has a large mass and the rest N-1 bodies $P_1, P_3, \ldots, P_N$ have small but not negligible masses. Thus, this system represents a planetary system with the body P_2 being the Sun and the bodies $P_1, P_3, \ldots, P_N$ the planets (or comets) or the body P_2 being a planet and the bodies $P_1, P_3, \ldots, P_N$ being its satellites.

The rotating frame mentioned above is defined as follows: The origin O coincides with the center of mass of the bodies P_1 and P_2 and the x axis contains always these bodies, the positive direction being from P_2 to P_1. This system is rotating with a non constant angular velocity and it can be proved (Hadjidemetriou 1975a, 1976b,c) that the motion of the N-body system in this rotating frame can be studied independently of the motion of the rotating system with respect to an inertial frame. This separation is possible because of the existence of the angular momentum integral which introduces the angular position of the rotating frame as an ignorable coordinate. Thus, a qualitative study of the motion can be made in the rotating frame only and this simplifies the analysis. The equations of motion in the rotating frame are given in the above mentioned papers. In what follows we shall restrict ourselves, to the study of the motion in the rotating frame only.

In this approach to the study of a planetary system no approximation is made and the gravitational effect of each planet on the other is completely taken into account. Also, this method applies equally well to planetary orbits with small and with large eccentricities and thus one can study planetary and cometary orbits by using the same method. And indeed, as we shall see in the following, there is a continuous transition from circular (planetary) orbits to elliptic (cometary) orbits. Also, any existing comensurabilities do not present any problem at all and the same method is used for all cases. Another advantage is that the smallness of the masses of the planets (or comets, or satellites) is not required and the same method can be applied to planetary systems with large masses of the planets. Thus, one can study the evolution of a planetary system, particularly its stability, by increasing the masses of the planets.

The purpose of this paper is to present an overall qualitative view of planetary systems, as obtained through the study of periodic orbits of the general N-body problem, based on the work made so far at the University of Thessaloniki. For this reason we shall not present detailed numerical computations. In some particular cases however which have a special interest, as the motion of the Sun-Jupiter-Saturn system or the three inner satellites of Jupiter, we present the exact numerical data. The applications are made for three and four bodies, but the same method can be applied for any number of bodies.

2. PLANETARY-TYPE ORBITS IN THE GENERAL THREE-BODY PROBLEM

(a). A family for zero masses of the two planets

In order to obtain approximate initial conditions for a periodic orbit we assume that the mass m_2 of the body P_2 (which we shall call "Sun") is equal to 1 and the masses of P_1 and P_3 (which we shall call planets) are equal to zero and that they describe circular orbits around P_2 in the same direction, in the plane. We can assume, without loss of generality, that the distance R_1 between P_2 and P_1 is equal to unity. Then, for any value of the distance $R_3>1$ between P_2 and P_3 we have a periodic motion in the rotating frame x0y, defined in the previous section. The ratio of the periods of the orbits of P_1 and P_3 around P_2 is equal to

$$T_1/T_3 = R_3^{-3/2} \tag{1}$$

and the period of the periodic motion in the rotating frame x0y is equal to

$$T = 2\pi/(1-T_1/T_3). \tag{2}$$

We note that T varies between $T=2\pi$(for $R=\infty$) to $T=\infty$ (for $R_3=1$). Thus, we have a degenerate monoparametric family of symmetric periodic orbits of three bodies, with respect to the x axis, where the two bodies P_1 and P_3 have zero masses. One can use the relative period T, given by (2), as a parameter along the family. We shall always take the two planets and the Sun to lie on the same straight line at $t=0$.

The initial conditions corresponding to the above family can be used as approximate initial conditions to obtain a symmetric periodic orbit in the rotating frame when the masses of P_1 and P_3 are increased. This continuation is possible, as shown by Hadjidemetriou (1976a), for all members of the degenerate family except those corresponding to a resonance of the form

$$T_1/T_3 = n/(n+1), \tag{3}$$

because in that case the period T is a multiple of 2π. This has as a consequence the generation of an infinite number of families of periodic orbits, all corresponding to the same (nonzero) masses of the three bodies. In this continuation the ratio m_1/m_3 may have any prespecified value.

This method has been applied by Hadjidemetriou (1976a) to obtain families of periodic planetary-type orbits for the case $m_1=m_3=0.001$ and $m_2=0.998$. This work was extended by Delibaltas (1976) who obtained families of periodic orbits for the case where the three bodies have masses equal to the mass of Sun, Jupiter and Saturn, respectively. Also, the evolution of these families is studied, when the masses of the planets are increased.

To summarize the results obtained so far, we note at this point that a symmetric periodic orbit can be specified by its initial conditions x_{10}, x_{30}, $\dot{y}_{30}$ in the rotating frame x0y, since $y_{30}=\dot{x}_{10}=\dot{x}_{30}=0$ provided a certain normalization scheme is used. We have used in our calculations the normalization

$$G=1, \quad m_1+m_2+m_3 = 1, \quad \dot{\vartheta}_o = 1, \tag{4}$$

where $\dot{\vartheta}_o$ is the initial value of the angular velocity of x0y, (Hadjidemetriou and Christides, 1975). Thus, we see that a family of periodic orbits, for fixed masses of all the bodies, can be represented by a continuous curve in the space $x_{10}x_{30}\dot{y}_{30}$. In this paper, in order to present the results in the simplest possible way, we shall use the projection of the above curve in the $x_{10}x_{30}$ plane only. Evidently, the above mentioned degenerate family of periodic orbits is represented in this plane by the straight line $x_{10}=1$.

We also note that we can take, without loss of generality, $R_3>1$, which implies that $|x_{30}|>1$, i.e. the orbit of P_3 is outside the orbit of P_1. If we had taken $0<R_3<1$ we would obtain the same family but with the roles of P_1 and P_3 interchanged, i.e. P_1 is the outer planet and P_3 the inner planet.

(b). Families for nonzero masses of the planets

In Fig. 1 the straight line $x_{10}=1$ represents the degenerate family for zero masses of the two planets and the points A_1, A_2, A_3,..., are the resonant orbits 1/2, 2/3, 3/4,..., respectively. These points have an

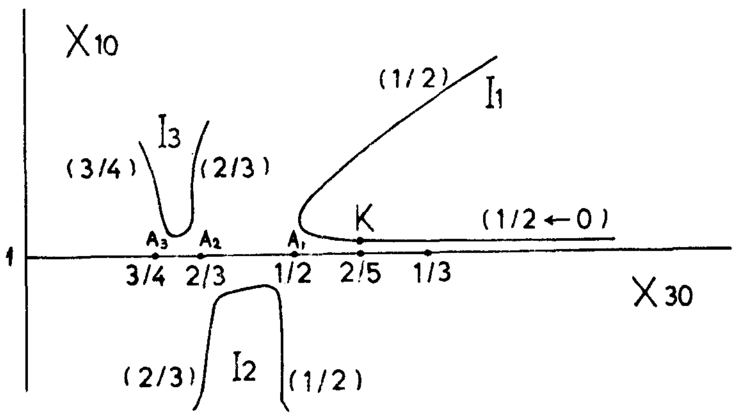

Fig. 1: The degenerate family $x_{10}=1$ and the families I_1, I_2, I_3,... generated from it by increasing the masses of the planets (schematically). The points A_1, A_2, A_3 represent resonant orbits of the form 1/2, 2/3, 3/4, respectively, of the degenerate family. The point K, at the resonance 2/5, represents the Sun-Jupiter-Saturn system. All these families correspond to a fixed value of m_2 and m_1/m_3. The picture is qualitatively the same for all values of m_1/m_3.

https://doi.org/10.1017/S0252921100062370 Published online by Cambridge University Press

accumulation point at $x_{30}=1$. When this degenerate family is extended by increasing the masses of the two planets, it breaks down to an infinity of families, all corresponding to the same fixed masses for the three bodies. Each family lies approximately between two consequtive resonant orbits of the form $n/(n+1)$ and $(n+1)/(n+2)$, respectively $(n=1,2,3,...)$.

The continuation of the degenerate family, as given in Fig.1, is qualitatively the same for all ratios of the masses of the two planets. The part of the families I_1, I_2, I_3,... which is nearly parallel to the line $x_{10}=1$ corresponds to almost circular orbits of the two planets around the Sun and the rest part corresponds to an elliptic orbit of at least one planet. An interesting result is that all along the branch of a family which is not parallel to the line $x_{10}=1$, there is an almost constant resonance T_1/T_3 of the osculating periods of the two planets. These constant values of the resonance are shown in Fig.1, in the parentheses near each branch. The association of this resonance with the resonance of the degenerate orbits A_1, A_2... is evident.

The actual Sun-Jupiter-Saturn system corresponds to the resonant orbit K of Figure 1 for the resonance 2/5. The initial conditions of a periodic orbit closely representing the Sun-Jupiter-Saturn system are given (Hadjidemetriou 1976a) by

$$x_{10}=0.99915744, \quad x_{30}=-1.84094099, \quad \dot{y}_{30}=1.10378309, \qquad (5)$$

for the masses

$$m_1=0.0009508, \quad m_3=0.9987640, \quad m_3=0.0002852, \qquad (6)$$

according to the normalization (4). The ratio T_1/T_3 at $t=0$ is equal to 0.406 and the osculating elements of the orbits of the two planets vary during one period as follows:

$$1.000550 \leq a_1 \leq 1.000950, \qquad 1.8367 \leq a_3 \leq 1.8492$$

$$0.000611 \leq e_1 \leq 0.000677, \qquad 0.00075 \leq e_3 \leq 0.00368 \; .$$

We note that the eccentricities are smaller than in the actual case. Perhaps the addition of more bodies in the system has as a consequence an increase in the eccentricities. This seems to be confirmed by numerical results in the 4-body problem (the resonant branches of families B,C in Fig.5, for the resonance 2/5 are found to have eccentricities of the order of 10^{-2}).

In the families of the type I_1 the upper branch (not parallel to the line $x_{10}=1$) corresponds to an almost circular orbit of the outer planet, P_3, and an elliptic orbit of the inner planet, P_1, (more appropriately called now a comet). The eccentricity increases as we proceed to larger values of x_{30} along this branch of the family. This picture is qualitatively the same for equal masses of the two planets and also for the actual masses (6) of Jupiter and Saturn and it seems that it is the same for any value of m_1/m_3. However, we have important differences as far as stability is concerned, as we shall describe below.

As far as the families I_2 and I_3 are concerned, we note that these are mostly almost resonant families, corresponding to the resonances 1/2 and 2/3 for I_2, and 2/3 and 3/4 for I_3. These resonances appear in the parts of the families not parallel to the line $x_{10}=1$. The transition from one resonance to the other in each family is along the lower part of the family in Fig.1, which is nearly parallel to the line $x_{10}=1$. This latter segment which does not correspond to resonant motion is small, and becomes smaller and smaller as we proceed to the families I_4, I_5... (not shown in Fig.1), corresponding to higher resonances.

A general remark is that all the resonant orbits of the form n/(n+1), n=1,2,3,... correspond to elliptic motion of at least one planet, though the eccentricities are in most cases small and only towards the end of the branches the eccentricities have large values. The values of the elements of the orbits of families I_1, I_2, I_3 are presented in Hadjidemetriou (1976a) and Delibaltas (1976).

(b). Stability of planetary-type orbits

We shall discuss families of the type I_1, I_2, I_3 (Figure 1). From the available numerical results (Hadjidemetriou 1976a, Delibaltas 1976) we can draw the following conclusions:

- The stability character of an orbit is not necessarily associated with commensurabilities in the periods of the two planets. Indeed, almost all resonant periodic orbits of family I_1 with nearly circular orbits of the two planets are stable, for all values m_1/m_3. Also the resonant orbits 1/2 in I_1, 2/3 in I_2 and 3/4 in I_3, for $m_1 \geq m_3$ are stable while the resonant orbits 1/2 and 2/3 in I_3 are unstable.
- The resonant periodic orbit corresponding to the commensurability 1/3 is the only unstable resonant motion with nearly circular orbits of the two planets. This seems to be true for all values of m_1/m_3, even for vanishingly small values of the masses of the two planets.

For finite masses of the two planets there is a small instability region corresponding to the resonance 1/3. In particular, the resonant orbit 2/5, corresponding to the actual Sun-Jupiter-Saturn system is found to be outside this unstable region.

- The upper branch of the family I_1, corresponding to an almost circular orbit of the outer planet P_3 and an elliptic orbit of the inner planet P_1 is stable only when the outer planet is the more massive one (we remind that in the degenerate family in 2(a) $R_3>1$). When the mass of the inner planet (comet) becomes larger than the mass of the outer planet, the system becomes unstable. The exact point of transition from stability to instability along the upper branch of I_1 depends on the ratio m_1/m_3.

As mentioned above, the upper branch of I_3 corresponds to a circular orbit of the outer planet P_3, and an elliptic orbit of the inner planet P_1. Also, the period of P_1 is just larger than half the period of P_3. All these facts suggest clearly that this upper branch of I_1 can be considered as the Jupiter family of comets. Consequently, the Jupiter family of comets could not exist if the masses of the comets were larger than the mass of Jupiter.

Another interesting aspect concerning stability is the study of the evolution of the families shown in Fig.1 by increasing the masses of the two planets. For example, we can take a certain set of families as in Fig. 1, for a fixed ratio of the masses m_1/m_3 and continue all these families by increasing the masses of the two planets, keeping the ratio m_1/m_3 fixed. The study of the stability of the obtained families will give interesting information for the generation and evolution of a planetary system as a whole. Of course, this procedure must be

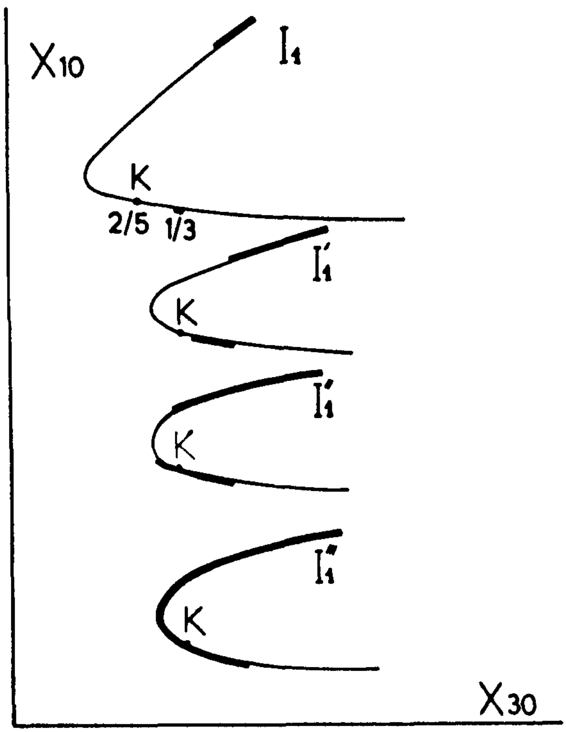

Fig. 2: The evolution of the family I_1 by increasing the mass of both planets, for a fixed ratio of the masses (schematically). The orbit K is the resonant orbit 2/5 and the region in bold line represents unstable motion. It is assumed that $m_3 < m_1$. If $m_3 \geq m_1$ then the unstable region in the upper branch of I_1 does not appear when m_1, $m_3 \ll 1$.

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repeated for several values of m_1/m_3 in order to obtain a clear picture of the whole problem.

We have done this in detail for the case $m_1=m_3$ (Hadjidemetriou 1976a) and also, to a lesser extent, for the case where the values of m_1 and m_3 are those of Jupiter and Saturn, respectively. The results are shown, qualitatively, in Fig.2. Only the family I_1 has been used for this continuation.

We note that when the masses of the two planets are small there is a very small unstable region corresponding to the resonance 1/3. An additional unstable region in the upper branch of I_1 exists when $m_3<m_1$. For the actual masses of Jupiter and Saturn and also for the masses of these two planets equal to 0.001 (in normalized units), the resonant orbit 2/5 is outside the unstable region and consequently it is stable. As the masses of the two planets increase, their ratio being fixed, we obtain the families I_1', I_1'', I_1'''. etc. We note that the unstable region due to the resonance 1/3 extends and also the unstable region in the upper branch of I_1 extends (this latter unstable region would not be present for very small values of m_1, m_2 if $m_1<m_3$ but would appear as m_1, m_3 increase). Eventually, the unstable region due to the resonance 1/3 extends and covers the resonant orbit 2/5 which corresponds to the Sun-Jupiter-Saturn system. For still larger values of the masses, the above mentioned two unstable regions merge and we are left with a family whose stable region corresponds to nearly circular orbits of the two planets not very near to each other.

It was also found that this continuation can be carried out until the mass of the Sun becomes equal to zero, i.e. we end up to the circular restricted three-body problem.

The transition from stability to instability for the actual Sun-Jupiter-Saturn system, by incrasing the masses of both planets (keeping their ratio fixed) is shown in Tables I and II. In Table I we present a part of a family (family A) of periodic orbits corresponding to the masses

$$m_1=0.0342888, \qquad m_2=0.9554260, \qquad m_3=0.0102852 \qquad (7)$$

and in Table II a part of family B, corresponding to the masses

$$m_1=0.0409564, \qquad m_2=0.9467594, \qquad m_3=0.0122952. \qquad (8)$$

The masses of the planets in family A are about 36 times the masses of Jupiter and Saturn and those of family B are about 43 times the masses of these planets. As a parameter along the family we have used the ratio $T_{JUPITER}/T_{SATURN}$ of the osculating elements of the two planets at t=0 and we present the stability index b_1 (of Hadjidemetriou 1975b), which is the first which becomes unstable, as a function of T_{JUP}/T_{SAT}. The value of b_1 when $T_{JUP}/T_{SAT}=2/5$ is obtained by linear interpolation between the adjacent values.

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TABLE I
A part of the family A, for the masses (7).

T_{JUP}/T_{SAT}	b_1	
.414	1.865	Stable
.402	1.973	"
.400	1.985	"
.398	1.996	"
.394	2.015	Unstable
.390	2.029	"
.379	2.043	"

TABLE II
A part of the family B, for the masses (8).

T_{JUP}/T_{SAT}	b_1	
.426	1.878	Stable
.419	1.947	"
.415	1.974	"
.412	1.998	"
.410	2.008	Unstable
.405	2.032	"
.400	2.049	"
.399	2.052	"
.391	2.061	"

From the numerical results presented we can find that the transition from stability to instability for the Sun-Jupiter-Saturn system (defined as that system corresponding to the resonance 2/5) takes place when the masses of the two planets are increased by about 38 times the actual masses. We must note however that the numerical value of this factor is quite sensitive to the definition of the "Sun-Jupiter-Saturn" system. For example, if we allow for a variation in the value of T_{JUP}/T_{SAT}, in the definition of the "Sun-Jupiter-Saturn" system, this factor will change appreciably.

The above stability analysis is based on a linear theory. In order to study the nonlinear effects we have computed the intersections of a perturbed orbit to the Sun-Jupiter-Saturn system, given by (7), with the plane $y_3=0$, by a method described in Hadjidemetriou (1975b). At each point of intersection, in the same direction, we have computed the osculating semimajor axes and eccentricities of the two planets. The results are shown in Fig.3 for the semimajor axes and in Fig.4 for the eccentricities. We note that these points lie on smooth curves in such a way that every third point of intersection lies on the same curve. There does not seem to exist any secular change in the semimajor axes. This result does not contradict Poisson's theorem on the invariability

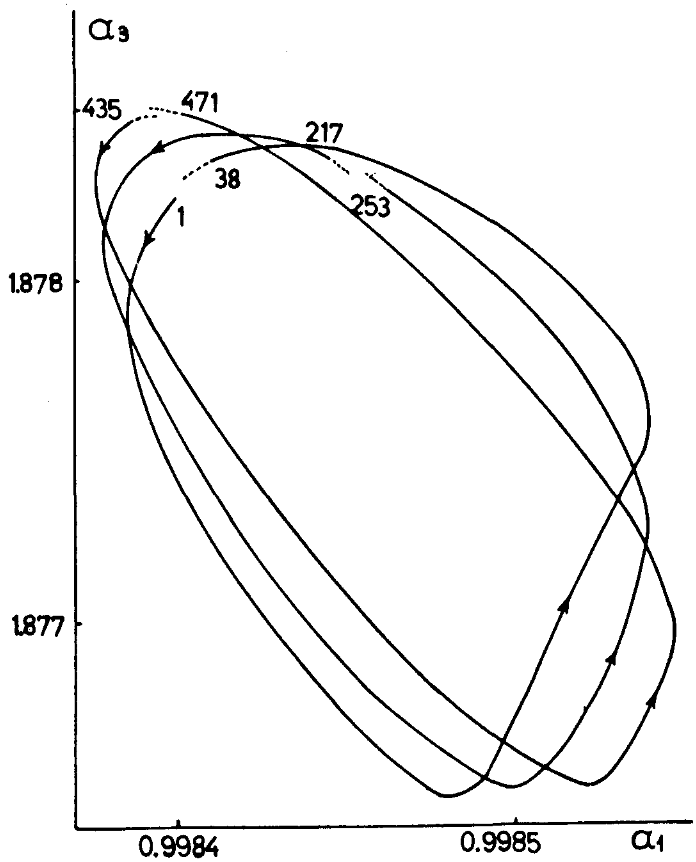

Fig. 3: Curves defined by the values of a_1 and a_3 at the consecutive points of intersection with $y_3=0$ (in the same direction) for a perturbed orbit to the periodic orbit (8) for the isoenergetic perturbation $\Delta\dot{x}_{10}=0.04$, $\Delta\dot{x}_{30}=0$, $\Delta\dot{x}_{10}=0.025$, $\Delta\dot{x}_{30}=0.04$. Every third point of intersection is shown only. The computations correspond to about 500 periods. Only the curves defined by the points 1-38, 217-253, 435-471 are shown.

of the semimajor axes (Hagihara, 1961, p.101). As far as the eccentricities are concerned, we noted appreciable changes. However, although we did not carry out the computations very far, we believe that these changes are also quasiperiodic, with very long periods. The obtained points are an indication (but not a proof) that the orbit is stable to all orders. These results can be also concidered as an indication that additional integrals exist, at least locally (see also Hagihara, 1961, p.107).

The stability studied above is with respect to perturbations in the plane of motion. As far as vertical stability is concerned (i.e. with respect to perturbations normal to the plane of motion) it was found by

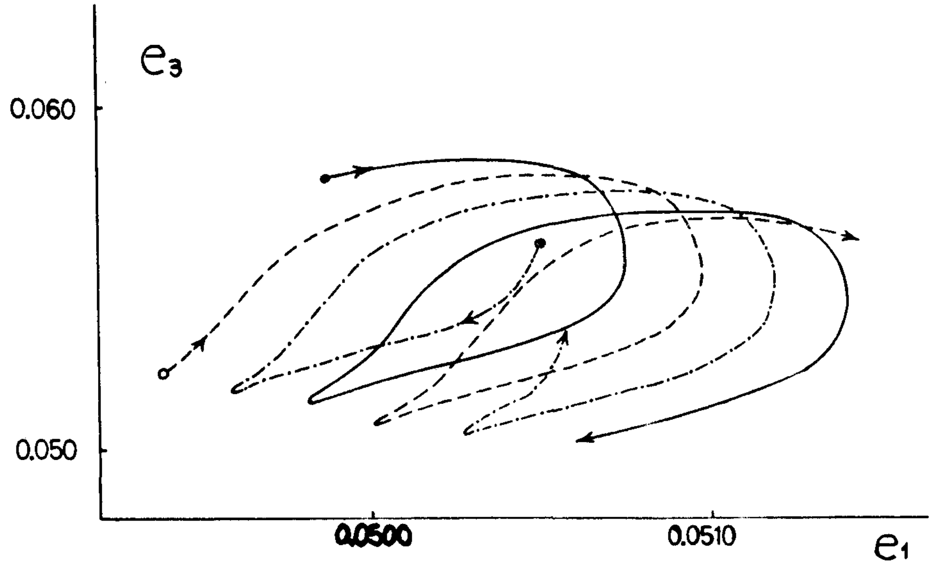

Fig.4: The same as in Fig.3 for the eccentricities e_1 and e_3. The three smooth curves mentioned in the text are shown. Only 60 points have been used in this Figure. The value of e_1 for the 500 periods increased from 0.050 to 0.058 and the value of e_3 decreased from 0.058 to 0.031.

Delibaltas (1976) that the family I_1, for the masses (6) of Jupiter and Saturn, is vertically stable. Thus, there are no critical orbits which would generate three-dimensional periodic orbits. This means that the extension of planetary type orbits of the kind I_1 to three dimensions is not possible. Hence, the actual three-dimensional motion of the Sun-Jupiter-Saturn system must be considered as a perturbed motion to a planar periodic orbit.

3. PLANETARY-TYPE ORBITS IN THE GENERAL 4-BODY PROBLEM

(a). Families for zero masses of the three planets

We consider the mass of the body P_2 to be equal to unity and the masses of the bodies P_1, P_3, P_4 equal to zero, and assume that the bodies P_1, P_3, P_4 (called planets) describe circular orbits in the same plane around the body P_2 (called Sun), in the same direction. Let

$$\omega_1 = 2\pi/T_1, \qquad \omega_3 = 2\pi/T_3, \qquad \omega_4 = 2\pi/T_4, \tag{9}$$

be the angular velocities of rotation of the three planets whose periods are T_1, T_3, T_4, respectively. We define a rotating frame xOy such that the origin is at P_2 and the x axis contains always the body P_1 and normalize the unit of length in such a way that the radius R_1 of P_1 is

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equal to unity. We note now that if we take an arbitrary radius R_3 for the orbit of P_3 and select the radius R_4 of P_4 in such a way that

$$(\omega_3-\omega_1)/(\omega_4-\omega_1) = p/q, \tag{10}$$

where p,q are integers, the system is periodic with respect to the rotating frame x0y with a period equal to

$$T = \frac{q}{1-T_1/T_4} T_1. \tag{11}$$

Using (9) and (10) we can find that the ratio T_1/T_4 is expressed in terms of T_1/T_3 by the relation

$$\frac{T_1}{T_4} = 1 + \frac{q}{p} (\frac{T_1}{T_3} - 1). \tag{12}$$

If now we normalize the units in such a way that $(P_2P_1)=R_1=1$ and keep p/q fixed ($p<q$), we have for each value of $R_3>1$, or equivalently of T_1/T_3, a periodic motion of the four bodies. In this normalization we have $T_1=2\pi$. Thus, we obtain a degenerate family of periodic orbits of the four bodies, with the ratio T_1/T_3 (or the distance R_3) as a parameter. This family is characterized by a particular value of the ratio p/q and consequently we have several families, one for each value of p/q. Thus, we can characterize a degenerate family of 4 bodies by its ratio p/q. We note that if $R_3>1$ and $p/q<1$, then $T_1/T_4<T_1/T_3$ which implies that the radii of P_1, P_3, P_4 increase in this order, i.e. P_4 is the outer planet, P_3 the intermediate planet and P_1 the inner planet. In this case the ratio T_1/T_3 varies between the values 0 (for $R_3=\infty$) and 1 (for $R_3=1$). It can be verified that there is no loss of generality in selecting $R_1=1$, $R_3>1$ and $p/q<1$. For the other possible values of R_1, R_3 and p/q we would obtain the same family, but with the roles of P_1, P_3, P_4 interchanged in their hierarchigal order.

We must also note that, apart from all the above parameters, a particular orbit depends also on the relative positions of the 3 planets with respect to the Sun, at t=0. In the present study we consider symmetric periodic orbits with respect the x0y only. For this reason we have taken all three planets to lie on the x axis at t=0. Moreover, we have restricted ourselves to positive values for x_1, x_3, x_4 respectively, at t=0. If, other things being the same, we had taken $x_1>0$, $x_3<0$ and $x_4>0$ ($x_1<|x_3|<x_4$) then we would obtain a different family.

It can be proved (Hadjidemetriou 1976b,c) that the above mentioned degenerate families of periodic orbits can be continued as monoparametric families of symmetric periodic orbits of the general planar 4-body problem, in a rotating frame of reference, by increasing the masses of the planets. In this way we can obtain a family for fixed masses. In this continuation we may have any value for the ratio $m_1:m_3:m_4$ of the masses of the three planets. The continuation is unique for all the orbits of the degenerate family except for those orbits whose period is a multiple of 2π. In this latter case the continuation theorem is not applicable and this results to a situation similar to that in the three-body problem (Figure 1). And we may note from (11) and (12) that there is an infinity of such resonant orbits when T_1/T_3 varies between 1 and zero. Thus, a continuous degenerate family corresponding to a ratio p/q is extended to an infinity of families of periodic orbits for fixed non-zero masses of the three planets.

As an example we give below, in Table III, the values of T_1/T_4 and T as a function of T_1/T_3, for a degenerate family corresponding to p/q=2/3.

TABLE III
Some characteristic orbits of the degenerate family for p/q=2/3

N_o	T_1/T_3	T_1/T_4	T	corresponding family
1	1/3	0	$3\times 2\pi$	A (1–3)
2	7/15	1/5	$(15/4)\times 2\pi$	A
3	1/2	1/4	$4\times 2\pi$	A
4	5/9	1/3	$(9/2)\times 2\pi$	B (4–5)
5	3/5	2/5	$5\times 2\pi$	B
6	2/3	1/2	$6\times 2\pi$	C

We note that the 1st, 3rd, 5th and 6th cases in Table III correspond to resonant periodic orbits of the corresponding degenerate family ($m_1=m_3=m_4=0$), where the period is a multiple of 2π. Evidently, there is an infinite number of such resonant orbits as T_1/T_3 increases to unity. As a consequence of the existence of these orbits in the degenerate family mentioned above, we obtain an infinite number of families of periodic orbits (for a fixed value p/q) when the masses of the three planets are increased. Each one of these latter families can be thought of as lying "between" two consecutive resonant orbits of the degenerate case with periods $n\times 2\pi$ and $(n+1)\times 2\pi$, respectively (n=3,4...).

The degenerate family p/q=2/3 can be associated with the three inner Galilean satellites of Jupiter, because the family for nonzero masses of P_1, P_3 and P_4 generated from the points of the degenerate family between the 3rd and 5th point in Table III has a branch which approximates the actual motion of Jupiter's satellites. (i.e. the periods are in the ratio 1:2:4).

(b). Planetary-type families of periodic orbits of four bodies with nonzero masses

Detailed calculations of planetary-type orbits in the general 4-body problem will be given elsewhere (Hadjidemetriou and Michalodimitrakis 1976). We shall present here the main qualitative features of the calculations obtained so far, for the family corresponding to p/q=2/3.

We extended the degenerate family p/q=2/3 for the masses

$$m_1=0.0000379946, \quad m_2=0.9998570204,$$
$$m_3=0.0000249964, \quad m_3=0.0000799886 . \tag{13}$$

These masses, normalized so that $m_1+m_2+m_3+m_4=1$, correspond to the mass of Jupiter (P_2) and its three inner satellites (P_1, P_3, P_4) (Reek, 1958). As mentioned above, this degenerate family will be extended, for the masses (13), to an infinity of families. We have computed three of these families, the family A lying between the degenerate resonant orbits

$$(T_1/T_3=1/3, \quad T_1/T_4=0) \quad \text{and} \quad (T_1/T_3=1/2, \quad T_1/T_4=1/4),$$

the family B lying between

$$(T_1/T_3=1/2, \quad T_1/T_4=1/4) \quad \text{and} \quad (T_1/T_3=3/5, \ T_1/T_4=2/5)$$

and the family C lying between

$$(T_1/T_3=3/5, \quad T_1/T_4=2/5) \quad \text{and} \quad (T_1/T_3=2/3, \ T_1/T_4=1/2),$$

as can be seen from table III.

To present the results, we note (Hadjidemetriou 1976b,c) that a symmetric periodic orbit for N=4 can be specified, in the rotating frame of reference whose origin coincides with the center of mass of P_1, P_2 and its x axis contains always these bodies, by the initial conditions

$$x_{10}, \quad x_{30}, \quad x_{40}, \quad \dot{y}_{30}, \quad \dot{y}_{40}, \tag{14}$$

provided a certain normalization scheme is used. Thus, a family for fixed masses is represented by a continuous curve in the space x_{10} x_{30} x_{40} $\dot{y}_{30}$ $\dot{y}_{40}$. To simplify things, we shall use, for qualitative purposes, the projections of this curve in the planes $x_{10}x_{30}$, and $x_{10}x_{40}$ only. Evidently, the degenerate family p/q=2/3 will be presented in the above plane by the straight line $x_{10}=1$, according to the normalization mentioned before.

In Fig. 5 we present qualitative results for the families A,B,C obtained for the masses (13). In all cases, we have taken

$$x_{10}>0, \quad x_{30}>0, \quad x_{40}>0 \quad \text{at } t=0 \quad (\text{also } \ x_{10}<x_{30}<x_{40}).$$

The osculating eccentricities of the orbits of P_1, P_3 and P_4 are very small and for all orbits of all these families the ratio of relative frequencies $(\omega_3-\omega_1)/(\omega_4-\omega_1)$ is found to be equal to 2/3 to an accuracy of three decimal places. We have also found that the satellites I, II, III are in their pericenter (P) or apocenter (A) at t=0, along the families A, B, C, as shown below. The branches of

https://doi.org/10.1017/S0252921100062370 Published online by Cambridge University Press

these families are designated by the resonance T_1/T_3 (see Fig. 5a).

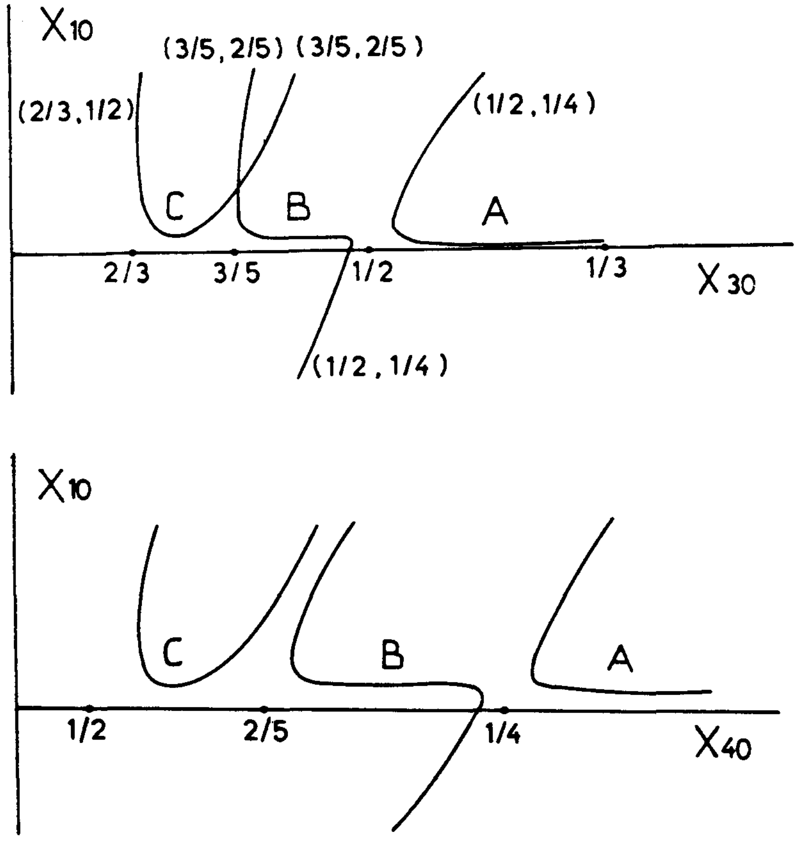

Fig. 5a,b: The families A,B,C (schematically). The lower branch (almost parallel to the line $x_{10}=1$) of family A in the plane $x_{10}x_{40}$ (Fig. 5b) extends to infinity while the lower branch of A in the plane $x_{10}x_{30}$ (Fig.5a) stops at the point corresponding to the resonance 1/3. The ratios in the parentheses near each branch of the families in Fig.5a denote the ratios T_1/T_3, T_1/T_4, respectively.

Family	branch	I	II	III
A	1/3	P	P	P
A	1/2	P	P	A
B	1/2	A	A	P
B	3/5	P	P	A
C	3/5	P	A	P
C	2/3	P	P	A

The lower branch of family B corresponds to a resonance 1/2, 1/4 between P_1-P_3 and P_1-P_4, respectively and has many similarities with the motion of the three inner satellites of Jupiter. We have selected the periodic orbit with initial conditions

$$x_{10}=0.99429726, \quad x_{30}=1.59200305, \quad x_{40}=2.44984472$$
$$\dot{y}_{30}=-0.80992184, \quad \dot{y}_{40}=-1.80974958 \tag{15}$$

as the closest orbit to the motion of Jupiter's satellites. The osculating elements of the orbits of the three satellites vary between the following limits:

$$0.9978 \leq a_1 \leq 0.9979, \qquad 0.01680 \leq e_1 \leq 0.01690,$$
$$1.5509 \leq a_3 \leq 1.5517, \qquad 0.02602 \leq e_3 \leq 0.02629,$$
$$2.4585 \leq a_4 \leq 2.4596, \qquad 0.00364 \leq e_4 \leq 0.00395.$$

At $t=0$ the bodies P_1 and P_3 are at apocenter and the body P_4 at pericenter.

The motion given by (15) has many similarities with the actual motion of Jupiter's satellites but does not coincide with it. This is so because the relative positions of P_1, P_3, P_4 with respect to P_2 are not those of the actual case. A periodic orbit representing closely the actual case has been obtained in the same way as (15), by the continuation of a degenerate orbit corresponding to $T_1/T_3=1/2$ and $T_1/T_4=1/4$, by increasing the masses to the values given by (13), if we take $x_{10}>0$, $x_{40}>0$ and $x_{30}<0$ (instead of $x_{30}>0$ we had in (15)). This orbit is given by

$$x_{10}=1.000594320, \quad x_{30}=-1.595000148, \quad x_{40}=2.572200612,$$
$$\dot{y}_{30}=0.801696725, \quad \dot{y}_{40}=-1.948632207$$

and its osculating elements vary between the limits

$$1.002595 \leq a_1 \leq 1.002693, \qquad 0.001982 \leq e_1 \leq 0.002012,$$
$$1.601014 \leq a_3 \leq 1.601357, \qquad 0.003780 \leq e_3 \leq 0.003949,$$
$$2.572558 \leq a_4 \leq 2.573604, \qquad 0.000154 \leq e_4 \leq 0.000368.$$

The satellites I and III are in conjuction and II in opposition when they are all in their perijoves, and this corresponds to the actual case. A periodic orbit for the three inner satellites of Jupiter has been found by de Sitter (Brower and Clemence, 1961, p.82, Hagihara, 1961, p.123) and was used as an intemediary orbit to obtain the ephemeris of the satellites.

As a general remark we can say that a large part of the families correspond to almost resonant motion of the three small bodies. This resonant motion corresponds to the part of the families in Figure 5 which is not parallel to the line $x_{10}=1$. The osculating eccentricities of the orbits of the planets along these branches are greater than zero, but in most cases remain small, of the other of 10^{-2}. This value is in agreement with the eccentricities in the solar system, in particular the Sun-Jupiter-Saturn system. (Compare this with the remark made in section 2(c) for the eccentricities of the Sun-Jupiter-Saturn system, considered there as a 3-body motion).

4. DISCUSSION

The method developed in this paper could provide us with useful information on the generation and evolution of planetary systems. This is so because by studying families of planetary-type orbits for several mass-ratios we can obtain an overall view of the problem. In this way we find which configurations are unstable, so that they are excluded as possible configurations for planetary systems existing in nature.

In the families of planetary-type orbits we have obtained both for three and for four bodies, we note that a large part of the obtained families correspond to resonant motion. Moreover, the appearance of resonances increases as the number of bodies increases. For N=3 the resonant orbits are those corresponding to the part of the families which are not parallel to the straight line $x_{10}=1$ and also some isolated orbits in the part which is almost parallel to $x_{10}=1$ (Figure 1). The former resonant orbits are associated with nonzero eccentricities of the orbits of the two planets (though in most cases the eccentricities are not large). The addition of a fourth body has as a consequence the increase of resonant cases, no matter how small the mass of the fourth body is. In fact, all orbits obtained from a degenerate family for N=4, corresponding to a fixed ratio p/q, when the masses of the planets assume nonzero values, are resonant in the sense that the ratio of the relative angular velocities in the rotating frame xOy $(\omega_3-\omega_1)/(\omega_4-\omega_1)$ is almost constant, equal to p/q, for all members of the family. Besides, the addition of a fourth body results in the appearance of more resonances, in the absolute orbits of the planets, not originally present in the (simple) periodic orbits of the three-body system. This can be seen from the comparison of Figs.1 and 5 (and Table III). The resonances in the ratios of the absolute motions of the three planets in the 4-body system appear in all the branches of the families in Fig.5 which are not parallel to the line $x_{10}=1$. For example, for N=3 we have the resonances 1/2, 2/3, 3/4 and when a fourth body is added, we can see from Figure 5, for p/q=2/3, that the resonances 1/4, 2/5, 3/5 are also present. In the same way, the addition of a fifth body will increase further the resonant cases.

Taking into account all the above, we can attempt an interpretation of the appearence of commensurabilities in our Solar System. Indeed, if we assume that the motion of the Solar System, or at least its most important components, must be near a periodic orbit, then it is very likely that many commensurabilities will exist, as a large part of the families of periodic planetary-type orbits in the N-body problem are resonant orbits.

As far as stability is concerned, we may note that if we restrict ourselves to N=3, i.e. we study the Sun-Jupiter-Saturn system only and ignore the other planets, then there exists an infinity of stable configurations for the actual masses of these bodies. It would be of interest to study whether the addition of more bodies make the system more stable or, on the contrary, limit the stable configurations of the Solar System. The stability of the 4-body systems has not yet been studied and will appear elsewhere (Hadjidemetriou and Michalodimitrakis, 1976).

There seems to be some confusion on the role which the resonances among the planets play in the stability of the Solar System (e.g. Moser 1973). The stability analysis of periodic planetary-type orbits of 3 bodies has shown that the commensurabilities do not play a very important role in the stability of the system. For example, the commensurable orbits 1/2 in family I_1 are stable for $m_1=m_3=0.001$, $m_2=0.998$ and the commensurable orbits 1/2 in family I_2 for these masses are unstable. Also, the commensurable orbits 1/2 of family I_1 for the masses (9) are unstable. Thus, the presence of small divisors (e.g. Hagihara, 1961, p.111) does not seem to play an important role in stability as, for the same commensurability, the stability depends on the relative dimensions of the orbits of the planets (i.e. on the position on the family) and on the relative masses of the planets.

We would also like to comment on the meaning of (linear) instability. Usually, instability is associated with escape of at least one body, and this seems to be the rule in most cases in other problems. In the planetary orbits however for N=3 we could not establish such a close connection between instability and escape. For small masses of the planets, a perturbed orbit to an unstable periodic orbit did not lead to escape but to random, bounded, motion. Of course, one can always argue that escape will eventually happen if the computations are carried further in time, but this remains an open question. We do have a case for N=3 where escape takes place after some hundred revolutions (Hadjidemetriou 1976a), but the masses of the planets were rather large ($m_1=m_3=0.05$, $m_2=0.90$).

Finally, we may note that the method of periodic orbits may provide a new approach to the accurate computation of the planets and satellites of the Solar System. Instead of using the two-body approach as a reference orbit to compute the perturbations we may use for this purpose a periodic motion for several members of the Solar System, and calculate the perturbations for the actual motion starting from this periodic orbit. The periodic motion can be obtained to a high degree of accuracy numerically and analytically in the form of Fourier series. In this way we avoid the small divisors which complicate the classical approach to the solution of the planetary problem, as all the resonances will be included in the reference periodic orbit itself, whose numerical computation does not depend on any resonance present and the gravitational effect of one planet on the other is completely taken into account.

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