11

The conformal constraint equations

This chapter analyses the intrinsic equations implied by the conformal Einstein field equations on non-null hypersurfaces. These equations are known as the ***conformal constraint equations***. They play an essential role in the construction of initial data sets for the conformal field equations and in the identification of boundary conditions. Not surprisingly, these conformal constraint equations are closely related to the standard ***Einstein constraint equations*** – consequently, this chapter starts by considering the properties of the latter.

The solvability and behaviour of solutions to the conformal constraint equations is closely related to the nature of the underlying three-dimensional manifold on which the equations are imposed. As a consequence, this chapter also provides a discussion of general properties of asymptotically Euclidean and asymptotically hyperboloidal 3-manifolds from a conformal point of view. The systematic analysis of the constraint equations relies on methods of elliptic partial differential equations. Hence, this chapter provides a discussion of some of the basic notions of this theory.

An important aspect of the conformal constraint equations – the so-called *propagation of the constraints* – is discussed in Chapter 13. The analysis of the constraint equations on null hypersurfaces is treated in Chapter 18.

11.1 General setting and basic formulae

Let $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ denote a spacetime satisfying the Einstein field equations. In what follows, it will be assumed that $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ can be conformally extended to an unphysical spacetime $(\mathcal{M}, \boldsymbol{g})$. Accordingly, there exists an embedding $\phi : \tilde{\mathcal{M}} \rightarrow \mathcal{M}$ and a conformal factor Ξ such that $\phi^*\boldsymbol{g} = \Xi^2\tilde{\boldsymbol{g}}$. Now, let $\tilde{\mathcal{S}}$ denote a three-dimensional submanifold of $\tilde{\mathcal{M}}$ and let $\varphi : \tilde{\mathcal{S}} \rightarrow \tilde{\mathcal{M}}$ denote the associated embedding. As the composition $\phi \circ \varphi : \tilde{\mathcal{S}} \rightarrow \mathcal{M}$ is also an embedding, the three-dimensional manifold $\tilde{\mathcal{S}}$ can be regarded, in turn, as a submanifold of $\mathcal{M}$. As discussed in Section 2.7.3, the spacetime metric $\tilde{\boldsymbol{g}}$ induces a metric $\tilde{\boldsymbol{h}}$ on $\tilde{\mathcal{S}}$ via

$\tilde{\boldsymbol{h}} = \varphi^*\tilde{\boldsymbol{g}}$. Similarly, regarding $\tilde{\mathcal{S}}$ as a hypersurface on $\mathcal{M}$, the unphysical metric $\boldsymbol{g}$ also induces a metric $\boldsymbol{h}$ via the pull-back $\boldsymbol{h} = (\phi \circ \varphi)^*\boldsymbol{g}$. A calculation shows that

$$\boldsymbol{h} = (\phi \circ \varphi)^*\boldsymbol{g} = (\varphi^* \circ \phi^*)\boldsymbol{g} = \varphi^*(\Xi^2|_{\tilde{\mathcal{S}}}\tilde{\boldsymbol{g}}) = \Omega^2\varphi^*\tilde{\boldsymbol{g}}$$

where $\Omega \equiv \Xi^2|_{\tilde{\mathcal{S}}}$ is the restriction of Ξ to the hypersurface $\tilde{\mathcal{S}}$. Following the conventions of previous chapters, $\boldsymbol{h} = \Omega^2\varphi^*\tilde{\boldsymbol{g}}$ will often be written as

$$\boldsymbol{h} = \Omega^2\tilde{\boldsymbol{g}}.$$

Now, let $\tilde{\boldsymbol{\nu}}$ and $\boldsymbol{\nu}$ denote, respectively, the $\tilde{\boldsymbol{g}}$-unit and $\boldsymbol{g}$-unit normals of $\tilde{\mathcal{S}}$ and define

$$\epsilon \equiv \tilde{\boldsymbol{g}}(\tilde{\boldsymbol{\nu}}, \tilde{\boldsymbol{\nu}}) = \boldsymbol{g}(\boldsymbol{\nu}, \boldsymbol{\nu}).$$

In accordance with the signature convention $(+ - --)$, the hypersurface $\tilde{\mathcal{S}}$ is spacelike if $\epsilon = 1$ and timelike if $\epsilon = -1$. It follows that

$$\boldsymbol{\nu} = \Xi\tilde{\boldsymbol{\nu}}, \qquad \boldsymbol{\nu}^\sharp = \Xi^{-1}\tilde{\boldsymbol{\nu}}^\sharp$$

or, using index notation, $\nu_a = \Xi\tilde{\nu}_a$ and $\nu^a = \Xi^{-1}\tilde{\nu}^a$. In what follows, the indices of objects in $\tilde{\mathcal{M}}$ are raised/lowered using the metric $\tilde{\boldsymbol{g}}$, while the indices of objects on $\mathcal{M}$ are moved using $\boldsymbol{g}$.

11.1.1 The transformation formulae for the extrinsic curvature

Having discussed the relation between the 3-metrics and the unit normals to $\tilde{\mathcal{S}}$, one is in the position to consider the relation between the extrinsic curvatures $\tilde{\boldsymbol{K}}$ and $\boldsymbol{K}$. Given *spatial vectors* $\boldsymbol{u}$, $\boldsymbol{v} \in T(\tilde{\mathcal{S}})$ – so that $\langle \tilde{\boldsymbol{\nu}}, \boldsymbol{u} \rangle = \langle \tilde{\boldsymbol{\nu}}, \boldsymbol{v} \rangle = 0$ – one has that

$$\tilde{\boldsymbol{K}}(\boldsymbol{u}, \boldsymbol{v}) = \langle \tilde{\boldsymbol{\nabla}}_{\boldsymbol{u}}\tilde{\boldsymbol{\nu}}, \boldsymbol{v} \rangle, \qquad \boldsymbol{K}(\boldsymbol{u}, \boldsymbol{v}) = \langle \boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{\nu}, \boldsymbol{v} \rangle;$$

see Equation (2.43). Recalling that $\boldsymbol{\nabla} - \tilde{\boldsymbol{\nabla}} = \boldsymbol{S}(\boldsymbol{\Upsilon})$ one readily has that

$$\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{\nu} = \tilde{\boldsymbol{\nabla}}_{\boldsymbol{u}}\boldsymbol{\nu} - \boldsymbol{S}(\boldsymbol{\Upsilon}, \boldsymbol{\nu}; \boldsymbol{u});$$

the minus sign arises from the fact that $\boldsymbol{\nu}$ is a covector. In abstract index notation $\boldsymbol{S}(\boldsymbol{\Upsilon}, \boldsymbol{\nu}; \boldsymbol{u})$ is given by $S_{ab}{}^{cd}\Upsilon_c\nu_d u^b$ from where a short calculation gives that

$$\begin{aligned} S_{ab}{}^{cd}\Upsilon_c\nu_d u^b &= S_{ab}{}^{cd}\tilde{\nabla}_c\Xi\tilde{\nu}_d u^b, \\ &= \tilde{\nabla}_b\Xi\tilde{\nu}_a u^b - \tilde{g}_{ab}u^b\tilde{g}^{cd}\tilde{\nabla}_c\Xi\tilde{\nu}_d \\ &= (u^c\tilde{\nabla}_c\Xi)\tilde{\nu}_a - \Xi\Sigma\tilde{g}_{ac}u^c, \end{aligned}$$

where

$$\Sigma \equiv g^{ab}\nabla_a\Xi\nu_b = \boldsymbol{g}^\sharp(\mathbf{d}\Xi, \boldsymbol{\nu}) = \Xi^{-1}\tilde{\boldsymbol{g}}^\sharp(\mathbf{d}\Xi, \tilde{\boldsymbol{\nu}})$$

is the derivative of Ξ in the direction of the $\boldsymbol{g}$-unit normal to $\tilde{\mathcal{S}}$. Accordingly, one has that

$$\boldsymbol{S}(\boldsymbol{\Upsilon}, \boldsymbol{\nu}; \boldsymbol{u}) = \boldsymbol{u}(\Xi)\tilde{\boldsymbol{\nu}} - \Xi\Sigma\tilde{\boldsymbol{g}}(\boldsymbol{u}, \cdot),$$

from where, recalling that $\boldsymbol{u}$, $\boldsymbol{v} \in T(\tilde{\mathcal{S}})$ so that $\tilde{\boldsymbol{g}}(\boldsymbol{u}, \boldsymbol{v}) = \tilde{\boldsymbol{h}}(\boldsymbol{u}, \boldsymbol{v})$, it follows that

$$\begin{aligned}
\boldsymbol{K}(\boldsymbol{u}, \boldsymbol{v}) &= \langle \tilde{\boldsymbol{\nabla}}_{\boldsymbol{u}}\boldsymbol{\nu}, \boldsymbol{v}\rangle - \boldsymbol{u}(\Xi)\langle \tilde{\boldsymbol{\nu}}, \boldsymbol{v}\rangle + \Omega\Sigma\langle \tilde{\boldsymbol{g}}(\boldsymbol{u}, \cdot), \boldsymbol{v}\rangle \\
&= \Omega\langle \tilde{\boldsymbol{\nabla}}_{\boldsymbol{u}}\tilde{\boldsymbol{\nu}}, \boldsymbol{v}\rangle + \Omega\Sigma\tilde{\boldsymbol{h}}(\boldsymbol{u}, \boldsymbol{v}) \\
&= \Omega\left(\tilde{\boldsymbol{K}}(\boldsymbol{u}, \boldsymbol{v}) + \Sigma\tilde{\boldsymbol{h}}(\boldsymbol{u}, \boldsymbol{v})\right),
\end{aligned}$$

where to pass from the first to the second line it has been used that $\langle \tilde{\boldsymbol{\nu}}, \boldsymbol{v}\rangle = 0$ as $T(\tilde{\mathcal{S}})$.

Summarising, the calculations in the previous paragraphs show that

$$h_{ij} = \Omega^2\tilde{h}_{ij}, \tag{11.1a}$$

$$K_{ij} = \Omega\big(\tilde{K}_{ij} + \Sigma\tilde{h}_{ij}\big). \tag{11.1b}$$

These are the basic transformation formulae for the remainder of this chapter. Taking the trace of the transformation formula for the extrinsic curvature, Equation (11.1b), it follows that

$$\Omega K = \tilde{K} + 3\Sigma,$$

where $\tilde{K} \equiv \tilde{h}^{ij}\tilde{K}_{ij}$ and $K \equiv h^{ij}K_{ij}$ – these scalars are sometimes called, respectively, the physical and unphysical ***mean curvature*** of $\tilde{\mathcal{S}}$. The scalars Σ, K admit a geometric interpretation: if $\Sigma = K = 0$, then, necessarily, $\tilde{K} = 0$ and the hypersurface $\tilde{\mathcal{S}}$ is ***maximal*** in $\tilde{\mathcal{M}}$ with respect to both the metrics $\tilde{\boldsymbol{g}}$ and $\boldsymbol{g}$ – that is, it encloses a maximum volume for a given area.

11.1.2 Decompositions in electric and magnetic parts

A key ingredient in the analysis of the conformal constraint equations is the ***decomposition in electric and magnetic parts*** of tensors with antisymmetric pairs of indices. Let $\mathcal{S}$ denote a hypersurface on a spacetime $(\mathcal{M}, \boldsymbol{g})$, and let $\boldsymbol{\nu}$ denote the unit normal to the hypersurface. The ***projector*** to $\mathcal{S}$ is the tensor ${h_a}^b$ given by

$${h_a}^b \equiv {\delta_a}^b - \epsilon\nu_a\nu^b.$$

It follows that

$${h_a}^b\nu_b = 0, \qquad {h_a}^b{h_b}^c = {h_a}^c.$$

Furthermore, using the properties of the spacetime volume form ϵ_{abcd} – see Section 2.5.3 – one can deduce that

$${h_a}^{[c}{h_b}^{d]} = -\frac{1}{2}\epsilon_{abe}\epsilon^{cde}, \tag{11.2}$$

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where $\epsilon_{abe} \equiv \epsilon_{fabe}\nu^f$ is the ***three-dimensional volume form***.

Now, let F_{ab} denote an antisymmetric tensor of rank 2 and let $F^*_{ab} \equiv -\frac{1}{2}\epsilon_{ab}{}^{cd}F_{cd}$ denote its Hodge dual. Its ***electric*** and ***magnetic parts*** are defined, respectively, to be

$$F_a \equiv F_{cb}\nu^b h_a{}^c, \qquad F^*_a \equiv F^*_{cb}\nu^b h_a{}^c.$$

It can be verified that

$$F_a\nu^a = F^*_a\nu^a = 0, \qquad h_a{}^b F_b = F_a, \qquad h_a{}^b F^*_b = F^*_a,$$

so that the electric and magnetic parts are said to be ***spatial*** tensors. Together, F_a and F^*_a encode the same information as the original tensor F_{ab}. In order to see this, one writes

$$\begin{aligned} F_{ab} &= F_{cd}\delta_a{}^c\delta_b{}^d = F_{cd}(h_a{}^c + \epsilon\nu_a\nu^c)(h_b{}^d + \epsilon\nu_b\nu^d) \\ &= F_{cd}h_a{}^c h_b{}^d + \epsilon F_{cd}h_a{}^c\nu_b\nu^d + \epsilon F_{cd}h_b{}^d\nu_a\nu^c \\ &= 2\epsilon F_{[a}\nu_{b]} + F_{cd}h_a{}^c h_b{}^d. \end{aligned} \tag{11.3}$$

The term $F_{cd}h_a{}^c h_b{}^d$ is, in turn, manipulated using the identity (11.2) as follows:

$$\begin{aligned} F_{cd}h_a{}^c h_b{}^d = F_{cd}h_a{}^{[c}h_b{}^{d]} &= -\frac{1}{2}F_{cd}\epsilon^{fcde}\nu_f\epsilon_{abe} \\ &= F^*_{ef}\nu^f\epsilon_{ab}{}^e = F^*_e\epsilon_{ab}{}^e. \end{aligned} \tag{11.4}$$

Thus, combining Equations (11.3) and (11.4), one concludes that

$$F_{ab} = 2\epsilon F_{[a}\nu_{b]} + F^*_e\epsilon^e{}_{ab}.$$

The decomposition in electric and magnetic parts can be extended to tensors W_{abcd} with the same symmetries as the Weyl tensor; such tensors are sometimes known as ***Weyl candidates***. By analogy to the rank-2 case one defines the $\boldsymbol{\nu}$-electric and $\boldsymbol{\nu}$-magnetic parts of W_{abcd} to be

$$W_{ac} \equiv W_{ebfd}\nu^b\nu^d h_a{}^e h_c{}^f, \qquad W^*_{ac} \equiv W^*_{ebfd}\nu^b\nu^d h_a{}^e h_c{}^f,$$

with $W^*_{abcd} \equiv -\frac{1}{2}\epsilon_{cd}{}^{ef}W_{abef}$ denoting the *right Hodge dual* of W_{abcd}. In the subsequent discussion it is convenient to consider

$$W_{abc} \equiv W_{efgh}\nu^f h_a{}^e h_b{}^g h_c{}^h.$$

It can be verified that

$$W^*_{ab} = -\frac{1}{2}W_{acd}\epsilon_b{}^{cd}.$$

As in the rank-2 case, the tensors W_{ab} and W^*_{ab} (or, alternatively, W_{ab} and W_{abc}) encode the same information as W_{abcd}. The argument to show this equivalence is similar to that of the rank-2 case:

$$\begin{aligned}W_{abcd} &= W_{efgh}\delta_a{}^e\delta_b{}^f\delta_c{}^g\delta_d{}^h \\ &= W_{efgh}(h_a{}^e+\epsilon\nu_a\nu^e)(h_b{}^f+\epsilon\nu_b\nu^f)(h_c{}^g+\epsilon\nu_c\nu^g)(h_d{}^h+\epsilon\nu_d\nu^h) \\ &= W_{efgh}h_a{}^eh_b{}^fh_c{}^gh_d{}^h+\epsilon W_{cab}\nu_d-\epsilon W_{deb}h_a{}^e\nu_c+\epsilon W_{acd}\nu_b \\ &\quad +W_{ac}\nu_b\nu_d-W_{ad}\nu_b\nu_c-\epsilon W_{bcd}\nu_a-W_{bc}\nu_a\nu_d+W_{bd}\nu_a\nu_c. \end{aligned} \tag{11.5}$$

From the definition of the magnetic part W^*_{ab} it follows that

$$W_{abc} = \epsilon^e{}_{bc}W^*_{ae}. \tag{11.6}$$

Moreover, using that

$$\epsilon_{abc}\epsilon^{def} = -6\delta_a{}^{[d}\delta_b{}^e\delta_c{}^{f]}, \tag{11.7}$$

it follows that

$$\begin{aligned}W_{efgh}h_a{}^eh_b{}^fh_c{}^gh_d{}^h &= \frac{1}{4}W_{efgh}\epsilon^{efz}\epsilon_{abz}\epsilon^{ghx}\epsilon_{cdx} \\ &= {}^*W^*_{rzsx}\nu^r\nu^s\epsilon_{ab}{}^z\epsilon_{cd}{}^x = -W_{zx}\epsilon_{ab}{}^z\epsilon_{cd}{}^x \\ &= W_{ca}h_{bd}+W_{db}h_{ac}-W_{cb}h_{ad}-W_{da}h_{bc}. \end{aligned} \tag{11.8}$$

Combining Equations (11.5), (11.6) and (11.8) one obtains the desired decomposition of W_{abcd} in terms of W_{ab} and W^*_{ab}:

$$W_{abcd} = 2\epsilon(l_{b[c}W_{d]a}-l_{a[c}W_{d]b})-2(\nu_{[c}W^*_{d]e}\epsilon^e{}_{ab}+\nu_{[a}W^*_{b]e}\epsilon^e{}_{cd}), \tag{11.9}$$

where $l_{ab}\equiv h_{ab}-\epsilon\nu_a\nu_b$. A similar computation renders

$$W^*_{abcd} = 2\nu_{[a}W_{b]e}\epsilon^e{}_{cd}-4W_{e[a}\epsilon_{b]}{}^e{}_{[c}\nu_{d]}-4\nu_{[a}W^*_{b][c}\nu_{d]}-W^*_{ef}\epsilon^e{}_{ab}\epsilon^f{}_{cd}. \tag{11.10}$$

Expressions in terms of an adapted frame

The decomposition discussed in the previous paragraphs acquires a particularly simple form when supplemented with a frame $\{\boldsymbol{e_a}\}$ adapted to the hypersurface $\mathcal{S}$. For such a frame, the projection of a particular index with respect to the normal corresponds to replacement of the corresponding frame index with $\perp$ while the spatial part of a tensor is given by the replacement of the spacetime frame indices $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}, \ldots$ with the spatial frame indices $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}, \ldots$ In particular, the three-dimensional volume form satisfies $\epsilon_{\boldsymbol{ijk}} = \epsilon_{\perp\boldsymbol{ijk}}$, and the electric and magnetic parts of the antisymmetric tensor F_{ab} are represented, respectively, by

$$F_{\boldsymbol{i}} = F_{\boldsymbol{i}\perp}, \qquad F^*_{\boldsymbol{i}} = F^*_{\boldsymbol{i}\perp}.$$

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In the case of the Weyl candidate W_{abcd} one has that the tensors W_{ab}, W^*_{ab} and W_{abc} correspond to

$$W_{\boldsymbol{ij}} = W_{\boldsymbol{i}\perp\boldsymbol{j}\perp}, \qquad W^*_{\boldsymbol{ij}} = W^*_{\boldsymbol{i}\perp\boldsymbol{j}\perp}, \qquad W_{\boldsymbol{ijk}} = W_{\boldsymbol{i}\perp\boldsymbol{jk}}.$$

11.2 Basic notions of elliptic equations

Elliptic differential operators arise naturally in the study of the constraint equations of general relativity on spacelike hypersurfaces. In view of this, some basic properties of elliptic operators on Riemannian manifolds are briefly discussed.

Let $(\mathcal{S}, \boldsymbol{h})$ denote a Riemannian three-dimensional manifold with $\boldsymbol{h}$ a negative definite metric. A ***linear differential operator*** of order M over $\mathcal{S}$ is a map between tensor bundles

$$\mathbf{L} : \mathfrak{T}_{i_1\cdots i_S}(\mathcal{S}) \to \mathfrak{T}_{k_1\cdots k_N}(\mathcal{S}), \qquad S,\, N \in \mathbb{N},$$

of the form

$$(\mathbf{L}v)_{k_1\cdots k_N} \equiv \sum_{r=0}^{M} a^{j_1\cdots j_r i_1\cdots i_S}{}_{k_1\cdots k_N} D_{j_1}\cdots D_{j_r} v_{i_1\cdots i_S}, \tag{11.11}$$

for a smooth $v_{i_1\cdots i_S} \in \mathfrak{T}_{i_1\cdots i_S}(\mathcal{S})$ and where the coefficients $a^{j_1\cdots j_r i_1\cdots i_S}{}_{k_1\cdots k_N}$ are smooth functions over $\mathcal{S}$. The ***principal part*** of $\mathbf{L}$ consists of the terms in Equation (11.11) with the highest order derivatives, that is,

$$a^{j_1\cdots j_M i_1\cdots i_S}{}_{k_1\cdots k_N} D_{j_1}\cdots D_{j_M} v_{i_1\cdots i_S}.$$

Closely related to the principal part is the ***symbol of*** $\mathbf{L}$, $\boldsymbol{\sigma}_{\mathbf{L}}(\boldsymbol{\xi})$, defined pointwise on $\mathcal{S}$, for $\boldsymbol{\xi} \in T^*|_p(\mathcal{S})$ as the linear map

$$\boldsymbol{\sigma}_{\mathbf{L}}(\boldsymbol{\xi}) : T_{i_1\cdots i_S}|_p(\mathcal{S}) \to T_{k_1\cdots k_N}|_p(\mathcal{S}),$$

given by

$$(\boldsymbol{\sigma}_{\mathbf{L}}(\boldsymbol{\xi})v)_{k_1\cdots k_N} \equiv a^{j_1\cdots j_M i_1\cdots i_S}{}_{k_1\cdots k_N} \xi_{j_1}\cdots \xi_{j_M} v_{i_1\cdots i_S}.$$

Observe that the symbol is obtained by the formal replacement of the derivatives $D_i \mapsto \xi_i$ in the principal part of the operator. The symbol $\boldsymbol{\sigma}_{\mathbf{L}}(\boldsymbol{\xi})$ determines the nature of the differential operator. In particular, $\mathbf{L}$ is said to be ***underdetermined elliptic*** at $p \in \mathcal{S}$ if $\boldsymbol{\sigma}_{\mathbf{L}}(\boldsymbol{\xi})$ is *surjective* for all $\boldsymbol{\xi} \neq 0$; $\mathbf{L}$ is ***overdetermined elliptic*** at $p \in \mathcal{S}$ if $\boldsymbol{\sigma}_{\mathbf{L}}(\boldsymbol{\xi})$ is *injective*. Finally, $\mathbf{L}$ is ***elliptic*** if $\boldsymbol{\sigma}_{\mathbf{L}}(\boldsymbol{\xi})$ is *bijective*, that is, if it is injective and surjective. If the coefficients $a^{j_1\cdots j_r i_1\cdots i_S}{}_{k_1\cdots k_N}$ in the operator (11.11) depend not only on the point on $\mathcal{S}$ but also on the derivatives $D_{j_1}\cdots D_{j_l}$, $l < r$, then $\mathbf{L}$ is said to

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be ***quasilinear***. The definitions of (underdetermined, overdetermined) elliptic differential operators extend in a natural way to the quasilinear case.

The paradigmatic example of an elliptic operator is the ***Laplace operator*** of the metric $\boldsymbol{h}$:

$$\Delta_{\boldsymbol{h}}\phi \equiv h^{ij} D_i D_j \phi, \qquad \phi \in \mathfrak{X}(\mathcal{S}).$$

In this case the operator is equal to its principal part. Moreover, its symbol is given by $h^{ij}\xi_i\xi_j < 0$ for $\xi_i \neq 0$ (as a consequence of negative-definiteness), from where it follows that the symbol is a bijection and, hence, $\Delta_{\boldsymbol{h}}$ is an elliptic operator. Particular examples of overdetermined and underdetermined elliptic operators are discussed in Section 11.3.3.

Associated to the differential operator $\mathbf{L}$ in (11.11) one has its ***formal adjoint*** $\mathbf{L}^*$ given by

$$(\mathbf{L}^* u)^{i_1\cdots i_S} \equiv \sum_{r=0}^{M} (-1)^r D_{j_1}\cdots D_{j_r}(a^{j_1\cdots j_r i_1\cdots i_S}{}_{k_1\cdots k_N} u^{k_1\cdots k_N}),$$

for smooth $u^{k_1\cdots k_N} \in \mathfrak{T}^{k_1\cdots k_N}(\mathcal{S})$. The above expression comes from the identity between ***inner products***

$$\int_{\mathcal{S}} (\mathbf{L}v)_{k_1\cdots k_N} u^{k_1\cdots k_N} \mathbf{d}\mu_{\boldsymbol{h}} = \int_{\mathcal{S}} v_{i_1\cdots i_S} (\mathbf{L}^* u)^{i_1\cdots i_S} \mathbf{d}\mu_{\boldsymbol{h}}, \tag{11.12}$$

which is obtained by repeated integration by parts. In the previous expression, $\mathbf{d}\mu_{\boldsymbol{h}}$ denotes the volume element of $\boldsymbol{h}$. For simplicity, in the identity (11.12) it is assumed that $\mathcal{S}$ is a compact manifold so that the integrals are well defined. Important for the subsequent discussion is the fact (verifiable using the definitions given in the previous paragraphs) that $\mathbf{L}$ is an underdetermined elliptic operator if and only if $\mathbf{L}^*$ is overdetermined elliptic. Moreover, if $\mathbf{L}$ is underdetermined elliptic, then $\mathbf{L} \circ \mathbf{L}^*$ is elliptic.

The interested reader is referred to appendix II in Choquet-Bruhat (2008) for further details on the theory of elliptic equations. An alternative summary can be found in the appendix of Besse (2008).

11.3 The Hamiltonian and momentum constraints

Before proceeding to analyse the conformal constraint equations, it is convenient to discuss the intrinsic equations implied by the Einstein field equations

$$\tilde{R}_{ab} - \frac{1}{2}\tilde{R}\tilde{g}_{ab} + \lambda\tilde{g}_{ab} = \tilde{T}_{ab}$$

on a non-null hypersurface of a spacetime $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ – the so-called Einstein constraint equations.

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11.3.1 Derivation of the Einstein constraint equations

Starting from the *Gauss-Codazzi identity*, Equation (2.47) and contracting with $\tilde{h}^{\boldsymbol{ik}}$ one obtains

$$\begin{aligned}\tilde{r}_{\boldsymbol{jl}} + \tilde{K}\tilde{K}_{\boldsymbol{jl}} - \tilde{K}^{\boldsymbol{k}}{}_{\boldsymbol{j}}\tilde{K}_{\boldsymbol{kl}} &= \tilde{h}^{\boldsymbol{ik}}\tilde{R}_{\boldsymbol{ijkl}} \\ &= \eta^{\boldsymbol{ab}}\tilde{R}_{\boldsymbol{ajbl}} - \epsilon\tilde{R}_{\perp\boldsymbol{j}\perp\boldsymbol{l}} \\ &= \tilde{R}_{\boldsymbol{jl}} - \epsilon\tilde{R}_{\perp\boldsymbol{j}\perp\boldsymbol{l}}.\end{aligned}$$

Contracting this last equation with $\tilde{h}^{\boldsymbol{jl}}$ one finally obtains

$$\begin{aligned}\tilde{r} + \tilde{K}^2 - \tilde{K}_{\boldsymbol{jl}}\tilde{K}^{\boldsymbol{jl}} &= \tilde{h}^{\boldsymbol{jl}}\tilde{R}_{\boldsymbol{jl}} - \epsilon\tilde{h}^{\boldsymbol{jl}}\tilde{R}_{\perp\boldsymbol{j}\perp\boldsymbol{l}} \\ &= \eta^{\boldsymbol{ab}}\tilde{R}_{\boldsymbol{ab}} - \epsilon\tilde{R}_{\perp\perp} - \epsilon\eta^{\boldsymbol{ab}}\tilde{R}_{\perp\boldsymbol{a}\perp\boldsymbol{b}} \\ &= \tilde{R} - 2\epsilon\tilde{R}_{\perp\perp}.\end{aligned}$$

Similarly, starting from the *Codazzi-Mainardi identity, Equation* (2.48), and contracting with $\tilde{h}^{\boldsymbol{ij}}$ one has that

$$\begin{aligned}\tilde{D}^{\boldsymbol{j}}\tilde{K}_{\boldsymbol{kj}} - \tilde{D}_{\boldsymbol{k}}\tilde{K} &= \tilde{h}^{\boldsymbol{ij}}\tilde{R}_{\boldsymbol{i}\perp\boldsymbol{jk}} \\ &= \eta^{\boldsymbol{ij}}\tilde{R}_{\boldsymbol{i}\perp\boldsymbol{jk}} = \tilde{R}_{\perp\boldsymbol{k}},\end{aligned}$$

where to pass from the first to the second line one uses that $\tilde{R}_{\perp\perp\boldsymbol{jk}} = 0$.

Using the Einstein field equations in the frame component form

$$\tilde{R}_{\boldsymbol{ab}} - \frac{1}{2}\eta_{\boldsymbol{ab}}\tilde{R} + \lambda\eta_{\boldsymbol{ab}} = \tilde{T}_{\boldsymbol{ab}}$$

one obtains the so-called ***Einstein constraint equations***

$$\tilde{r} + \tilde{K}^2 - \tilde{K}_{\boldsymbol{jl}}\tilde{K}^{\boldsymbol{jl}} = 2(\lambda - \epsilon\tilde{\varrho}), \tag{11.13a}$$

$$\tilde{D}^{\boldsymbol{j}}\tilde{K}_{\boldsymbol{kj}} - \tilde{D}_{\boldsymbol{k}}\tilde{K} = \tilde{j}_{\boldsymbol{k}}, \tag{11.13b}$$

where

$$\tilde{\varrho} \equiv \tilde{T}_{\perp\perp}, \qquad \tilde{j}_{\boldsymbol{k}} \equiv \tilde{T}_{\perp\boldsymbol{k}}$$

are, respectively, the ***energy density*** and the components of the ***energy flux vector*** of the energy-momentum tensor in the direction of $\tilde{\boldsymbol{\nu}}$. Equations (11.13a) and (11.13b) are known, respectively, as the ***Hamiltonian constraint*** and the ***momentum constraint***. The tensorial version of Equations (11.13a) and (11.13b) is given by

$$\tilde{r} + \tilde{K}^2 - \tilde{K}_{jl}\tilde{K}^{jl} = 2(\lambda - \epsilon\tilde{\varrho}), \qquad \tilde{D}^j\tilde{K}_{kj} - \tilde{D}_k\tilde{K} = \tilde{j}_k. \tag{11.14}$$

Finally, it is observed that in index-free notation the constraint equations can be written as

$$r[\tilde{\boldsymbol{h}}] + (\mathbf{tr}_{\tilde{\boldsymbol{h}}}\tilde{\boldsymbol{K}})^2 - |\tilde{\boldsymbol{K}}|^2_{\tilde{\boldsymbol{h}}} = 2(\lambda - \epsilon\tilde{\varrho}), \qquad \mathbf{div}_{\tilde{\boldsymbol{h}}}\tilde{\boldsymbol{K}} - \mathbf{grad}\,\mathbf{tr}_{\tilde{\boldsymbol{h}}}\tilde{\boldsymbol{K}} = \tilde{\boldsymbol{j}}.$$

In what follows, a collection $(\tilde{\mathcal{S}}, \tilde{\boldsymbol{h}}, \tilde{\boldsymbol{K}}, \tilde{\varrho}, \tilde{\boldsymbol{j}})$ such that the negative definite metric $\tilde{\boldsymbol{h}}$ and the symmetric rank-2 tensor $\tilde{\boldsymbol{K}}$ satisfy the Einstein constraints (11.14) with $\epsilon = 1$ on the three-dimensional manifold $\tilde{\mathcal{S}}$ will be known as an ***initial data set*** for the Einstein field equations. If $\tilde{\varrho} = 0$ and $\tilde{\boldsymbol{j}} = 0$, one speaks of a ***vacuum initial data set***.

An important class of initial data sets is that for which $\tilde{\boldsymbol{K}} = 0$ and $\tilde{\boldsymbol{j}} = 0$, so that one is left only with the Hamiltonian constraint in the form

$$r[\tilde{\boldsymbol{h}}] = 2(\lambda - \tilde{\rho}).$$

Such an initial data set is called ***time reflection symmetric*** (or ***time symmetric*** for short); it follows from the properties of the *Einstein reduced equations* that for this type of initial data one has $\partial_t h_{\alpha\beta} = 0$ on the initial hypersurface $\tilde{\mathcal{S}}$ so that the resulting solution to the Einstein field equations is invariant under the replacement $t \mapsto -t$.

11.3.2 The conformal Hamiltonian and momentum constraint equations

Regarding, as in Section 11.1, the three-dimensional manifold $\tilde{\mathcal{S}}$ as a hypersurface on both $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ and $(\mathcal{M}, \boldsymbol{g})$, it follows from a computation using the transformation rules (11.1a) and (11.1b) together with the transformation rules for the Ricci scalar, Equation (5.16c), that Equation (11.14) can be reexpressed in terms of unphysical quantities as:

$$\begin{aligned} &2\Omega D_i D^i \Omega - 3 D_i \Omega D^i \Omega + \frac{1}{2}\Omega^2 r - 3\epsilon \Sigma^2 \\ &\qquad + \frac{1}{2}\Omega^2 \big(K^2 - K_{ij}K^{ij}\big) + 2\epsilon \Omega \Sigma K = \lambda - \epsilon \Omega^4 \varrho, \qquad (11.15a) \\ &\Omega^3 D^i\big(\Omega^{-2} K_{ik}\big) - \Omega\big(D_k K - 2\Omega^{-1} D_k \Sigma\big) = \Omega^3 j_k, \qquad (11.15b) \end{aligned}$$

where

$$\varrho \equiv \Omega^{-4}\tilde{\varrho}, \qquad j_k \equiv \Omega^{-3}\tilde{j}_k, \qquad (11.16)$$

denote, respectively, the ***unphysical energy density*** and the ***flux vector***.

11.3.3 The Hamiltonian and momentum constraint as an elliptic system

The Einstein constraint Equations (11.14) on a spacelike manifold $\tilde{\mathcal{S}}$ (i.e. $\epsilon = 1$) have been studied extensively in the literature; see, for example, Bartnik and Isenberg (2004) for a review of the topic and see also Choquet-Bruhat (2008), chapter 7, and Choquet-Bruhat and York (1980). In this section an adaptation of the so-called ***conformal method*** of Licnerowicz, Choquet-Bruhat and York to analyse the conformal Hamiltonian and momentum constraints (11.15a) and (11.15b) will be discussed; see, for example, York (1971, 1972). This approach

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works directly on a compact *unphysical manifold* $\mathcal{S}$ which is a conformal extension of the physical manifold $\tilde{\mathcal{S}}$. The key idea in this analysis is to show that these constraint equations imply an elliptic system of equations for suitable conformal fields. Proceeding in this way, one also obtains an insight into the nature of the *freely specifiable data* in the Einstein constraints. The use of a compact manifold $\mathcal{S}$ simplifies some of the technical aspects of the analysis. This approach to the Einstein constraint equations has been advocated in Friedrich (1988, 1998c, 2004, 2013), Dain and Friedrich (2001) and Beig and O'Murchadha (1991, 1994).

Following the discussion of the previous paragraph, let $(\mathcal{S}, \boldsymbol{h})$ denote a compact Riemannian manifold with $\boldsymbol{h}$ negative definite and set $\epsilon = 1$ so that $\mathcal{S}$ can be regarded as a spacelike hypersurface of an unphysical spacetime $(\mathcal{M}, \boldsymbol{g})$. In what follows, for simplicity, it is assumed that the matter fields ϱ and $\boldsymbol{j}$ are known on $\mathcal{S}$.

The first step to transform Equations (11.15a) and (11.15b) into an elliptic system is given by the transformation law of the three-dimensional Ricci scalar, Equation (5.17), which suggests introducing a conformal factor ϑ satisfying $\Omega = \vartheta^{-2}$. By substituting this definition into Equation (11.15a) one finds that

$$\Delta_{\boldsymbol{h}}\vartheta - \frac{1}{8}r[\boldsymbol{h}]\vartheta = \frac{1}{8}(K_{ij}K^{ij} - K^2)\vartheta + \frac{1}{4}(\vartheta^{-3}\varrho - \vartheta^5\lambda) + \frac{3}{4}\Sigma^2\vartheta^5 - \frac{1}{2}\vartheta^3\Sigma K, \tag{11.17}$$

where, as before, $\Delta_{\boldsymbol{h}} \equiv h^{ij}D_iD_j$ and the notation $r[\boldsymbol{h}]$ has been used to make explicit the dependence of the Ricci scalar on the metric $\boldsymbol{h}$. Following the standard use in the literature, this equation will be known as the ***Licnerowicz equation***. If the fields $\boldsymbol{h}$ (and hence $r[\boldsymbol{h}]$), K_{ij}, K, ϱ and Σ are known, this last equation can be read as a *non-linear elliptic equation* determining ϑ. For future use, it is convenient to define the ***Yamabe operator*** $\mathbf{L}_{\boldsymbol{h}} : \mathfrak{X}(\mathcal{S}) \to \mathfrak{X}(\mathcal{S})$ as

$$\mathbf{L}_{\boldsymbol{h}}\vartheta \equiv \Delta_{\boldsymbol{h}}\vartheta - \frac{1}{8}r[\boldsymbol{h}]\vartheta, \tag{11.18}$$

so that Equation (11.17) can be rewritten as

$$\mathbf{L}_{\boldsymbol{h}}\vartheta = \frac{1}{8}(K_{ij}K^{ij} - K^2)\vartheta + \frac{1}{4}(\varrho - \lambda)\vartheta^{-3} + \frac{1}{2}\Sigma\vartheta^3\left(K - \frac{1}{6}\vartheta^2\Sigma\right).$$

The Yamabe operator has nice conformal transformation properties; see Equation (11.23) below.

Equation (11.15b) suggests that the extrinsic curvature K_{ij} should be split into a trace-free part multiplied by a power of the conformal factor and a pure trace part. In this spirit one writes

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$$K_{ij} = \vartheta^{-4}\psi_{ij} + \frac{1}{3}Kh_{ij}, \qquad h^{ij}\psi_{ij} = 0,$$

which, substituted into (11.15b), yields

$$D^i\psi_{ij} = \frac{2}{3}\vartheta^6 D_j(\vartheta^{-2}K) - 2\vartheta^{-6}D_j\Sigma + j_j.$$

In view of the latter, it is convenient to reintroduce the physical trace $\tilde{K} = \Omega K = \vartheta^{-2}K$ so that one obtains

$$D^i\psi_{ij} = \frac{2}{3}\vartheta^6 D_j\tilde{K} - 2\vartheta^{-4}D_j\Sigma + j_j. \tag{11.19}$$

This last equation is to be read as an equation for the trace-free tensor ψ_{ij}. If $\tilde{K}$ is a constant and $\Sigma = 0$, then Equations (11.17) and (11.19) decouple.

Following the discussion of Section 11.2 it can be verified that the principal part of Equation (11.19) is *underdetermined elliptic*. To transform Equation (11.19) into an elliptic equation one makes use of a so-called ***York splitting***; see York (1973). One considers an ansatz for ψ_{ij} of the form

$$\psi_{ij} = D_i\varsigma_j + D_j\varsigma_i - \frac{2}{3}h_{ij}D_k\varsigma^k + \psi'_{ij}, \tag{11.20}$$

where ς_i is some covector on $\mathcal{S}$ and ψ'_{ij} is a freely specifiable symmetric and trace-free tensor. The operator $(\boldsymbol{\mathcal{L}}_{\boldsymbol{h}}\varsigma)_i$ defined by

$$(\boldsymbol{\mathcal{L}}_{\boldsymbol{h}}\varsigma)_i \equiv D_i\varsigma_j + D_j\varsigma_i - \frac{2}{3}h_{ij}D_k\varsigma^k,$$

is called the ***conformal Killing operator***. It can be verified to be the formal adjoint of the divergence operator acting on symmetric trace-free tensors. Substituting the ansatz (11.20) into Equation (11.19) one obtains

$$\Delta_{\boldsymbol{h}}\varsigma_j + D^iD_j\varsigma_i - \frac{2}{3}D_jD_k\varsigma^k = \frac{2}{3}\vartheta^6 D_j\tilde{K} - 2\vartheta^{-4}D_j\Sigma + j_j - D^i\psi'_{ij}. \tag{11.21}$$

The symbol of this equation can be seen to be

$$(\boldsymbol{\sigma}_{\mathbf{div}\circ\boldsymbol{\mathcal{L}}}(\boldsymbol{\xi})\boldsymbol{\varsigma})_j = \xi^i\xi_i\varsigma_j + \xi^i\xi_j\varsigma_i - \frac{2}{3}\xi_j\xi^k\varsigma_k.$$

Contracting with ξ^j one immediately finds that

$$(\boldsymbol{\sigma}_{\mathbf{div}\circ\boldsymbol{\mathcal{L}}}(\boldsymbol{\xi})\boldsymbol{\varsigma})_j\xi^j = (\xi_i\xi^i)(\varsigma_k\varsigma^k) + \frac{1}{3}(\varsigma_i\xi^i)^2 > 0 \qquad \text{for} \qquad \xi_i,\, \varsigma_j \neq 0.$$

Thus, it follows that (11.21) is a linear elliptic equation for the covector ς_i. The freely specifiable data for this equation is the symmetric trace-free tensor ψ'_{ij}. As in the case of Equation (11.19) it decouples from the Licnerowicz Equation (11.17) if $\tilde{K}$ is constant and $\Sigma = 0$. The analysis of the coupled system (11.17)–(11.19) is much more challenging; see, for example, Holst et al. (2008a,b).

Gauge freedom

The conformal method described in the previous paragraphs has a ***conformal gauge freedom***. More precisely, if ϕ is a positive function on $\mathcal{S}$, then a direct computation shows that the transitions

$$h_{ij} \mapsto \phi^4 h_{ij}, \quad \psi_{ij} \mapsto \phi^{-2}\psi_{ij}, \quad \Omega \mapsto \phi^2\Omega, \quad K_{ij} \mapsto \phi^2 K_{ij}, \tag{11.22a}$$

$$\Sigma \mapsto \phi^2\Sigma, \quad \varrho \mapsto \phi^{-8}\varrho, \quad j_i \mapsto \phi^{-6} j_i, \tag{11.22b}$$

yield another solution to the conformal constraint Equations (11.15a) and (11.15b) with the same physical data $(\tilde{\boldsymbol{h}}, \tilde{\boldsymbol{K}})$. This gauge freedom can be exploited to simplify certain specific computations. In particular, letting $\boldsymbol{h}' = \phi^4 \boldsymbol{h}$, a calculation using the transformation laws for conformal transformations shows that

$$\phi^{-5}\big(\Delta_{\boldsymbol{h}} - \frac{1}{8} r[\boldsymbol{h}]\big)\vartheta = \big(\Delta_{\boldsymbol{h}'} - \frac{1}{8} r[\boldsymbol{h}']\big)(\phi^{-1}\vartheta), \tag{11.23}$$

that is,

$$\phi^{-5}\mathbf{L}_{\boldsymbol{h}}[\vartheta] = \mathbf{L}_{\boldsymbol{h}'}(\phi^{-1}\vartheta).$$

11.3.4 The Yamabe problem

A classic question of Differential Geometry is the so-called ***Yamabe problem*** which, given a compact three-dimensional Riemannian manifold $(\mathcal{S}, \boldsymbol{h})$, asks whether it is possible to conformally rescale the (smooth) metric $\boldsymbol{h}$ to a metric with constant Ricci scalar; see Yamabe (1960). This problem requires finding a positive conformal factor ω and a constant $r_\bullet$ satisfying the equation

$$\Delta_{\boldsymbol{h}}\omega = \frac{1}{8}\big(r[\boldsymbol{h}]\omega - r_\bullet \omega^5\big), \tag{11.24}$$

which follows from the transformation equation for the three-dimensional Ricci scalar Equation (5.17). The Yamabe problem has been solved in the affirmative; see Trudinger (1968), Aubin (1976) and Schoen (1984). In particular, one has the following (e.g. Lee and Parker (1987); O'Murchadha (1988)):

Theorem 11.1 (***resolution of the Yamabe problem***) *Let $\boldsymbol{h}$ be a smooth Riemannian metric on a compact manifold $\mathcal{S}$. There exists a smooth, positive definite function ω on $\mathcal{S}$ such that $r[\omega^4 \boldsymbol{h}]$ is constant.*

Theorem 11.1 allows the classification of Riemannian metrics according to whether they can be rescaled to a metric with constant Ricci scalar which is positive, negative or zero – a given metric $\boldsymbol{h}$ cannot be rescaled to two different metrics with constant curvature of different signs. Thus, the resulting ***Yamabe classes*** are conformal invariants. As will be seen in Section 11.5, this observation plays a role in the construction of initial data sets on compact manifolds. Remarkably, the analogous *Yamabe problem on non-compact manifolds* turns out not to be true as shown by a number of counterexamples; see, for example, Zhiren (1988).

11.4 The conformal constraint equations

Having analysed the standard Einstein constraint equations, focus is now on the constraint equations implied by the conformal Einstein field equations. These equations can be regarded as an extension of the conformal Hamiltonian and momentum constraints (11.15a) and (11.15b).

11.4.1 The derivation of the equations

In this section the frame version of the conformal Einstein field equations, Equations (8.32a) and (8.32b), are considered. By making use of an orthonormal frame adapted to the geometry of the hypersurface under consideration, as described in Section 2.7.3, the split of the equations follows almost directly.

In what follows, let $(\mathcal{M}, \boldsymbol{g})$ denote an *unphysical spacetime* and let $\mathcal{S}$ denote a hypersurface thereof. As in Section 11.1.1, let Σ denote the covariant derivative in the direction of the $\boldsymbol{g}$-unit normal. The evaluation of a spacetime frame index in the direction of the unit normal (i.e. the values $\mathbf{0}$ or $\mathbf{3}$ depending on the causal character of $\mathcal{S}$) will be indicated by the symbol ${}_\perp$.

The constraints implied by $Z_{\boldsymbol{ab}}$. Given

$$Z_{\boldsymbol{ab}} \equiv \nabla_{\boldsymbol{a}}\nabla_{\boldsymbol{b}}\Xi + \Xi L_{\boldsymbol{ab}} - s\eta_{\boldsymbol{ab}} - \frac{1}{2}\Xi^3 T_{\{\boldsymbol{ab}\}}, \tag{11.25}$$

the information of the conformal equation $Z_{\boldsymbol{ab}} = 0$ which is intrinsic to the hypersurface $\mathcal{S}$ is encoded in the components

$$Z_{\boldsymbol{ij}} = 0, \qquad Z_{\perp\boldsymbol{i}} = 0. \tag{11.26}$$

In order to obtain explicit intrinsic expressions for these equations it is observed that

$$\nabla_{\boldsymbol{a}}\nabla_{\boldsymbol{b}}\Xi \equiv \boldsymbol{e_a}{}^a \boldsymbol{e_b}{}^b \nabla_a\nabla_b\Xi = \boldsymbol{e_a}(\boldsymbol{e_b}(\Xi)) - \Gamma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}\boldsymbol{e_c}(\Xi).$$

Hence, in particular, one has that

$$\begin{aligned}\nabla_{\boldsymbol{i}}\nabla_{\boldsymbol{j}}\Xi &= \boldsymbol{e_i}(\boldsymbol{e_j}(\Xi)) - \Gamma_{\boldsymbol{i}}{}^{\boldsymbol{c}}{}_{\boldsymbol{j}}\boldsymbol{e_c}(\Xi) \\ &= \boldsymbol{e_i}(\boldsymbol{e_j}(\Xi)) - \Gamma_{\boldsymbol{i}}{}^{\boldsymbol{k}}{}_{\boldsymbol{j}}\boldsymbol{e_k}(\Xi) - \Gamma_{\boldsymbol{i}}{}^{\perp}{}_{\boldsymbol{j}}\boldsymbol{e}_\perp(\Xi) \\ &= \boldsymbol{e_i}(D_{\boldsymbol{j}}\Xi) - \gamma_{\boldsymbol{i}}{}^{\boldsymbol{k}}{}_{\boldsymbol{j}}D_{\boldsymbol{k}}\Xi + \epsilon K_{\boldsymbol{ij}}\Sigma \\ &= D_{\boldsymbol{i}}D_{\boldsymbol{j}}\Xi + \epsilon K_{\boldsymbol{ij}}\Sigma,\end{aligned} \tag{11.27}$$

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where, in the last term of the last line, Equation (2.45) for the extrinsic curvature has been used. A similar computation shows that

$$\begin{aligned}\nabla_{\boldsymbol{i}}\nabla_{\perp}\Xi &= \boldsymbol{e_i}(\boldsymbol{e_\perp}(\Xi)) - \Gamma_{\boldsymbol{i}}{}^{\boldsymbol{c}}{}_{\perp}\boldsymbol{e_c}(\Xi)\\ &= \boldsymbol{e_i}(\Sigma) - \Gamma_{\boldsymbol{i}}{}^{\boldsymbol{k}}{}_{\perp}\boldsymbol{e_k}(\Xi) - \Gamma_{\boldsymbol{i}}{}^{\perp}{}_{\perp}\Sigma\\ &= D_{\boldsymbol{i}}\Sigma - \epsilon K_{\boldsymbol{i}}{}^{\boldsymbol{k}}D_{\boldsymbol{k}}\Xi, \end{aligned} \tag{11.28}$$

where, in the third line, it has been used that $\Gamma_{\boldsymbol{a}}{}^{\perp}{}_{\perp} = 0$ as a consequence of the metricity of the connection and the fact that $K_{\boldsymbol{i}}{}^{\boldsymbol{k}} = \Gamma_{\boldsymbol{i}}{}^{\boldsymbol{k}}{}_{\perp}$.

Substituting the above expressions into Equation (11.26) and taking into account definition (11.25) one obtains the constraint equations

$$\begin{aligned}&D_{\boldsymbol{i}}D_{\boldsymbol{j}}\Omega = -\epsilon K_{\boldsymbol{ij}}\Sigma - \Omega L_{\boldsymbol{ij}} + sh_{\boldsymbol{ij}} + \frac{1}{2}\Omega\left(T_{\boldsymbol{ij}} - \frac{1}{4}Th_{\boldsymbol{ij}}\right),\\ &D_{\boldsymbol{j}}\Sigma = K_{\boldsymbol{j}}{}^{\boldsymbol{k}}D_{\boldsymbol{k}}\Omega - \Omega L_{\boldsymbol{j}} + \frac{1}{2}\Omega^3 j_{\boldsymbol{j}},\end{aligned}$$

where

$$L_{\boldsymbol{i}} \equiv L_{\boldsymbol{i}\perp}, \qquad \Omega \equiv \Xi|_{\mathcal{S}}.$$

The constraints implied by $Z_{\boldsymbol{a}}$. Given

$$Z_{\boldsymbol{a}} \equiv \nabla_{\boldsymbol{a}}s + L_{\boldsymbol{ac}}\nabla^{\boldsymbol{c}}\Xi - \frac{1}{2}\Xi^2\nabla^{\boldsymbol{c}}\Xi T_{\{\boldsymbol{ac}\}} - \frac{1}{6}\Xi^3\nabla^{\boldsymbol{c}}T_{\{\boldsymbol{ca}\}}, \tag{11.29}$$

the intrinsic information of the equation $Z_{\boldsymbol{a}} = 0$ is encoded in the components

$$Z_{\boldsymbol{i}} = 0. \tag{11.30}$$

Now, the spatial components of the term $L_{\boldsymbol{ab}}\nabla^{\boldsymbol{b}}\Omega$ in Equation (11.29) can be expanded as

$$\begin{aligned}L_{\boldsymbol{ib}}\nabla^{\boldsymbol{b}}\Omega &= L_{\boldsymbol{ib}}\eta^{\boldsymbol{ba}}\nabla_{\boldsymbol{a}}\Omega\\ &= L_{\boldsymbol{i}\perp}\eta^{\perp\perp}\nabla_{\perp}\Omega + L_{\boldsymbol{ik}}\eta^{\boldsymbol{kl}}\nabla_{\boldsymbol{l}}\Omega\\ &= \epsilon L_{\boldsymbol{i}}\Sigma + L_{\boldsymbol{ik}}D^{\boldsymbol{k}}\Omega.\end{aligned}$$

By similar arguments one concludes that

$$\begin{aligned}\nabla^{\boldsymbol{c}}\Xi T_{\{\boldsymbol{ic}\}} &= \epsilon\Sigma j_{\boldsymbol{i}} - T_{\boldsymbol{ik}}D^{\boldsymbol{k}}\Omega - \frac{1}{4}D_{\boldsymbol{i}}T,\\ \nabla^{\boldsymbol{c}}T_{\{\boldsymbol{ic}\}} &= \epsilon\nabla_{\perp}j_{\boldsymbol{i}} + D^{\boldsymbol{k}}T_{\boldsymbol{ki}} - \frac{1}{4}D_{\boldsymbol{i}}T.\end{aligned}$$

It is important to observe in $\nabla^{\boldsymbol{c}}T_{\{\boldsymbol{ic}\}}$ the presence of the term $\nabla_{\perp}j_{\boldsymbol{i}}$ which requires further information about the matter model in order to be cast in a form intrinsic to the hypersurface $\mathcal{S}$. In the case of trace-free matter, one has that $\nabla^{\boldsymbol{c}}T_{\{\boldsymbol{ic}\}} = 0$, so that no further considerations are required.

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From the discussion in the previous paragraphs it follows that Equation (11.30) can be reexpressed as

$$D_i s = -\epsilon L_i \Sigma - L_{ik} D^k \Omega + \frac{1}{2}\Omega^2 \big(\epsilon \Sigma j_i - T_{ik} D^k \Omega - \frac{1}{4} D_i T\big) + \frac{1}{6}\Omega^3 \big(\epsilon \nabla_\perp j_i + D^k T_{ki} - \frac{1}{4} D_i T\big).$$

The constraints implied by Δ_{cdb}. Given

$$\Delta_{cdb} \equiv \nabla_c L_{db} - \nabla_d L_{cb} - \nabla_a \Xi d^a{}_{bcd} - \Xi T_{cdb}, \qquad (11.31)$$

the information intrinsic to the hypersurface $\mathcal{S}$ of the conformal equation $\Delta_{cdb} = 0$ is encoded in the components

$$\Delta_{ijk} = 0, \qquad \Delta_{ij\perp} = 0. \qquad (11.32)$$

A calculation similar to that leading to Equations (11.27) and (11.28) yields

$$\nabla_i L_{jk} = D_i L_{jk} + \epsilon K_{ik} L_j,$$
$$\nabla_i L_j = D_i L_j + K_i{}^k L_{kj}.$$

Given the components d_{abcd} of the rescaled Weyl tensor with respect to the adapted frame $\{e_a\}$, it is convenient to define

$$d_{ij} \equiv d_{i\perp j\perp}, \qquad d_{ijk} \equiv d_{i\perp jk}.$$

Following the discussion of Section 11.1.2, d_{ij} corresponds to the components of the electric part of the rescaled Weyl tensor, while d_{ijk} encodes the information of the magnetic part. It can be verified that

$$d_{ij} = d_{ji}, \qquad d^i{}_i = 0, \qquad d_{ijk} = -d_{ikj}, \qquad d_{[ijk]} = 0, \qquad (11.33a)$$
$$d_{ijkl} = 2(h_{i[k} d_{l]j} + h_{j[l} d_{k]i}). \qquad (11.33b)$$

It follows from the latter expressions, together with (11.31), that the constraints (11.32) can be reexpressed as

$$D_i L_{jk} - D_j L_{ik} = -\epsilon \Sigma d_{ijk} + D^l \Omega d_{lkij} - \epsilon (K_{ik} L_j - K_{jk} L_i) + \Omega T_{ijk},$$
$$D_i L_j - D_j L_i = D^l \Omega d_{lij} + K_i{}^k L_{jk} - K_j{}^k L_{ik} + \Omega J_{ij},$$

where

$$J_{jk} \equiv T_{jk\perp}.$$

The constraints implied by Λ_{bcd}. Given

$$\Lambda_{bcd} \equiv \nabla_a d^a{}_{bcd} - T_{cdb},$$

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as a consequence of the decomposition of the Weyl tensor in electric and magnetic parts, it follows that the information of the equation $\Lambda_{bcd} = 0$ which is intrinsic to the hypersurface $\mathcal{S}$ is contained in the components

$$\Lambda_{\perp ij} = 0, \qquad \Lambda_{\perp j \perp} = 0. \tag{11.34}$$

Observing that

$$\nabla^a d_{aijk} = \eta^{ab}\big(e_a(d_{bijk}) - \Gamma_a{}^c{}_b d_{cijk} - \Gamma_a{}^c{}_i d_{bcjk} - \Gamma_a{}^c{}_j d_{bick} - \Gamma_a{}^c{}_k d_{bijc}\big),$$

one concludes, by arguments similar to those used to obtain Equations (11.27) and (11.28), that

$$\nabla_a d^a{}_{\perp jk} = D^i d_{ijk} + \epsilon(K^i{}_k d_{ji} - K^i{}_j d_{ki}),$$
$$\nabla_a d^a{}_{\perp j\perp} = D^i d_{ij} - K^{ik} d_{ijk}.$$

It follows from the previous discussion that the constraint Equations (11.34) can be reexpressed as

$$D^i d_{ijk} = \epsilon(K^i{}_j d_{ki} - K^i{}_k d_{ji}) + J_{jk},$$
$$D^i d_{ij} = K^{ik} d_{ijk} + J_j,$$

where

$$J_{jk} \equiv T_{jk\perp}, \qquad J_j \equiv T_{j\perp\perp}.$$

The explicit form of J_{jk} and J_j depends on the matter model under consideration. In the case of the electromagnetic field, they can be expressed in terms of the electric and magnetic parts of the Faraday tensor and their spatial derivatives.

The constraint $\mathrm{Z} = 0$**.** Recall that

$$Z \equiv 6\Xi s - 3\nabla_c \Xi \nabla^c \Xi + \frac{1}{4}\Xi^4 T - \lambda.$$

As discussed in Section 8.2.4 the equation $Z = 0$ is, in fact, a constraint equation whose propagation is ensured by the other conformal field equations; see Lemma 8.1. Following the procedure employed in the decomposition of the other conformal equations, it can be expressed in terms of quantities intrinsic to the hypersurface $\mathcal{S}$ as

$$\lambda = 6\Omega s - 3\epsilon\Sigma^2 - 3D_k\Omega D^k\Omega + \frac{1}{4}\Omega^4 T.$$

11.4.2 The Gauss-Codazzi and Codazzi-Mainardi equations in terms of conformal fields

The intrinsic equations discussed in the previous section are supplemented by the Gauss-Codazzi and Codazzi-Mainardi equations, Equations (2.47) and (2.48)

$$R_{ijkl} = r_{ijkl} + K_{ik}K_{jl} - K_{il}K_{jk},$$
$$R_{i\perp jk} = D_j K_{ki} - D_k K_{ji},$$

expressed in terms of conformal fields. As a consequence of the decomposition of the four-dimensional Riemann tensor R_{abcd} in terms of the Weyl and Schouten tensor, Equation (2.21b), one has that

$$R_{abcd} = \Xi d_{abcd} + \eta_{ac}L_{db} - \eta_{ad}L_{cb} + L_{ac}\eta_{db} - L_{ad}\eta_{cb},$$

while the three-dimensional Riemann tensor r_{ijkl} can be expressed in terms of the three-dimensional Schouten tensor l_{ij} as

$$r_{ijkl} = h_{ik}l_{lj} - h_{il}l_{kj} + h_{jl}l_{ki} - h_{jk}l_{li}, \qquad l_{ij} \equiv r_{ij} - \frac{1}{4}rh_{ij};$$

see Equation (2.40). A direct calculation using the above expressions yields the two additional constraint equations

$$D_j K_{ki} - D_k K_{ji} = \Omega d_{ijk} + h_{ij}L_k - h_{ik}L_j,$$
$$l_{ij} = \Omega d_{ij} + L_{ij} - K_k{}^k\big(K_{ij} - \frac{1}{4}Kh_{ij}\big) + K_{ki}K_j{}^k - \frac{1}{4}K_{kl}K^{kl}h_{ij}.$$

These equations provide the link between the spatial curvature tensor l_{ij} and the spacetime curvature as described by d_{ab}, d_{abc}, L_{ab} and L_a.

11.4.3 Summary of the equations and basic properties of the conformal constraint equations

As a summary of the discussion of the previous sections, the conformal constraint equations are collected:

$$D_iD_j\Omega = -\epsilon\Sigma K_{ij} - \Omega L_{ij} + sh_{ij} + \frac{1}{2}\Omega^3\left(T_{ij} - \frac{1}{4}Th_{ij}\right), \tag{11.35a}$$

$$D_i\Sigma = K_i{}^k D_k\Omega - \Omega L_i + \frac{1}{2}\Omega^3 j_i, \tag{11.35b}$$

$$\begin{aligned} D_i s = {} & -\epsilon L_i\Sigma - L_{ik}D^k\Omega + \frac{1}{2}\Omega^2\left(\epsilon\Sigma j_i - T_{ik}D^k\Omega - \frac{1}{4}D_iT\right) \\ & + \frac{1}{6}\Omega^3\left(\epsilon\nabla_\perp j_i + D^kT_{ki} - \frac{1}{4}D_iT\right), \end{aligned} \tag{11.35c}$$

$$\begin{aligned} D_iL_{jk} - D_jL_{ik} = {} & -\epsilon\Sigma d_{kij} + D^l\Omega d_{lkij} \\ & - \epsilon(K_{ik}L_j - K_{jk}L_i) + \Omega T_{ijk}, \end{aligned} \tag{11.35d}$$

$$D_iL_j - D_jL_i = D^l\Omega d_{lij} + K_i{}^kL_{jk} - K_j{}^kL_{ik} + \Omega T_{ij\perp}, \tag{11.35e}$$

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$$D^k d_{kij} = \epsilon\big(K^k{}_i d_{jk} - K^k{}_j d_{ik}\big) + J_{ij}, \tag{11.35f}$$

$$D^i d_{ij} = K^{ik} d_{ijk} + J_j, \tag{11.35g}$$

$$\lambda = 6\Omega s - 3\epsilon\Sigma^2 - 3D_k\Omega D^k\Omega + \frac{1}{4}\Omega^4 T, \tag{11.35h}$$

$$D_j K_{ki} - D_k K_{ji} = \Omega d_{ijk} + h_{ij} L_k - h_{ik} L_j, \tag{11.35i}$$

$$l_{ij} = \Omega d_{ij} + L_{ij} - K(K_{ij} - \frac{1}{4}K h_{ij}) + K_{ki}K_j{}^k - \frac{1}{4}K_{kl}K^{kl}h_{ij}. \tag{11.35j}$$

Using the identity (11.7) and recalling that $d^*_{ij} = -\frac{1}{2}d_{ikl}\epsilon_j{}^{kl}$, Equations (11.35f) and (11.35g) can be rewritten in the alternative form

$$D^i d^*_{ij} = -\epsilon\epsilon_j{}^{kl} K^i{}_k d_{li} - \frac{1}{2}\epsilon_j{}^{kl} J_{kl}, \tag{11.36a}$$

$$D^i d_{ij} = \epsilon^l{}_{jk} K^{ik} d^*_{il} + J_j. \tag{11.36b}$$

The conformal constraint Equations (11.35a)–(11.35j) are not independent since integrability conditions have been used in their derivation. A list of various relations between the vacuum constraint equations can be found in Friedrich (1983). In particular, it can be shown that

$$D_i\left(6\Omega s - 3\epsilon\Sigma^2 - 3D_k\Omega D^k\Omega + \frac{1}{4}\Omega^4 T\right) = 0,$$

consistent with the fact that the left-hand side of Equation (11.35h) equals the cosmological constant λ.

For future reference, it is observed that from Equation (11.35j) it follows that

$$r_{ij} = \Omega d_{ij} + L_{ij} + L_k{}^k h_{ij} - KK_{ij} + K_{ik}K^k{}_j, \tag{11.37a}$$

$$r = 4L_k{}^k - K^2 + K_{ij}K^{ij}. \tag{11.37b}$$

The vacuum version of the conformal constraint equations is obtained by setting the *matter fields* T_{ij}, T, j_i, T_{ijk}, J_i, J_{ij} equal to zero. In the derivation of the conformal constraint Equations (11.35a)–(11.35j), it has been assumed that the connection $\boldsymbol{D}$ is the Levi-Civita connection of the intrinsic metric $\boldsymbol{h}$. Thus, by analogy to the full conformal field equations one also has the relations

$$\sigma_i{}^k{}_j = 0, \qquad \Pi^k{}_{lij} = \pi^k{}_{lij}, \tag{11.38}$$

where $\sigma_i{}^k{}_j$, $\Pi^k{}_{lij}$ and $\pi^k{}_{lij}$ denote, respectively, the components of the ***torsion***, the ***geometric curvature*** and the ***algebraic curvature*** of the connection $\boldsymbol{D}$. Explicitly, one has that

$$\sigma_i{}^k{}_j e_k \equiv [e_i, e_j] - (\gamma_i{}^k{}_j - \gamma_j{}^k{}_i)e_k,$$

$$\Pi^k{}_{lij} \equiv e_i(\gamma_j{}^k{}_l) - e_j(\gamma_i{}^k{}_l) + \gamma_m{}^k{}_l(\gamma_j{}^m{}_i - \gamma_i{}^m{}_j) + \gamma_j{}^m{}_l\gamma_i{}^k{}_m - \gamma_i{}^m{}_l\gamma_j{}^k{}_m,$$

$$\pi_{klij} \equiv h_{ik}l_{lj} - h_{il}l_{kj} + h_{jl}l_{ki} - h_{jk}l_{li}.$$

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Given a collection of matter fields on $\mathcal{S}$,

$$\mathbf{m}_\star \equiv (T_{ij}, T, \varrho, j_i, \nabla_\perp j_i, J_i, J_{ij}),$$

by a ***solution to the conformal constraint equations*** on $\mathcal{S}$ it will be understood a collection

$$\mathbf{u}_\star \equiv (\Omega, \Sigma, s, \boldsymbol{e}_i, \gamma_i{}^k{}_j, K_{ij}, L_{ij}, L_i, d_{ij}, d_{ijk})$$

satisfying Equations (11.35a)–(11.35j) together with the supplementary conditions (11.38).

The relation between the conformal constraint Equations (11.35a)–(11.35j) and the conformal Hamiltonian and momentum constraints (11.15a)–(11.15b) is summarised in the following lemma.

Lemma 11.1 (***relation between the solutions to the Einstein constraints and the conformal constraints***) *A solution to the conformal constraints* (11.35a)–(11.35j) *for a collection* $\mathbf{m}_\star$ *of matter fields implies a solution to the conformal Hamiltonian and momentum constraints* (11.15a) *and* (11.15b)*. Conversely, a solution of* (11.15a) *and* (11.15b) *together with a collection of matter fields* $\mathbf{m}_\star$ *gives rise to a solution to* (11.35a)–(11.35j) *on the points of* $\mathcal{S}$ *for which* $\Omega \neq 0$.

Proof Using Equations (11.35a) and (11.35h) to eliminate $L_k{}^k$ one readily obtains the conformal Hamiltonian Equation (11.15a). Similarly, starting from Equation (11.35i) and using Equation (11.35b) to eliminate L_i one obtains the conformal momentum constraint (11.15b). Thus, any solution to the conformal constraints (11.35a)–(11.35j) implies a solution to the conformal Hamiltonian and momentum constraints, Equations (11.15a) and (11.15b).

Assume now one has a collection $(\Omega, \boldsymbol{h}, \boldsymbol{K}, \Sigma, \varrho, j_i)$ satisfying Equations (11.15a) and (11.15b) together with a collection $(T_{ij}, T, J_i, J_{ij}, \nabla_\perp j_i)$ consistent with the matter fields ϱ and j_i . Let now $\{\boldsymbol{e}_i\}$ denote an $\boldsymbol{h}$-orthonormal frame. Using this frame one can compute the components l_{ij} and K_{ij} of the three-dimensional Schouten tensor and of the extrinsic curvature. If $\Omega \neq 0$, one can use the conformal constraint (11.35h) to compute the field s. Next, one makes use of Equations (11.35a) and (11.35b) to compute L_{ij} and L_i. A computation using the commutator of the covariant derivative D_i shows that Equation (11.35c) is automatically satisfied. Once the components L_{ij} and L_i are known, one can use Equations (11.35i) and (11.35j), respectively, to compute d_{ijk} and d_{ij} – it can be verified that the resulting fields are trace free. A final computation using the three-dimensional Bianchi identity in the form

$$D^i r_{ij} = \frac{1}{4} D_j r,$$

together with the irreducible decomposition of the three-dimensional Riemann tensor r_{ijkl}, the decomposition of d_{ijkl} into the electric and magnetic parts and the commutator of D_i shows that Equations (11.35d)–(11.35g) are also automatically satisfied. Thus, the fields obtained constitute the required solution to the conformal constraint equations; see Friedrich (1983) for further details. □

Remark. In order to make assertions about the behaviour of solutions to the conformal constraint equations at points where $\Omega = 0$, the equations need to be supplemented with boundary conditions. Several different classes of boundary conditions on three-dimensional manifolds will be considered: ***compact manifolds***, ***asymptotically Euclidean manifolds*** and ***hyperboloidal manifolds***.

11.4.4 The conformal constraints at the conformal boundary

By construction, the conformal constraint equations can be evaluated in a regular manner at a non-null hypersurface belonging to the conformal boundary of spacetime. By definition such a hypersurface satisfies the conditions

$$\Omega = 0, \qquad \mathbf{d}\Omega \neq 0.$$

Following the convention introduced in Chapter 6 this hypersurface will be denoted by $\mathscr{I}$. The null case will be discussed in Chapter 18.

The defining properties of the hypersurface $\mathscr{I}$ lead to a number of simplifications in the conformal constraint equations. In particular, $\mathbf{d}\Omega$ is normal to $\mathscr{I}$ so that, in terms of a tetrad adapted to the hypersurface, one has $D_i\Omega = 0$. Assuming that the matter fields T_{ij}, T and T_{ijk} are smooth at $\mathscr{I}$ one finds that on the hypersurface the conformal constraints (11.35a)–(11.35j) imply the equations

$$s h_{ij} \simeq \epsilon \Sigma K_{ij}, \tag{11.39a}$$

$$D_i \Sigma \simeq 0, \tag{11.39b}$$

$$D_i s \simeq -\epsilon L_i \Sigma, \tag{11.39c}$$

$$D_i L_{jk} - D_j L_{ik} \simeq -\epsilon \Sigma d_{ijk} - \epsilon (K_{ik} L_j - K_{jk} L_i), \tag{11.39d}$$

$$D_i L_j - D_j L_i \simeq K_i{}^k L_{jk} - K_j{}^k L_{ik}, \tag{11.39e}$$

$$D^k d_{kij} \simeq \epsilon \big(K^k{}_i d_{jk} - K^k{}_j d_{ik} \big) + J_{ij}, \tag{11.39f}$$

$$D^i d_{ij} \simeq K^{ik} d_{ijk} + J_j, \tag{11.39g}$$

$$\lambda \simeq -3\epsilon \Sigma^2, \tag{11.39h}$$

$$D_j K_{ki} - D_k K_{ji} \simeq h_{ij} L_k - h_{ik} L_j, \tag{11.39i}$$

$$l_{ij} \simeq L_{ij} - K \left(K_{ij} - \frac{1}{4} K h_{ij} \right) + K_{ki} K_j{}^k - \frac{1}{4} K_{kl} K^{kl} h_{ij}, \tag{11.39j}$$

where $\simeq$ denotes equality at the conformal boundary. From Equations (11.39b) and (11.39h) it follows that Σ is a constant on $\mathscr{I}$ with a value given by

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$\Sigma = \sqrt{-\epsilon\lambda/3}$ – observe that if $\epsilon = 1$, then $\lambda < 0$, and if $\epsilon = -1$, then $\lambda > 0$, for the previous expression to make sense. Moreover, from Equation (11.39a) the extrinsic curvature of $\mathscr{I}$ is proportional to the intrinsic metric.

A procedure for constructing solutions to Equations (11.39a)–(11.39j) in the vacuum case (so that $J_{\boldsymbol{jk}} = 0$, $J_{\boldsymbol{j}} = 0$) has been given in Friedrich (1986a, 1995). The fundamental idea is to identify the function s and the 3-metric $\boldsymbol{h}$ on $\mathscr{I}$ as freely specifiable data. Instead of working directly with s, it is more convenient to use a smooth function $\varkappa \in \mathfrak{X}(\mathcal{S})$ such that

$$s \simeq \Sigma\varkappa. \tag{11.40}$$

It follows directly from Equations (11.39a), (11.39c) and (11.39j) that

$$K_{\boldsymbol{ij}} \simeq \epsilon\varkappa h_{\boldsymbol{ij}}, \qquad L_{\boldsymbol{i}} \simeq -\epsilon D_{\boldsymbol{i}}\varkappa, \qquad L_{\boldsymbol{ij}} \simeq l_{\boldsymbol{ij}} + \frac{1}{2}\varkappa^2 h_{\boldsymbol{ij}}. \tag{11.41}$$

Substituting these expressions into Equation (11.39d) one obtains, after some simplification, that

$$d_{\boldsymbol{ijk}} \simeq -\epsilon\Sigma^{-1} y_{\boldsymbol{ijk}} \tag{11.42}$$

where $y_{\boldsymbol{ijk}} \equiv D_{\boldsymbol{i}} l_{\boldsymbol{jk}} - D_{\boldsymbol{j}} l_{\boldsymbol{ik}}$ denote the components of the Cotton tensor of the metric $\boldsymbol{h}$; see Section 5.2.2. Alternatively, one can write

$$d^*_{\boldsymbol{ij}} \simeq -\epsilon\Sigma^{-1} y_{\boldsymbol{ij}},$$

with $y_{\boldsymbol{ij}} \equiv -\frac{1}{2} y_{\boldsymbol{klj}}\epsilon_{\boldsymbol{i}}{}^{\boldsymbol{kl}}$ the components of the ***Bach tensor***. It can be verified that the integrability conditions (11.39e), (11.39f) and (11.39i) are automatically satisfied by (11.41) and (11.42). Finally, by substituting into Equation (11.39g) one obtains that

$$D^{\boldsymbol{i}} d_{\boldsymbol{ij}} \simeq 0.$$

This is the only differential condition that has to be solved in this procedure. This can be done by means of a York splitting so as to obtain an elliptic equation for the components of a covector.

The discussion of the previous paragraph is summarised in the following:

Proposition 11.1 (***solutions to the conformal constraint equations at the conformal boundary***) *Given a three-dimensional metric* $\boldsymbol{h}$*, an* $\boldsymbol{h}$*-divergence-free and trace-free field* d_{ij} *and a smooth function* $\varkappa$*, the fields* s*,* $K_{\boldsymbol{ij}}$*,* $L_{\boldsymbol{i}}$*,* $L_{\boldsymbol{ij}}$*,* $d_{\boldsymbol{ijk}}$ *as given by Equations* (11.40)*,* (11.41) *and* (11.42) *constitute a solution to the vacuum conformal constraint equations with* $\Omega = 0$*.*

As will be seen in later chapters, a solution to Equations (11.39a)–(11.39j) constitutes, in the case of $\epsilon = 1$ (i.e. $\mathscr{I}$ spacelike), initial data at, say, past null infinity for de Sitter-like spacetimes. In the case $\epsilon = -1$ (i.e. $\mathscr{I}$ timelike), the solution gives boundary data for an anti-de Sitter-like spacetime.

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Remark. The procedure indicated in the previous paragraphs can be extended to the matter case if the field $J_{\boldsymbol{i}}$ is known.

Exploiting the conformal freedom

The conformal freedom inherent to the conformal field equations can be employed to express the solution to the conformal constraint equations at the conformal boundary in an even simpler form. Recall the discussion in Section 8.2.5 on the transformation properties of the various fields appearing in the conformal field equations. In particular, it follows from Equation (8.29b) that, under a rescaling of the form $\boldsymbol{g}' = \vartheta^2 \boldsymbol{g}$ which implies a rescaling

$$\boldsymbol{h}' \simeq \vartheta^2 \boldsymbol{h}$$

of the intrinsic metric of $\mathscr{I}$, the field s on $\mathscr{I}$ transforms as

$$s' \simeq \left(\vartheta^{-1} s + \vartheta^{-2} \nabla^a \vartheta \nabla_a \Xi\right).$$

In particular, *it is always possible to choose ϑ at $\mathscr{I}$ so that locally*

$$s' \simeq 0.$$

Accordingly, in this particular conformal gauge one has that Equation (11.40) implies $\varkappa' = 0$ and, moreover,

$$K'_{\boldsymbol{ij}} \simeq 0, \qquad L'_{\boldsymbol{i}} \simeq 0, \qquad L'_{\boldsymbol{ij}} \simeq l_{\boldsymbol{ij}}.$$

11.5 The constraints on compact manifolds

An important class of initial data sets for the Einstein field equations involves physical 3-manifolds $\tilde{\mathcal{S}}$ which are compact. This type of initial data set is of relevance in the discussion of cosmological models. In particular, in the vacuum case with negative cosmological constant one expects these initial data sets to give rise to de Sitter-like spacetimes. Initial data sets on compact manifolds have been studied extensively in the literature, and there is a good understanding of the required conditions on the free data in order to ensure existence of solutions to the Einstein constraint equations; see, for example, Isenberg (1995).

For this type of initial data one can set, without loss of generality, $\Omega = 1$ and $\Sigma = 0$ and let $\mathcal{S} = \tilde{\mathcal{S}}$. For simplicity of the presentation, in the remainder of this section the discussion is restricted to the vacuum case. Furthermore, it is assumed that the physical mean curvature $\tilde{K}$ is constant so that Equations (11.17) and (11.21) decouple from each other. The fundamental tool in the analysis of the solvability of the constraint equations is given by the ***maximum principle*** for the Laplacian of a Riemannian metric. A convenient formulation of this result is given by (see Isenberg (1995)):

Proposition 11.2 (***maximum principle for compact manifolds***) *Let $(\mathcal{S}, \boldsymbol{h})$ denote a Riemannian manifold with $\mathcal{S}$ compact. Given a smooth $\psi \in \mathfrak{X}(\mathcal{S})$ such that $\Delta_{\boldsymbol{h}}\psi$ has the same sign on the whole of $\mathcal{S}$, then ψ must be a constant.*

As a consequence of the above principle, the equation

$$\Delta_{\boldsymbol{h}}\psi = F(x, \psi),$$

with $\psi > 0$ has no solution if $F(x,\psi)$ does not change sign on $\mathcal{S}$ except for the case where $F(x,\psi) = 0$. Using this observation it is easy to see that certain combinations of free data cannot give rise to solutions of the constraint equations. As an example, consider time-symmetric data (i.e. $\tilde{\boldsymbol{K}} = 0$) with vanishing cosmological constant on a compact manifold $\mathcal{S}$. As a consequence of the conformal gauge freedom given in Equations (11.22a) and (11.22b) and of the *Yamabe theorem*, Theorem 11.1, one can assume that $r[\boldsymbol{h}]$ is a negative constant on $\mathcal{S}$ – such a metric is said to be of ***positive Yamabe class***. An example of this situation is $\mathbb{S}^3$ with its standard metric. One is then left with a Licnerowicz equation of the form

$$\Delta_{\boldsymbol{h}}\vartheta = \frac{1}{8}r[\boldsymbol{h}]\vartheta.$$

If ϑ is required to be positive everywhere on $\mathcal{S}$, it follows that $\Delta_{\boldsymbol{h}}\vartheta < 0$ everywhere so that no positive solution can exist since, as a consequence of the maximum principle, ϑ must be a constant so that $\Delta_{\boldsymbol{h}}\vartheta = 0$ which is a contradiction. To get around this situation one can consider initial data with a negative (i.e. de Sitter-like) cosmological constant. Keeping the time symmetry of the initial data and the condition $r[\boldsymbol{h}] < 0$, one obtains the Licnerowicz equation

$$\Delta_{\boldsymbol{h}}\vartheta = \frac{1}{8}r[\boldsymbol{h}]\vartheta - \frac{1}{4}\lambda\vartheta^3. \tag{11.43}$$

The right-hand side of this equation has no definite sign for positive ϑ, so there is no obstruction to the existence of solutions. In any case, a further argument (not discussed here) is required to show that Equation (11.43) does indeed have a solution.

The methods in Isenberg (1995) allow one to prove the following proposition:

Proposition 11.3 (***solvability of the Einstein constraints with cosmological constant on a compact manifold***) *Let $(\mathcal{S}, \boldsymbol{h})$ be a Riemannian manifold with $\mathcal{S} \approx \mathbb{S}^3$ and $\boldsymbol{h}$ conformal to a metric with constant negative Ricci scalar (positive Yamabe class). Then the vacuum Einstein constraints with de Sitter-like cosmological constant have a solution for an arbitrary choice of the seed metric $\boldsymbol{h}$, trace-free tensor ψ'_{ij} and constant physical mean curvature $\tilde{K}$.*

The initial data sets given by this proposition will be used to construct de Sitter-like spacetimes in Chapter 15.

11.6 Asymptotically Euclidean manifolds

Spacetimes with $\lambda = 0$ can be thought of as describing *isolated systems* for which the effects of cosmological expansion are neglected. An important class of these spacetimes consists of those solutions to the Einstein field equations which are asymptotically simple in the sense of Definition 7.1, that is, asymptotically simple and empty. Proposition 14.3 shows that these spacetimes are globally hyperbolic, suggesting a systematic procedure for their construction through suitable initial data prescribed on a Cauchy hypersurface.

In order to develop intuition, it is convenient to look at the *Minkowski spacetime* $(\mathbb{R}^4, \tilde{\boldsymbol{\eta}})$. A foliation of this spacetime is given by the hypersurfaces of constant time t. These hypersurfaces are Riemannian manifolds of the form $(\mathbb{R}^3, -\boldsymbol{\delta})$. It can be verified that these hypersurfaces are extrinsically flat; that is, their extrinsic curvature $\tilde{\boldsymbol{K}} = 0$ vanishes. Of course, these are not the only possible types of Cauchy hypersurfaces in this spacetime.

As a second example, consider the Schwarzschild spacetime. In terms of the so-called Schwarzschild ***isotropic radial coordinate***

$$\bar{r} \equiv \frac{1}{2}\big(r - m + \sqrt{r(r-2m)}\big),$$

the line element of the spacetime can be rewritten as

$$\tilde{\boldsymbol{g}}_{\mathscr{S}} = \left(\frac{1 - m/2\bar{r}}{1 + m/2\bar{r}}\right)^2 \mathbf{d}t \otimes \mathbf{d}t - \left(1 + \frac{m}{2\bar{r}}\right)^4 (\mathbf{d}\bar{r} \otimes \mathbf{d}\bar{r} + \bar{r}^2 \boldsymbol{\sigma}).$$

An example of a Cauchy hypersurface for this spacetime is given by the $t = 0$ hypersurface. One can verify that the intrinsic metric and the extrinsic curvature of this hypersurface are given, respectively, by

$$\tilde{\boldsymbol{h}}_{\mathscr{S}} = -\left(1 + \frac{m}{2\bar{r}}\right)^4 \boldsymbol{\delta}, \qquad \tilde{\boldsymbol{K}}_{\mathscr{S}} = 0. \tag{11.44}$$

The most general form of the above initial data set is obtained by performing a translation of the radial coordinate to obtain

$$\tilde{\boldsymbol{h}}_{\mathscr{S}} = -\left(1 + \frac{m}{2|y - y_0|}\right)^4 \boldsymbol{\delta}, \tag{11.45}$$

with $|y - y_0|^2 \equiv (y^1 - y_0^1)^2 + (y^2 - y_0^2)^2 + (y^3 - y_0^3)^2$ where $(y^\alpha) = (y^1, y^2, y^3)$ are standard Cartesian coordinates and $(y_0^\alpha) \in \mathbb{R}^3$ arbitrary. Observe that the metric $\tilde{\boldsymbol{h}}_{\mathscr{S}}$ is, in fact, conformally flat and that $\tilde{\boldsymbol{h}}_{\mathscr{S}} \to -\boldsymbol{\delta}$ as $\bar{r} \to \infty$. Moreover, one has that

$$\left(1 + \frac{m}{2\bar{r}}\right)^4 = 1 + \frac{2m}{\bar{r}} + O\left(\frac{1}{\bar{r}^2}\right). \tag{11.46}$$

To understand the behaviour as $\bar{r} \to 0$, it is observed that under the coordinate inversion $\breve{r} \equiv m^2/4\bar{r}$ one has that

$$\tilde{\boldsymbol{h}}_{\mathscr{S}} = -\left(1 + \frac{m}{2\breve{r}}\right)^4 (\mathbf{d}\breve{r} \otimes \mathbf{d}\breve{r} + \breve{r}^2 \boldsymbol{\sigma}).$$

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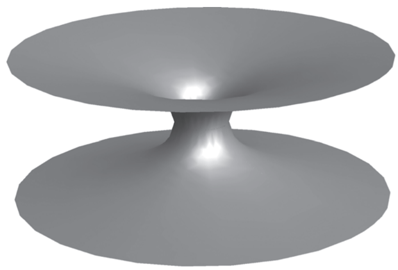

Figure 11.1 Embedding diagram of the Einstein-Rosen bridge in the standard time-symmetric Schwarzschild hypersurface. The diagram is obtained as the surface of revolution of the curve $z = \pm\ln(1+\sqrt{r^2-1})$; see Morris and Thorne (1988) for more details.

Thus, the behaviour of the metric $\tilde{\boldsymbol{h}}$ is identical for both $\bar{r}$, $\breve{r} \to \infty$. There is a discrete reflexion symmetry with respect to the two-dimensional surface $\{r = m/2\}$. Thus, the topology of the hypersurface is $\mathcal{S} \approx \mathbb{R}\times\mathbb{S}^2$. One says that $\mathcal{S}$ has a ***non-trivial topology*** with two asymptotically flat regions (see next section) joined by a so-called ***Einstein-Rosen bridge***. A representation of this is given in Figure 11.1.

An example of an initial data set with non-vanishing extrinsic curvature is given by the family of conformally flat initial data sets for the Schwarzschild spacetime with extrinsic curvature given by

$$\tilde{K}^{\alpha\beta} = \frac{A}{|y|^3}(3y^\alpha y^\beta + |y|^2\delta^{\alpha\beta}),$$

where $|y|^2 = \delta_{\alpha\beta}y^\alpha y^\beta$ and (y^α) are, again, standard Cartesian coordinates; see Beig and O'Murchadha (1998), Estabrook et al. (1973) and Reinhart (1973). It can be verified that $\tilde{K} = \tilde{h}^{\alpha\beta}\tilde{K}_{\alpha\beta} = 0$ as $h_{\alpha\beta}y^\alpha y^\beta = -|y|^2$. This hypersurface has the nontrivial topology of $\mathbb{R}\times\mathbb{S}^2$. However, in contrast to the time-symmetric case, the conformal factor ϑ cannot be written in a closed form. Nevertheless, the leading terms of its asymptotic expansion are the same as in Equation (11.46) with $|y|$ playing the role of the radial coordinate $\bar{r}$.

11.6.1 Definition in terms of physical fields

The hypersurfaces discussed in the previous paragraphs are examples of ***asymptotically Euclidean manifolds***. Given a three-dimensional manifold $\tilde{\mathcal{S}}$, an ***asymptotic end*** is a subset $\tilde{\mathcal{E}} \subset \tilde{\mathcal{S}}$ which is diffeomorphic to the complement of a closed ball on $\mathbb{R}^3$; that is,

$$\tilde{\mathcal{E}} \approx \left\{(y^\alpha) \in \mathbb{R}^3 \mid |y| > r_0\right\},$$

where r_0 is some positive real number and $|y|^2 \equiv \delta_{\alpha\beta}y^\alpha y^\beta$. By identifying $\tilde{\mathcal{E}}$ with the complement of a ball, the triple $\underline{y} = (y^\alpha)$ can be used as coordinates on the asymptotic end – so-called ***asymptotically Cartesian coordinates***. The hypersurfaces of the Minkowski and Schwarzschild spacetimes have, respectively, one and two asymptotic ends. More generally, a three-dimensional manifold $\tilde{\mathcal{S}}$ is said to have N ***asymptotically flat ends*** if there exists a compact subset of $\tilde{\mathcal{S}}$ such that its complement is the union of disjoint subsets $\tilde{\mathcal{E}}_k$, $k = 1, 2, \ldots, N$, each of which is an asymptotic end. In terms of the above concepts one can now introduce the key definition of this section:

Definition 11.1 (***asymptotically Euclidean manifolds***) *An initial data set for the vacuum Einstein field equations* $(\tilde{\mathcal{S}}, \tilde{\boldsymbol{h}}, \tilde{\boldsymbol{K}})$ *is said to be* ***asymptotically Euclidean*** *if* $\tilde{\mathcal{S}}$ *is a three-dimensional manifold with* N *asymptotically flat ends* $\tilde{\mathcal{E}}_k$*,* $k = 1, \ldots, N$ *such that on each* $\tilde{\mathcal{E}}_k$ *the 3-metric and the extrinsic curvature satisfy, in terms of asymptotically Cartesian coordinates on the end, the asymptotic behaviour*

$$\tilde{h}_{\alpha\beta} = -\left(1 + \frac{2m_k}{|y|}\right)\delta_{\alpha\beta} + O_2\left(\frac{1}{|y|^2}\right), \tag{11.47a}$$

$$\tilde{K}_{\alpha\beta} = O_1\left(\frac{1}{|y|^2}\right), \tag{11.47b}$$

where m_k*,* $k = 1, \ldots, N$ *are constants.*

The notation O_1 and O_2 in Equations (11.47a) and (11.47b) is explained in the Appendix to this chapter. More general notions of asymptotic flatness for three-dimensional manifolds have been considered in the literature; see, for example, Chaljub-Simon (1982), Chaljub-Simon and Choquet-Bruhat (1980), Choquet-Bruhat and York (1980) and Christodoulou and O'Murchadha (1981). Their precise formulation require the use of the notion of weighted Sobolev spaces; see, for example, appendix I of Choquet-Bruhat (2008) and Bartnik (1986). These definitions are tailored for the analysis of the elliptic equations arising from the constraint equations.

The asymptotic conditions in Definition 11.1 ensure the finiteness of the ***ADM-linear momentum*** and ***ADM-angular momentum***[1] of each asymptotic end. These asymptotic quantities are given, respectively, by the surface integrals

$$P_\alpha \equiv \frac{1}{8\pi} \lim_{r\to\infty} \int_{S_r} (\tilde{K}_{\alpha\beta} - \tilde{K}\tilde{h}_{\alpha\beta}) n^\beta \mathrm{d}S_{\tilde{\boldsymbol{h}}},$$

$$J_\alpha \equiv \frac{1}{8\pi} \lim_{r\to\infty} \int_{S_r} \tilde{\epsilon}_{\alpha\beta\gamma} y^\beta (\tilde{K}^{\gamma\delta} - \tilde{K}\tilde{h}^{\gamma\delta}) n_\delta \mathrm{d}S_{\tilde{\boldsymbol{h}}}$$

with

$$S_r \equiv \left\{(y^\alpha) \in \tilde{\mathcal{S}} \,|\, |y| = r\right\},$$

[1] ADM stands for Arnowitt-Deser-Miser, pioneers of the *Hamiltonian* formulation of general relativity; see Arnowitt et al. (1962) and Arnowitt et al. (2008) for a republication of this classical review.

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n^α its outward pointing normal and $\mathrm{d}S_{\tilde{\boldsymbol{h}}}$ the surface element induced by $\tilde{\boldsymbol{h}}$ on S_r. The constants m_k in Definition 11.1 correspond to the ***ADM mass*** of each of the asymptotic ends. They are also computable as surface integrals of the *sphere at infinity* via the expression

$$m = -\frac{1}{16\pi}\lim_{r\to\infty}\int_{S_r}\tilde{h}^{\alpha\beta}(\partial_\alpha\tilde{h}_{\beta\gamma} - \partial_\gamma\tilde{h}_{\alpha\beta})n^\gamma\mathrm{d}S_{\tilde{\boldsymbol{h}}}.$$

Strictly speaking, m is the *time component* of a 4-vector, the ***ADM 4-momentum***, whose *spatial components* are given by P^α; thus, it is more accurately described as an *energy*.

Definition 11.1 can be extended to initial data sets with matter. In these situations, decay conditions for the matter sources which are compatible with the decay for $\tilde{\boldsymbol{h}}$ and $\tilde{\boldsymbol{K}}$ are given by (11.47a) and (11.47b). Direct inspection of the constraint Equation (11.14) suggests that

$$\tilde{\varrho} = O\left(\frac{1}{|y|^3}\right), \qquad \tilde{j}_\alpha = O\left(\frac{1}{|y|^3}\right).$$

These conditions can be refined via a more careful analysis of the constraint equations.

It is possible to have an initial data set with several asymptotic ends, some of which are not asymptotically Euclidean. The simplest example is given by the extremal Reissner-Nordström spacetime; see Equation (6.43). The intrinsic metric of the hypersurface $t = 0$ is given, in terms of the extremal Reissner-Nordström ***isotropic radial coordinate*** $\bar{r} = r - m$, by

$$\tilde{\boldsymbol{h}} = -\left(1 + \frac{m}{\bar{r}}\right)^2\boldsymbol{\delta}. \tag{11.48}$$

Clearly

$$\left(1 + \frac{m}{\bar{r}}\right)^2 = 1 + \frac{2m}{\bar{r}} + O\left(\frac{1}{\bar{r}^2}\right) \qquad \text{as } \bar{r}\to\infty.$$

Thus, for large $\bar{r}$, the extremal Reissner-Nordström 3-metric (11.48) has an asymptotically Euclidean end. To discuss the behaviour as $\bar{r}\to 0$, consider the new radial coordinate $\check{r} = -\ln\bar{r}$, so that $\check{r}\to\infty$ as $\bar{r}\to 0$. It follows that in terms of this coordinate the metric (11.48) can be rewritten as

$$\tilde{\boldsymbol{h}} = -\big(m + e^{-\check{r}}\big)(\mathbf{d}\check{r}\otimes\mathbf{d}\check{r} + \boldsymbol{\sigma}).$$

This metric approaches a constant multiple of the standard metric of the cylinder $\mathbb{R}^+\times\mathbb{S}^2$ as $\check{r}\to\infty$. Accordingly, one speaks of a ***cylindrical asymptotic end***. A similar type of asymptotic behaviour can be found, for example, in hypersurfaces of the extremal Kerr spacetime; see, for example, Dain and Gabach-Clement (2011).

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11.6.2 Definition using conformal rescalings

The notion of asymptotically Euclidean manifolds can be strengthened by requiring the physical hypersurface $\tilde{\mathcal{S}}$ to have a conformal extension which is a ***point compactification***. This approach provides a more geometrical setting for the discussion of the asymptotic behaviour of the various fields, that is, independent of the use of particular *asymptotically Cartesian coordinates*. This point of view was first introduced by Geroch (1972b).

Definition 11.2 (***asymptotically Euclidean and regular manifolds***) *A three-dimensional Riemannian manifold* $(\tilde{\mathcal{S}}, \tilde{\boldsymbol{h}})$ *will be said to be* ***asymptotically Euclidean and regular*** *if there exists a three-dimensional, orientable, compact manifold* $(\mathcal{S}, \boldsymbol{h})$ *with points* $i_k \in \mathcal{S}$, $k = 1, \ldots, N$ *with* N *some integer, a diffeomorphism* $\varphi : \mathcal{S} \setminus \{i_1, \ldots, i_N\} \rightarrow \tilde{\mathcal{S}}$ *and a function* $\Omega \in C^2$ *such that:*

(i) $\Omega(i_k) = 0$, $\mathbf{d}\Omega(i_k) = 0$, $\mathbf{Hess}\,\Omega(i_k) = -2\boldsymbol{h}(i_k)$.
(ii) $\Omega > 0$ *on* $\mathcal{S} \setminus \{i_1, \ldots, i_N\}$.
(iii) $\boldsymbol{h} = \Omega^2 \varphi^* \tilde{\boldsymbol{h}}$ *on* $\mathcal{S} \setminus \{i_1, \ldots, i_N\}$ *with* $\boldsymbol{h} \in C^2(\mathcal{S}) \cap C^\infty(\mathcal{S} \setminus \{i_1, \ldots, i_N\})$.

More generally, a function $\Lambda^{1/2}$ such that Λ satisfies conditions (i) in the above definition is called an ***asymptotic distance function***. The function Λ does not need to be defined globally on $\mathcal{S}$.

When no confusion arises, condition (iii) in Definition 11.2 will simply be written as $\boldsymbol{h} = \Omega^2 \tilde{\boldsymbol{h}}$ so that $\mathcal{S} \setminus \{i_1, \ldots, i_N\}$ and $\tilde{\mathcal{S}}$ are identified. As will be seen in the following, for asymptotically Euclidean and regular manifolds, suitable neighbourhoods of the points i_k – the ***points at infinity*** – are mapped to the asymptotic ends of $\tilde{\mathcal{S}}$. Thus, one can use local differential geometry to discuss the asymptotic properties of the initial data set $(\tilde{\mathcal{S}}, \tilde{\boldsymbol{h}})$. The question of the differentiability of Ω and $\boldsymbol{h}$ at $i_1, \ldots, i_N$ will be addressed later in this subsection. Definition 11.2 is purely Riemannian; that is, it makes no reference to the extrinsic curvature. The behaviour of the extrinsic curvature at the points at infinity will be discussed in the subsequent paragraphs.

There is some ***conformal gauge freedom*** in Definition 11.2. A replacement of the form

$$\boldsymbol{h} \mapsto \phi^4 \boldsymbol{h}, \qquad \Omega \mapsto \phi^2 \Omega, \tag{11.49}$$

with $\phi(i_k) = 1$ gives rise to the same physical metric $\tilde{\boldsymbol{h}} = \Omega^{-4}\boldsymbol{h}$ and preserves the boundary conditions in point (i) of the definition. This gauge freedom can be used to select conformal metrics with special properties. For example, given a particular point at infinity i, and choosing ϕ such that

$$\Delta_{\boldsymbol{h}} \phi - \frac{1}{8} r[\boldsymbol{h}]\phi = 0 \qquad \text{on } \mathcal{B}_\varepsilon(i), \tag{11.50}$$

with $\mathcal{B}_\varepsilon(i)$ the ball of radius ε centred at i for some $\varepsilon > 0$, it follows from Equation (11.23) that

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$$r[\boldsymbol{h}'] = 0 \qquad \text{on } \mathcal{B}_\varepsilon(i).$$

A general property of elliptic equations with smooth coefficients is that they can always be solved locally; see, for example, Besse (2008) and Garabedian (1986). Thus, the requirement (11.50) can always be satisfied. In other words, the conformal metric $\boldsymbol{h}$ can always be chosen so that it vanishes in a neighbourhood of one of the points at infinity. In general, this statement is not true globally.

Normal coordinates around i

The consequences of Definition 11.2 are better analysed by means of ***normal coordinates***. Consider the set of $\boldsymbol{h}$-geodesics $\gamma_{\boldsymbol{v}} \subset \mathcal{S}$ starting at a particular point at infinity i (i.e. $\gamma_{\boldsymbol{v}}(0) = i$) with initial velocity $\boldsymbol{v} \in T|_i(\mathcal{S})$. Moreover, let $\mathcal{T}$ denote the subset of $T|_i(\mathcal{S})$ defined by

$$\mathcal{T} \equiv \big\{ \boldsymbol{v} \in T|_i(\mathcal{S}) \,\big|\, \gamma_{\boldsymbol{v}} \text{ is defined on an interval containing } [0,1] \big\}.$$

On the set $\mathcal{T}$ one can define the ***exponential map at*** i, $\exp_i : \mathcal{T} \to \mathcal{S}$, through the condition $\exp_i(\boldsymbol{v}) = \gamma_{\boldsymbol{v}}(1)$; that is, the exponential map sends the vector $\boldsymbol{v}$ to the point at a unit parameter distance along the unique geodesic through i with initial velocity $\boldsymbol{v}$. It can be shown that *there exists a neighbourhood $\mathcal{Q} \subset T|_i(\mathcal{S})$ of the vector $\mathbf{0}$ such that the exponential map at i gives a diffeomorphism onto a neighbourhood $\mathcal{U} \subset \mathcal{S}$ of i*; see, for example, O'Neill (1983) for a proof. If $\boldsymbol{v} \in \mathcal{Q}$ implies that $\lambda\boldsymbol{v} \in \mathcal{Q}$ for all $\lambda \in [0,1]$, then one says that $\mathcal{Q}$ is ***star shaped*** and $\mathcal{U} = \exp_i(\mathcal{Q})$ is called a ***normal neighbourhood of*** i. In particular, if $\mathcal{U} = \mathcal{B}_\varepsilon(i)$, the open ball of radius $\varepsilon > 0$ with respect to $\boldsymbol{h}$, one has a ***geodesic ball***.

In what follows, assume one has a normal neighbourhood $\mathcal{U}$ around i and that one is given an orthonormal basis $\{\boldsymbol{e_i}\}$ for $T|_i(\mathcal{S})$. Given $p \in \mathcal{U}$ and $\boldsymbol{v}$ such that $p = \exp(\boldsymbol{v})$, then writing $\boldsymbol{v} = x^{\boldsymbol{i}}\boldsymbol{e_i}$ one can use the components $\underline{x} = (x^{\boldsymbol{i}}) \in \mathbb{R}^3$ as coordinates for the point p – these are the ***normal coordinates*** determined by the basis $\{\boldsymbol{e_i}\}$. For consistency, the normal coordinates will be written as (x^α) rather than $(x^{\boldsymbol{i}})$. In terms of normal coordinates a geodesic through the origin has the form $x(\mathrm{s}) = (\mathrm{s}\, x^\alpha)$ where s is an affine parameter. As $\dot{\boldsymbol{x}} = (x^\alpha)$ and $\ddot{\boldsymbol{x}} = 0$, it follows from the geodesic equation that $\gamma_\beta{}^\alpha{}_\gamma(i) x^\beta x^\gamma = 0$ with $\gamma_\beta{}^\alpha{}_\gamma$ being the Christoffel symbols of the metric $\boldsymbol{h}$. As this has to hold for any geodesic on $\mathcal{U}$, one concludes that $\gamma_\beta{}^\alpha{}_\gamma(i) = 0$. It also follows that $\partial_\alpha h_{\beta\gamma} = 0$, so that one can write

$$h_{\alpha\beta} = -\delta_{\alpha\beta} + O(|x|^2) \qquad \text{close to } \; i, \tag{11.51}$$

where $|x|^2 \equiv \delta_{\alpha\beta} x^\alpha x^\beta$. Moreover, from the above construction it follows that

$$x^\alpha h_{\alpha\beta} = -\delta_{\alpha\beta} x^\alpha. \tag{11.52}$$

For future use it is observed that, in terms of normal coordinates and sufficiently close to i, one has

$$d\mu_{\boldsymbol{h}} = |x|^2 d\sigma + O(|x|^3), \tag{11.53}$$

where $d\mu_{\boldsymbol{h}}$ is the volume element of the metric $\boldsymbol{h}$ and $d\sigma$ denotes the area element of the unit 2-sphere $\mathbb{S}^2$.

For later use, it is convenient to define the ***(square of the) geodesic distance*** $\Gamma^2 \equiv |x|^2$. One has that Γ^2 is a smooth function of the normal coordinates. It can be verified that

$$h^{\alpha\beta} D_\alpha \Gamma D_\beta \Gamma = -4\Gamma \tag{11.54}$$

and that

$$\Gamma(i) = 0, \qquad D_\alpha \Gamma(i) = 0, \qquad D_\alpha D_\beta \Gamma(i) = -2h_{\alpha\beta}, \qquad D_\alpha D_\beta D_\gamma \Gamma(i) = 0.$$

Hence, Γ satisfies the boundary conditions (i) in Definition 11.2 so it is an asymptotic distance function. Observe that, in general, Γ is not defined globally on $\mathcal{S}$.

Remark. The results obtained using normal coordinates can be strengthened by exploiting the conformal freedom in (11.49). In particular, a conformal factor can always be found such that the Riemann curvature tensor of the resulting rescaled metric vanishes at i. In order to see this, given the metric $\boldsymbol{h}$, let $\Omega' \equiv e^f$ with $f \in \mathfrak{X}(\mathcal{S})$ such that

$$f = \frac{1}{2} x^\alpha x^\beta l_{\alpha\beta}(i) \quad \text{on } \mathcal{B}_\varepsilon(i),$$

where $l_{\alpha\beta}(i)$ denotes the components with respect to the normal coordinates of the three-dimensional Schouten tensor at i. A calculation shows that

$$\Omega'(i) = 1, \qquad D_\alpha \Omega'(i) = 0, \qquad D_\alpha D_\beta \Omega'(i) = l_{\alpha\beta}(i).$$

Hence, using the conformal transformation formula for the Schouten tensor (5.16b) one finds that the metric $\boldsymbol{h}' \equiv \Omega'^2 \boldsymbol{h}$ satisfies $l'_{\alpha\beta}(i) = 0$. As in dimension 3 the Riemann tensor is completely determined by the Schouten tensor one concludes that $r'_{\alpha\beta\gamma\delta}(i) = 0$ as claimed. The metric $\boldsymbol{h}'$ satisfies the *improved* expansion

$$h'_{\alpha\beta} = -\delta_{\alpha\beta} + O(|x|^3) \qquad \text{close to } i;$$

compare with (11.51).

The construction described in the previous paragraph is not the only possible way of exploiting the conformal gauge freedom. Depending on the particular analysis, other choices may be more convenient – for example, the *conformal normal gauge* introduced in Friedrich and Schmidt (1987) and Friedrich (1998c) or the *central harmonic gauge* used in Friedrich (2013).

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Asymptotically Euclidean data versus asymptotically Euclidean and regular data

It is useful to compare the two definitions of asymptotic flatness presented in this Section: Definitions 11.1 and 11.2. Condition (i) in Definition 11.2 restricts the form of the conformal factor Ω in a neighbourhood $\mathcal{B}_a(i_k)$ of a given point at infinity i_k. More precisely, one has that

$$\Omega = |x|^2 f(\underline{x}) \quad \text{near } i_k, \tag{11.55}$$

where f is continuous with $f(0) = 1$. Given the normal coordinates (x^α) on $\mathcal{B}_a(i_k)$, one can introduce inversion coordinates $y^\alpha \equiv x^\alpha/|x|^2$ so that

$$\tilde{h}_{\alpha\beta} = \Omega^{-2} h_{\alpha\beta} = -\delta_{\alpha\beta} + O(|y|^{-1}) \quad \text{as} \quad |y| \to \infty.$$

Thus, to recover the mass term in the expansion (11.47a) one requires further information about the fields Ω and $\boldsymbol{h}$.

With regards to the second fundamental form, it follows from the transformation rules discussed in Section 11.1.1 that

$$\tilde{K}_{\alpha\beta} = \Omega^{-1} K_{\alpha\beta} = \Omega \psi_{\alpha\beta}.$$

Hence, if the physical field $\tilde{K}_{\alpha\beta}$ satisfies the decay given by condition (11.47b), then

$$\tilde{K}_{\alpha\beta} = O(|x|^0), \qquad \psi_{\alpha\beta} = O(|x|^{-4}), \qquad \text{as } |x| \to 0.$$

Consequently, the decay conditions of Definition 11.1 imply a tensor $\psi_{\alpha\beta}$ which is singular at the points at infinity. To have a regular $\psi_{\alpha\beta}$ one needs the stronger decay condition $\tilde{K}_{\alpha\beta} = O(1/|y|^6)$. This decay excludes the possibility of a non-vanishing *ADM linear momentum* and *ADM angular momentum*.

The regularity at the points at infinity

The regularity requirements on Ω and $\boldsymbol{h}$ of Definition 11.2 are given with respect to some suitable coordinate system. A natural choice is the normal coordinates $\underline{x} = (x^\alpha)$ centred at the point at infinity – intuitively, one expects the regularity with respect to normal coordinates to be optimal. In these coordinates the function $|x|$ is not smooth at i as its second derivative is not well defined there. More generally, even powers of $|x|$ will be smooth, while odd ones will be only C^k, for some k.

Initial data sets for static vacuum spacetimes admit a conformal metric which is, in fact, analytic at the point at infinity; see Beig and Simon (1980b) and Beig and Schmidt (2000). Remarkably, this is not the case for stationary solutions which can be seen to be only C^2 at the point at infinity; see Dain (2001b). More precisely, *any asymptotically Euclidean data set for a stationary spacetime*

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(and in particular for the Kerr solution!) has a conformal metric of the form $\boldsymbol{h}$ *which, in a suitable neighbourhood* $\mathcal{B}_a(i_k)$ *of infinity, takes the form*

$$\boldsymbol{h} = \boldsymbol{h}' + |x|^3\boldsymbol{h}'',$$

with $\boldsymbol{h}'$ *and* $\boldsymbol{h}''$ *analytic tensors with respect to normal coordinates.*

11.6.3 Fundamental solutions and punctures

Consider now, for simplicity, an asymptotically Euclidean and regular manifold $(\mathcal{S}, \boldsymbol{h})$ with a single point at infinity i. Suppose, for ease of the presentation, that the conformal factor $\Omega = \vartheta^{-2}$ satisfies the ***Yamabe equation***

$$\Delta_{\boldsymbol{h}}\vartheta - \frac{1}{8}r[\boldsymbol{h}]\vartheta = 0 \qquad \text{on } \mathcal{S}\setminus\{i\}. \tag{11.56}$$

Condition (i) of Definition 11.2 implies a singular behaviour for the conformal factor ϑ. Indeed, from Equation (11.55) it follows that

$$\vartheta|x| \to 1 \qquad \text{as } |x| \to 0. \tag{11.57}$$

In order to develop a better understanding of the singular behaviour at i consider the integral

$$I_\varepsilon \equiv \int_{\mathcal{B}_\varepsilon(i)} \left(\Delta_{\boldsymbol{h}}\vartheta - \frac{1}{8}r[\boldsymbol{h}]\vartheta\right)\mathrm{d}\mu_{\boldsymbol{h}}$$

over an open ball $\mathcal{B}_\varepsilon(i)$ of a suitably small radius $\varepsilon > 0$ centred at i. To simplify the evaluation of the integral it is assumed that the metric $\boldsymbol{h}$ has been chosen such that $r[\boldsymbol{h}] = 0$ on $\mathcal{B}_\varepsilon(i)$; as seen in Section 11.6.2 this is is always possible locally. Using the *divergence theorem* (see the Appendix to this chapter), one has that

$$I_\varepsilon = \int_{\mathcal{B}_\varepsilon(i)} \Delta_{\boldsymbol{h}}\vartheta\mathrm{d}\mu_{\boldsymbol{h}} = -\int_{\partial\mathcal{B}_\varepsilon(i)} \langle \mathbf{d}\vartheta, \boldsymbol{n}\rangle \mathrm{d}S_{\boldsymbol{h}},$$

where $\boldsymbol{n}$ is the outward-pointing unit normal to $\partial\mathcal{B}_\varepsilon(i)$ and $\mathrm{d}\sigma_{\boldsymbol{h}}$ is the surface element of $\partial\mathcal{B}_\varepsilon(i)$ implied by $\boldsymbol{h}$. From the expansion (11.53) it follows for sufficiently small ε that $\mathrm{d}S_{\boldsymbol{h}} = \varepsilon^2\mathrm{d}\sigma + o(\varepsilon^2)$ with $\mathrm{d}\sigma$ the surface element of $\mathbb{S}^2$. Moreover, as a consequence of (11.57) one has

$$\langle \mathbf{d}\vartheta, \boldsymbol{n}\rangle = -\frac{1}{\varepsilon^2} + o(\varepsilon^{-2}).$$

Putting everything together one concludes that

$$I_\varepsilon = -\int_{\partial\mathcal{B}_\varepsilon(i)} \mathrm{d}\sigma + o(\varepsilon) \longrightarrow 4\pi \qquad \text{as } \varepsilon \to 0,$$

so that

$$\int_{\mathcal{S}} \Delta_{\boldsymbol{h}}\vartheta\mathrm{d}\mu_{\boldsymbol{h}} = 4\pi.$$

The latter implies that one can write

$$\Delta_{\boldsymbol{h}}\vartheta = 4\pi\delta(i),$$

where $\delta(i)$ denotes the ***Dirac's delta distribution*** with support at the point i; see the Appendix to this chapter for more details and references. To obtain the expression for a generic metric with a non-vanishing Ricci scalar in a neighbourhood of i one makes use of the transformation law for the Yamabe equation, Equation (11.23), to obtain

$$\left(\Delta_{\boldsymbol{h}'} - \frac{1}{8}r[\boldsymbol{h}']\right)(\phi^{-1}\vartheta) = 4\pi\phi^{-5}\delta(i) \qquad \text{with } \boldsymbol{h}' = \phi^4\boldsymbol{h}.$$

As $\delta(i)$ has support only on i and $\phi(i) = 1$ one finally concludes that

$$\left(\Delta_{\boldsymbol{h}'} - \frac{1}{8}r[\boldsymbol{h}']\right)\vartheta' = 4\pi\delta(i) \qquad \text{with } \vartheta' = \phi^{-1}\vartheta.$$

This expression provides an alternative description of the singular behaviour of solutions to the Yamabe equation which satisfy the boundary condition (i) of Definition 11.2. The previous discussion can be generalised to manifolds $(\mathcal{S}, \boldsymbol{h})$ with several points at infinity. For example, if $\mathcal{S} = \mathbb{S}^3$ and $\boldsymbol{h} = -\boldsymbol{\hbar}$ the standard metric of $\mathbb{S}^3$, then the Yamabe equation

$$\left(\Delta_{-\boldsymbol{\hbar}} - \frac{1}{8}r[-\boldsymbol{\hbar}]\right)\vartheta = 4\pi\Big(\delta(i_N) + \delta(i_S)\Big),$$

where $\delta(i_N)$ and $\delta(i_S)$ are supported, respectively, at the north and south poles of $\mathbb{S}^3$, describes the conformal factor ϑ for time-symmetric data for the Schwarzschild spacetime. Letting $\phi \equiv 1 + m/2r$, it follows from combining the first equation in (11.44) with the conformal factor ω compactifying $\mathbb{R}^3$ into $\mathbb{S}^3$ given in (6.5) that

$$\tilde{\boldsymbol{h}} = -\Omega^2\boldsymbol{\hbar}, \qquad \Omega = \omega\phi^{-2}.$$

Setting $\alpha = 1$ in Equation (6.5), one has that

$$\Omega = \frac{2\sin^2\dfrac{\psi}{2}}{\left(1 + \dfrac{m}{2}\tan\dfrac{\psi}{2}\right)^2}.$$

One can verify that Ω and $\mathbf{d}\Omega$ vanish at $\psi = 0,\, \pi$ (the north and south poles of $\mathbb{S}^3$). Moreover, one has

$$\Omega = \psi^2 + O(\psi^3), \qquad \Omega = (\psi - \pi)^2 + O((\psi - \pi)^3),$$

so that the fundamental solution $\vartheta = \Omega^{-1/2}$ has the expected singular behaviour.

Conformal decompactification of initial data sets on compact manifolds

From a geometric point of view, the purpose of introducing a conformal factor ϑ which is singular at the point at infinity is to produce a ***conformal decompactification*** of the manifold $\mathcal{S}$. As an example, consider a *vacuum maximal initial data set* $(\tilde{\mathcal{S}}, \tilde{\boldsymbol{h}}, \tilde{\boldsymbol{K}})$ with $\tilde{\mathcal{S}}$ compact. Under these assumptions one has that the Einstein constraints (11.14) reduce to

$$r[\tilde{\boldsymbol{h}}] = -\tilde{K}_{ij}\tilde{K}^{ij}, \qquad \tilde{D}^i\tilde{K}_{ij} = 0, \qquad \tilde{K} = \tilde{h}^{ij}\tilde{K}_{ij} = 0.$$

If given a point $i \in \tilde{\mathcal{S}}$ one can find a solution $\bar{\vartheta}$ to the equation

$$\Delta_{\tilde{\boldsymbol{h}}}\bar{\vartheta} - \frac{1}{8}r[\tilde{\boldsymbol{h}}]\bar{\vartheta} - \frac{1}{8}r[\tilde{\boldsymbol{h}}]\bar{\vartheta}^{-7} = 4\pi\delta(i),$$

it follows from a calculation involving the conformal transformation properties of the various fields that

$$\bar{\boldsymbol{h}} \equiv \bar{\vartheta}\tilde{\boldsymbol{h}}, \qquad \bar{\boldsymbol{K}} \equiv \vartheta^{-2}\tilde{\boldsymbol{K}},$$

gives rise to an asymptotically Euclidean and regular solution to the Einstein constraints

$$r[\bar{\boldsymbol{h}}] = -\bar{K}_{ij}\bar{K}^{ij}, \qquad \bar{D}^i\bar{K}_{ij} = 0, \qquad \bar{K} \equiv \bar{h}^{ij}\bar{K}_{ij} = 0.$$

As pointed out in O'Murchadha (1988), this construction can be used to argue that, in a certain sense, there are more asymptotically flat initial data sets than initial data sets on compact surfaces.

The Yamabe invariant

The possibility of conformally decompactifying a compact Riemannian manifold $(\mathcal{S}, \boldsymbol{h})$ to obtain a physical manifold $(\tilde{\mathcal{S}}, \tilde{\boldsymbol{h}})$ which is asymptotically Euclidean and regular depends on being able to solve the equation

$$\left(\Delta_{\boldsymbol{h}} - \frac{1}{8}r[\boldsymbol{h}]\right)\vartheta = 4\pi\delta(i). \tag{11.58}$$

The discussion in Section 11.5 suggests that this may not be possible for all cases. To explore this further, consider a *test function* $\phi \in \mathfrak{X}(\mathcal{S})$. A calculation shows that

$$\begin{aligned}
\int_{\mathcal{S}} |D(\vartheta\phi)|^2 \mathrm{d}\mu_{\boldsymbol{h}} &= \int_{\mathcal{S}} \big(\vartheta^2|D\phi|^2 + \phi^2|D\vartheta|^2\big)\mathrm{d}\mu_{\boldsymbol{h}} + \int_{\mathcal{S}} \vartheta D_i\vartheta D^i\phi^2 \mathrm{d}\mu_{\boldsymbol{h}} \\
&= \int_{\mathcal{S}} \big(\vartheta^2|D\phi|^2 + \phi^2|D\vartheta|^2\big)\mathrm{d}\mu_{\boldsymbol{h}} - \int_{\mathcal{S}} D^i(\vartheta D_i\vartheta)\phi^2 \mathrm{d}\mu_{\boldsymbol{h}} \\
&= \int_{\mathcal{S}} \vartheta^2|D\phi|^2 \mathrm{d}\mu_{\boldsymbol{h}} - \int_{\mathcal{S}} \vartheta\phi^2\Delta_{\boldsymbol{h}}\vartheta \mathrm{d}\mu_{\boldsymbol{h}},
\end{aligned}$$

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where the second equality follows by integration by parts on a compact manifold of the last integral in the first line. As $|D\phi|^2 = D_i\phi D^i\phi < 0$ and $\vartheta > 0$ it follows that

$$
\begin{aligned}
-\int_{\mathcal{S}} |D(\vartheta\phi)|^2 \mathrm{d}\mu_{\boldsymbol{h}} &> \int_{\mathcal{S}} \vartheta\phi^2\Delta_{\boldsymbol{h}}\vartheta \mathrm{d}\mu_{\boldsymbol{h}} \\
&> 4\pi\int_{\mathcal{S}} \vartheta\phi^2\delta(i)\mathrm{d}\mu_{\boldsymbol{h}} + \frac{1}{8}\int_{\mathcal{S}} \vartheta^2\phi^2 r[\boldsymbol{h}] d\mu_{\boldsymbol{h}} \\
&> 4\pi\vartheta(i)\phi^2(i) + \frac{1}{8}\int_{\mathcal{S}} r[\boldsymbol{h}]\vartheta^2\phi^2\mathrm{d}\mu_{\boldsymbol{h}}, \\
&> \frac{1}{8}\int_{\mathcal{S}} r[\boldsymbol{h}]\vartheta^2\phi^2\mathrm{d}\mu_{\boldsymbol{h}},
\end{aligned}
$$

where the last inequality follows from the fact that ϕ is an arbitrary test function. Thus setting $\zeta \equiv \vartheta\phi$ one concludes that

$$
-\inf_{\zeta\in\mathfrak{X}(\mathcal{S})}\int_{\mathcal{S}} \big(8|D\zeta|^2 + r[\boldsymbol{h}]\zeta^2\big)\mathrm{d}\mu_{\boldsymbol{h}} > 0,
$$

where inf denotes the infimum, that is, the biggest lower bound. *The latter is a necessary condition for the existence of a solution to Equation* (11.58). Under some further technical assumptions, it can be shown to be a sufficient condition; see, for example, Friedrich (2011). The above expression can be reformulated in a conformal way by adding a suitable normalisation factor. Accordingly, one defines the ***Yamabe invariant (number)*** of $\boldsymbol{h}$ as

$$
Y[\boldsymbol{h}] \equiv -\inf_{\zeta\in\mathfrak{X}(\mathcal{S})} \frac{\displaystyle\int_{\mathcal{S}} \big(8h^{ij}D_i\zeta D_j\zeta + r[\boldsymbol{h}]\zeta^2\big)\mathrm{d}\mu_{\boldsymbol{h}}}{\displaystyle\left(\int_{\mathcal{S}} \zeta^6\mathrm{d}\mu_{\boldsymbol{h}}\right)^{1/3}}.
$$

The conformal invariance of the above expression follows from the transformation properties of the three-dimensional Ricci scalar and of the volume element. Accordingly, the Yamabe number is, in fact, a property of the conformal class $[\boldsymbol{h}]$. In particular, if $(\mathcal{S},\boldsymbol{h})$ is such that $Y[\boldsymbol{h}] > 0$, then there exists $\bar{\boldsymbol{h}} \in [\boldsymbol{h}]$ such that $r[\bar{\boldsymbol{h}}] < 0$ on $\mathcal{S}$; see Lee and Parker (1987).

11.6.4 Constructing solutions to the constraint equations using fundamental solutions

As already mentioned, fundamental functions of the Yamabe equation on compact manifolds allows one to obtain solutions to the Hamiltonian and momentum constraints by means of a procedure of *conformal decompactification*. In this section an overview of some of the technical details of this construction is provided.

In the first instance, attention is restricted to the time-symmetric case. Furthermore, it is assumed that there is only one point at infinity. Given a

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compact Riemannian manifold $(\mathcal{S}, \boldsymbol{h})$ and a point at infinity i, the construction of a time-symmetric initial data set $(\tilde{\mathcal{S}}, \tilde{\boldsymbol{h}})$ requires a global solution to Equation (11.58). As already observed, the function $\Gamma = |x|$, defined only in a neighbourhood $\mathcal{B}_a(i)$ with $a > 0$, satisfies the required boundary conditions for a solution to Equation (11.58). Indeed, it can be shown that the solution ϑ satisfies

$$\vartheta = \Gamma^{-1} + \frac{m}{2} + O(\Gamma), \qquad \text{near } i,$$

where m is a constant; see Lee and Parker (1987). The above expansion is also valid for any other choice of asymptotic distance function – the constant m is independent of the particular choice. As Γ^2 is a smooth function on its domain of definition, it can be extended to a smooth function (to be denoted again by Γ^2) on the whole of the compact manifold $\mathcal{S}$; see the Appendix to this chapter for further discussion. To obtain the global solution to Equation (11.58), one considers the ansatz

$$\vartheta = \Gamma^{-1} + \frac{m}{2} + W, \tag{11.59}$$

with W some smooth function on $\mathcal{S}$. To make effective use of this ansatz it is assumed, without loss of generality, that the conformal metric $\boldsymbol{h}$ satisfies $r_{\alpha\beta\gamma\delta}(i) = 0$ so that one can write

$$h_{\alpha\beta} = -\delta_{\alpha\beta} + \bar{h}_{\alpha\beta}, \qquad \bar{h}_{\alpha\beta} = O(|x|^3).$$

Hence, using the identity

$$\Delta_{\boldsymbol{h}}\vartheta = \frac{1}{\sqrt{-\det \boldsymbol{h}}}\partial_\alpha\left(\sqrt{-\det \boldsymbol{h}}\, h^{\alpha\beta}\partial_\beta\vartheta\right),$$

it follows that

$$\begin{aligned}\mathbf{L}_{\boldsymbol{h}} &= \Delta_{\boldsymbol{h}} - \frac{1}{8}r[\boldsymbol{h}]\\ &= \Delta_{-\boldsymbol{\delta}} + \bar{\mathbf{L}} + r[\boldsymbol{h}],\end{aligned}$$

with

$$\bar{\mathbf{L}} \equiv \bar{h}^{\alpha\beta}\partial_\alpha\partial_\beta + b^\alpha\partial_\alpha, \qquad \bar{h}^{\alpha\beta} = O(|x|^3), \qquad b^\alpha = O(|x|^2)$$

and $r[\boldsymbol{h}] = O(|x|)$. Using the above expressions one can compute that

$$\mathbf{L}_{\boldsymbol{h}}\left(\frac{1}{\Gamma}\right) = \Delta_{-\boldsymbol{\delta}}\left(\frac{1}{\Gamma}\right) + \bar{f} \qquad \text{with } \bar{f} = O(|x|^0).$$

Now, a calculation similar to the one discussed in Section 11.6.3 shows that

$$\Delta_{-\boldsymbol{\delta}}\left(\frac{1}{\Gamma}\right) = 4\pi\delta(i),$$

so that substitution of ansatz (11.59) into $\mathbf{L}_{\boldsymbol{h}}\vartheta = 4\pi\delta(i)$ leads to the *regular equation*

$$\Delta_{\boldsymbol{h}} W - \frac{1}{8} r[\boldsymbol{h}] W = f \qquad \text{with } f = O(|x|^0), \tag{11.60}$$

for which a suitable existence theory is readily available. A unique smooth solution to Equation (11.60) exists if the Yamabe number of the metric $\boldsymbol{h}$ satisfies $Y[\boldsymbol{h}] > 0$; see Beig and O'Murchadha (1991) and Friedrich (1998c). A further argument using the *maximum principle* shows that ϑ – as given by Equation (11.59) – with W solving Equation (11.60) is positive on $\mathcal{S} \setminus \{i\}$ and gives the unique global solution to Equation (11.58). It follows that $(\tilde{\mathcal{S}}, \vartheta^4 \boldsymbol{h})$ is an asymptotically Euclidean and regular time-symmetric initial data set.

Data with a non-vanishing extrinsic curvature

The procedure to solve the Yamabe equation described in the previous section can be extended to the case of an initial data set with a trace-free extrinsic curvature. One first needs a solution to the momentum constraint. Several procedures to construct solutions to the maximal momentum constraint (and in particular of the elliptic Equation (11.21)) have been considered in the literature; see, for example, Beig and O'Murchadha (1996), Chaljub-Simon (1982) and Dain and Friedrich (2001). In particular, it is well understood how to specify the free datum ψ'_{ij} in Equation (11.21) so as to ensure non-vanishing ADM linear momentum and ADM angular momentum.

In what follows, assume that Equation (11.21) has been solved for a particular choice of the free datum ψ'_{ij}. Substituting the transverse and trace-free tensor ψ_{ij} obtained from the York splitting (11.20) into the Licnerowicz Equation (11.17) yields the equation

$$\Delta_{\boldsymbol{h}} \vartheta - \frac{1}{8} r[\boldsymbol{h}] \vartheta = \frac{1}{8} \psi_{ij} \psi^{ij} \vartheta^{-7}.$$

As in the case of the Yamabe equation, one can incorporate the singular behaviour of the conformal factor required to decompactify the compact manifold $\mathcal{S}$ via a Dirac's delta. This leads to the equation

$$\Delta_{\boldsymbol{h}} \vartheta - \frac{1}{8} r[\boldsymbol{h}] \vartheta = 4\pi\delta(i) + \frac{1}{8} \psi_{ij} \psi^{ij} \vartheta^{-7}. \tag{11.61}$$

To construct a solution to this equation one first considers a solution $\vartheta_\bullet$ to Equation (11.58) – such solution exists if $Y[\boldsymbol{h}] > 0$. One uses $\vartheta_\bullet$ to write the ansatz

$$\vartheta = \vartheta_\bullet + V$$

with V a smooth function to be determined. Equation (11.61) yields

$$\Delta_{\boldsymbol{h}} V - \frac{1}{8} r[\boldsymbol{h}] V = \frac{1}{8} \psi_{ij} \psi^{ij} \vartheta_\bullet^{-7} (1 + \vartheta_\bullet^{-1} V)^{-7}. \tag{11.62}$$

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Observe that if $\psi_{ij} = O(|x|^{-4})$, then, in principle, $\frac{1}{8}\psi_{ij}\psi^{ij}\vartheta_\bullet^{-7} = O(|x|^{-1})$ so that the right-hand side of Equation (11.62) is still singular. This singularity is, nevertheless, mild, and suitable existence results are available; see theorem 12 in Dain and Friedrich (2001) and also the appendix in Beig and O'Murchadha (1994). The solution ϑ is smooth, and, again, it can be verified that it satisfies $\vartheta > 0$ on $\mathcal{S} \setminus \{i\}$.

11.7 Hyperboloidal manifolds

In certain applications of the conformal field equations it is convenient to consider initial data sets prescribed on hypersurfaces similar to the hyperboloids of the Minkowski spacetime discussed in Section 6.2.4. Hyperboloidal 3-manifolds arise in the construction of asymptotically simple spacetimes with vanishing cosmological constant and in the construction of anti-de Sitter like spacetimes.

11.7.1 Hyperboloidal initial data sets

For the sake of the presentation, the discussion in this section is restricted to the vacuum case with vanishing cosmological constant. Based on the intuition gained through the analysis of hyperboloids in the Minkowski spacetime one has the following definition (see Friedrich (1983) and Kánnár (1996a)):

Definition 11.3 (***hyperboloidal initial data sets***) *A triple $(\tilde{\mathcal{S}}, \tilde{\boldsymbol{h}}, \tilde{\boldsymbol{K}})$ satisfying the vacuum Einstein constraint equations is called a* ***hyperboloidal initial data set*** *if:*

(i) There exists a conformal compactification whereby $\tilde{\mathcal{S}}$ is diffeomorphically identified with the interior of a manifold $\mathcal{S}$ with boundary $\partial\mathcal{S}$ such that $\mathcal{S}$ is diffeomorphic to the closed unit ball in $\mathbb{R}^3$ (whence $\partial\mathcal{S}$ is diffeomorphic to $\mathbb{S}^2$).

(ii) There exist functions Ω and Σ on $\mathcal{S}$ such that $\Omega > 0$ on $\tilde{\mathcal{S}}$ and $\Omega = 0$ and $\Sigma > 0$ on $\partial\mathcal{S}$.

(iii) The conformal fields

$$\boldsymbol{h} = \Omega^2\tilde{\boldsymbol{h}}, \qquad \boldsymbol{K} = \Omega(\tilde{\boldsymbol{K}} + \Sigma\tilde{\boldsymbol{h}}),$$

extend smoothly to $\mathcal{S}$. Moreover, one has that $\boldsymbol{h}^\sharp(\mathbf{d}\Omega, \mathbf{d}\Omega) = \Sigma^2$ on $\partial\mathcal{S}$.

The simplest type of hyperboloidal initial data sets consists of the case where the physical extrinsic curvature is *pure trace*; that is, one has

$$\tilde{\boldsymbol{K}} = \frac{1}{3}\tilde{K}\tilde{\boldsymbol{h}}. \tag{11.63}$$

As a consequence of the momentum constraint and assuming (11.63) it follows that $\tilde{K}$ must be a constant. From the conformal Hamiltonian constraint (11.15a)

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one concludes that

$$4\Omega D_i D^i \Omega - 6 D_i \Omega D^i \Omega + 2\Omega^2 r = \tilde{K}^2. \tag{11.64}$$

In order to encode the right behaviour of the conformal factor Ω at $\partial\mathcal{S}$ one introduces a so-called ***boundary defining function*** ρ, that is, a real function over $\mathcal{S}$ satisfying

$$\rho|_{\partial\mathcal{S}} = 0, \qquad \mathbf{d}\rho|_{\partial\mathcal{S}} \neq 0.$$

Given a Riemannian manifold $(\mathcal{S}, \boldsymbol{h})$, such a function can always be constructed. Making use of the ansatz $\Omega = \rho\vartheta^{-2}$ with $\vartheta > 0$ on $\mathcal{S}$, it follows from (11.64) that

$$\rho^2 \Delta_{\boldsymbol{h}}\vartheta - \rho D_i\rho D^i\vartheta + \left(\frac{3}{2}D_i\rho D^i\rho - \frac{1}{8}r[\boldsymbol{h}]\vartheta\rho^2\theta - \frac{1}{2}\rho\vartheta^2\Delta_{\boldsymbol{h}}\rho\right)\vartheta = -\frac{1}{8}\tilde{K}^2\vartheta^{-5}. \tag{11.65}$$

The latter is an elliptic equation for ϑ which becomes singular at $\partial\mathcal{S}$ as its principal part vanishes at this set.

The properties of solutions to Equation (11.65) have been analysed in Andersson et al. (1992). One has the following:

Theorem 11.2 (***existence of hyperboloidal initial data sets***) *Let $(\mathcal{S}, \boldsymbol{h})$ be a smooth Riemannian manifold with boundary $\partial\mathcal{S}$. Then, there exists a unique positive solution ϑ to Equation (11.65). Moreover, the following are equivalent:*

(i) The function ϑ and the tensors

$$L_{ij} \equiv -\frac{1}{\Omega}D_{\{i}D_{j\}}\Omega + \frac{1}{12}\left(r + \frac{2}{3}K^2\right)h_{ij}, \tag{11.66a}$$

$$d_{ij} \equiv \frac{1}{\Omega^2}D_{\{i}D_{j\}}\Omega + \frac{1}{\Omega}r_{\{ij\}}, \tag{11.66b}$$

determined on $\tilde{\mathcal{S}}$ by $\boldsymbol{h}$ and $\Omega = \rho\vartheta^{-2}$ extend smoothly to all of $\mathcal{S}$.

(ii) The Weyl tensor $C^a{}_{bcd}$ computed from the data on $\mathcal{S}$ vanishes on $\partial\mathcal{S}$.

(iii) The conformal class $[\boldsymbol{h}]$ is such that the extrinsic curvature of $\partial\mathcal{S}$ with respect to its embedding in $(\mathcal{S}, \boldsymbol{h})$ is pure trace.

The expressions for the fields L_{ij} and d_{ij} correspond to the spatial part of the Schouten tensor and the electric part of the rescaled Weyl tensor as determined by the conformal constraint equations of Section 11.4.3. Observe that the expressions for the fields are formally singular at $\Omega = 0$, so that the conclusion of the theorem is non-trivial and ensures the existence of regular hyperboloidal data for the conformal field equations. Extensions of Theorem 11.2 to more general forms of the extrinsic curvature have been analysed in Andersson and Chruściel (1993, 1994).

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Initial data for anti-de Sitter-like spacetimes

By making the identification $\tilde{K}^2 \mapsto \lambda$ with $\lambda > 0$ in Equation (11.64), hyperboloidal initial data sets can be interpreted as initial data sets for anti-de Sitter-like spacetimes. Thus, all available knowledge about the existence of hyperboloidal initial data sets can be transferred to this setting. This idea has been investigated for a larger class of data than the one considered in this section in Kánnár (1996a).

11.8 Other methods for solving the constraint equations

The analysis of the Einstein constraint equations carried out in the previous sections relies on a systematic use of the conformal method of Licnerowicz, Choquet-Bruhat and York. There are, however, other alternative procedures, each providing a different insight into the properties of the solutions to the constraint equations; see, for example, Bartnik and Isenberg (2004). In this section, methods of particular relevance for the analysis of the conformal field equations are briefly discussed: the first one based on the so-called *extended constraint equations*, and the second one being the so-called *exterior gluing procedure*.

11.8.1 The extended constraint equations

Given a solution to the conformal constraint equations, Lemma 11.1 shows how to construct initial data for the conformal Einstein field equations. It is, nevertheless, of interest to directly obtain a solution to the conformal constraint equations without having to solve the Einstein constraint equations. A construction of this type is of importance as the expressions for the rescaled Weyl tensor and the Schouten tensor in terms of the conformal factor and the intrinsic 3-geometry of the hypersurface are formally singular at the points where $\Omega = 0$; see, for example, Equations (11.66a) and (11.66b) in Theorem 11.2. Currently available results in this direction are restricted to the case where the $\Omega = 1$; see Butscher (2002, 2007). Despite this limitation, they provide insight into the properties and structure of the conformal constraint equations and lead to a procedure for the construction of initial data sets by perturbative methods.

Assuming that the matter fields vanish, and setting $\Omega = 1$, $\Sigma = 0$ in the conformal constraint equations (11.35a)–(11.35j) one finds that the essential equations of the system can be rewritten in tensorial form as

$$D_j K_{ki} - D_k K_{ji} = \epsilon^l{}_{jk} d^*_{il}, \tag{11.67a}$$

$$D^k d_{ki} = K^{jk} \epsilon^l{}_{ki} d^*_{jl}, \tag{11.67b}$$

$$D^k d^*_{ki} = -\epsilon_i{}^{jl} K_j{}^k r_{kl}, \tag{11.67c}$$

$$r_{ij} = d_{ij} + K K_{ij} - K_i{}^k K_{kj}. \tag{11.67d}$$

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These equations are known as the ***extended Einstein constraints*** since a solution thereof implies a solution to the Einstein vacuum constraints; see Lemma 11.1 in this chapter and lemma 1 in Butscher (2007) for a more detailed discussion. The first three equations constitute an underdetermined elliptic system for the fields K_{ij}, d_{ij} and d^*_{ij}.

A direct computation shows that the formal adjoint of the operator in the principal part of Equation (11.67a) is the divergence with respect to the index $_j$. Applying this divergence to the equation and writing

$$K_{ij} = \psi_{ij} - \frac{1}{3}Kh_{ij} \qquad \text{with } \psi_{ij}h^{ij} = 0$$

one obtains the equation

$$D^jD_j\psi_{ki} - D^jD_k\psi_{ji} = \epsilon^l{}_{jk}D^jd^*_{il} + \frac{1}{3}\big(h_{ki}D^jD_jK - D_iD_kK\big).$$

If the fields d^*_{ij} and K are known, this equation can be verified to be an elliptic equation for the trace-free part ψ_{ij}.

Equations (11.67b) and (11.67c) can be transformed into fully elliptic equations by means of a York splitting of the fields d_{ij} and d^*_{ij}; see Section 11.3.3. Hence, writing

$$d_{ij} = D_iv_j + D_jv_i - \frac{2}{3}D_kv^kh_{ij} + d'_{ij},$$

$$d^*_{ij} = D_iu_j + D_ju_i - \frac{2}{3}D_ku^kh_{ij} + d^{*\prime}_{ij},$$

where d'_{ij} and $d^{*\prime}_{ij}$ are freely specifiable symmetric trace-free tensors, one obtains elliptic equations for the fields v_i and u_i whose principal part is identical to that of Equation (11.21). Finally, Equation (11.67d) can be transformed into an elliptic equation for the components of the 3-metric $\boldsymbol{h}$ by introducing ***harmonic coordinates*** $\underline{x} = (x^\alpha)$, $\Delta_{\boldsymbol{h}}x^\alpha = 0$; compare the analogous use of *wave coordinates* in the case of a Lorentzian metric to obtain the reduced Einstein field equation discussed in the Appendix to Chapter 13.

The system of elliptic equations for the fields K_{ij}, v_i, u_i, h_{ij} discussed in the previous paragraphs is called the ***auxiliary system***. Solutions to the auxiliary system could be obtained, in principle, by means of perturbative methods relying on the use of the *implicit function theorem* – see, for example, Ambrosetti and Prodi (1995) – if some background solution is known. The solutions thus obtained are not a priori solutions to the original Equations (11.67a)–(11.67d). Hence, in a second step, one needs to investigate the conditions under which a solution to the auxiliary system implies a solution to the extended Einstein constraints and, consequently, a solution to the vacuum Einstein constraints. This strategy has been investigated in Butscher (2002, 2007) to obtain asymptotically Euclidean solutions to the extended constraints which are close to data for the Minkowski spacetime. The particular details require the use of *weighted Sobolev spaces*

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to control the decay of the various fields. These methods can be adapted, in principle, to obtain data on $\mathbb{S}^3$ corresponding to perturbations of de Sitter initial data.

11.8.2 Exterior asymptotic gluing

The ***exterior asymptotic gluing*** is a method to construct solutions to the Einstein constraint equations by *gluing the interior region of an asymptotically Euclidean solution to the Einstein vacuum constraints to an asymptotic end of initial data for the Kerr spacetime or, in fact, of any stationary solution*; see Corvino (2000), Chruściel and Delay (2003), Corvino and Schoen (2006) and Corvino (2007). More precisely, given a smooth asymptotically Euclidean initial data set for the vacuum Einstein field equations $(\tilde{\mathcal{S}}, \tilde{\boldsymbol{h}}, \tilde{\boldsymbol{K}})$ and a given compact subset $\mathcal{U} \subset \tilde{\mathcal{S}}$ such that $\tilde{\mathcal{S}} \setminus \mathcal{U}$ is an asymptotic end, it is possible to show that there exists another smooth asymptotically Euclidean solution to the vacuum Einstein constraints $(\tilde{\mathcal{S}}, \bar{\boldsymbol{h}}, \bar{\boldsymbol{K}})$ which is identical to the original solution on $\mathcal{U}$ and coincides with initial data for the Kerr spacetime on $\tilde{\mathcal{S}} \setminus \bar{\mathcal{U}}$ for some $\bar{\mathcal{U}} \subset \tilde{\mathcal{S}}$. In addition, the initial data set $(\tilde{\mathcal{S}}, \bar{\boldsymbol{h}}, \bar{\boldsymbol{K}})$ contains an annular transition region in which the initial data can be controlled. In the case of time-symmetric initial data sets this method glues any interior region to an exterior region of a slice of the Schwarzschild spacetime.

The underlying idea in the asymptotic exterior gluing method is to exploit the underdetermined character of the Einstein constraints as a system of partial differential equations for the fields $(\tilde{\boldsymbol{h}}, \tilde{\boldsymbol{K}})$. Prior to the development of the asymptotic exterior gluing methods Cutler and Wald (1989) have shown that it is possible to make use of the standard conformal method to construct solutions to the time symmetric constraints containing a Minkowskian interior region and a Schwarzschildean exterior region joined together by an annular region containing a *purely magnetic solution to the Einstein-Maxwell constraints.*

As will be discussed in Chapter 20, initial data sets obtained by means of asymptotic exterior gluing play a key role in the construction of Minkowski-like asymptotically simple spacetimes. For simplicity, in the remainder of this section *attention is restricted to the time-symmetric case* for which the Einstein vacuum constraints reduce to $r[\tilde{\boldsymbol{h}}] = 0$. In the present context, one regards the Ricci scalar as a map between the space of Riemannian metrics over $\tilde{\mathcal{S}}$ and $\mathfrak{X}(\tilde{\mathcal{S}})$. Under certain circumstances this mapping is an isomorphism; that is, given a metric $\boldsymbol{h}$ and $f \in \mathfrak{X}(\tilde{\mathcal{S}})$ such that $r[\boldsymbol{h}] = f$ and given a further $g \in \mathfrak{X}(\tilde{\mathcal{S}})$ close enough to f, then there exists another metric $\bar{\boldsymbol{h}}$ close to $\boldsymbol{h}$ such that $r[\bar{\boldsymbol{h}}] = g$. This property of the *Ricci scalar operator* is the essential ingredient in the gluing procedure. As part of the gluing construction, one connects the inner region $(\mathcal{U}, \tilde{\boldsymbol{h}})$ and an asymptotic region $(\mathcal{E}, \tilde{\boldsymbol{h}}_{\mathscr{S}})$ with $\tilde{\boldsymbol{h}}_{\mathscr{S}}$ as given in Equation (11.45) for some choice (so far undetermined) of the constants m and (x_0^α) through an annular region. A positive definite symmetric tensor $\breve{\boldsymbol{h}}$ is defined on $\tilde{\mathcal{S}}$ by requiring it to be identical to $\tilde{\boldsymbol{h}}$ on $\mathcal{U}$ and to $\tilde{\boldsymbol{h}}_{\mathscr{S}}$ on $\mathcal{E}$, while on the asymptotic region it is chosen so that it interpolates smoothly between $\tilde{\boldsymbol{h}}$ and $\tilde{\boldsymbol{h}}_{\mathcal{S}}$. By construction $r[\breve{\boldsymbol{h}}] = 0$ in both $\mathcal{U}$ and $\mathcal{E}$, while $r[\breve{\boldsymbol{h}}] \neq 0$ in the transitional annular region. Nevertheless, by moving $\mathcal{U}$ suitably into the asymptotic region, one can make $r[\breve{\boldsymbol{h}}]$ small enough so that the isomorphism properties of the Ricci scalar operator can be used to ensure the existence of a tensor $\boldsymbol{k}$ with support on an annular region such that $\bar{\boldsymbol{h}} \equiv \breve{\boldsymbol{h}} + \boldsymbol{k}$ is a Riemannian metric with $r[\bar{\boldsymbol{h}}] = 0$ on $\tilde{\mathcal{S}}$.

The asymptotic exterior gluing construction requires a careful analysis of the properties of the ***linearised Ricci operator***

$$\mathscr{R}_{\boldsymbol{h}}[\bar{\boldsymbol{h}}] \equiv -\Delta_{\boldsymbol{h}}(\mathbf{tr}_{\boldsymbol{h}}(\bar{\boldsymbol{h}})) + \mathbf{div}_{\boldsymbol{h}}(\mathbf{div}_{\boldsymbol{h}}(\bar{\boldsymbol{h}})) - \boldsymbol{h}(\bar{\boldsymbol{h}}, \mathbf{Ric}[\bar{\boldsymbol{h}}]).$$

For a fixed metric $\boldsymbol{h}$, the latter is an underdetermined elliptic operator. It can be transformed into an elliptic system by composition with its formal adjoint

$$\mathscr{R}^*_{\boldsymbol{h}}(f) \equiv -(\Delta_{\boldsymbol{h}} f)\boldsymbol{h} + \mathbf{Hess}(f) - f\mathbf{Ric}[\boldsymbol{h}].$$

The composite elliptic operator $\mathscr{R}^*_{\boldsymbol{h}} \circ \mathscr{R}_{\boldsymbol{h}}$ is a fourth-order partial differential operator. Once the linearised problem is controlled, the non-linear problem is then solved by means of an iteration. To conclude, one needs to show that the metric $\bar{\boldsymbol{h}}$ is indeed a solution to $r[\bar{\boldsymbol{h}}] = 0$. It is in this part of the construction that the value of the constants m and (x_0^α) are fixed. A refined version of the original construction in Corvino (2000) has been given in Corvino (2007), from which the following result has been adapted:

Theorem 11.3 (***exterior asymptotic gluing construction***) *Let $(\tilde{\mathcal{S}}, \tilde{\boldsymbol{h}})$ denote an asymptotically Euclidean initial data set for the Einstein vacuum equations. Let $\mathcal{E} \subset \tilde{\mathcal{S}}$ be any asymptotically flat end of $\tilde{\mathcal{S}}$. Given $r_0 > 0$ let $\mathcal{E}_{r_0} \subset \mathcal{E}$ be an exterior region in $\mathcal{E}$ expressed in asymptotically Cartesian coordinates by $\mathcal{E}_{r_0} = \{(x^\alpha) \in \mathbb{R}^3 \,|\, |x| > r > r_0\}$. Suppose, furthermore, that in these coordinates the metric $\tilde{\boldsymbol{h}}$ has the form*

$$\tilde{h}_{\alpha\beta} = -\left(1 + \frac{2m}{|x|}\right)\delta_{\alpha\beta} + O_3(|x|^{-2}).$$

Let k be a non-negative integer. Then for any $\varepsilon > 0$ there exists $r_* > 0$ and a smooth metric $\bar{\boldsymbol{h}}$ satisfying $r[\bar{\boldsymbol{h}}] = 0$ and $||h_{\alpha\beta} - \bar{h}_{\alpha\beta}||_{C^k(\mathcal{E})} < \varepsilon$ so that $\bar{\boldsymbol{h}}$ is equal to $\tilde{\boldsymbol{h}}$ on $\mathcal{U} = \tilde{\mathcal{S}} \setminus \mathcal{E}_{r_*}$ and identical to an asymptotically flat end of a standard Schwarzschild slice on $\mathcal{E}_{2r_*}$.

The precise definition of the ***supremum norm*** $||\quad||_{C^k(\mathcal{E})}$ is discussed in the Appendix to this chapter. A schematic depiction of the construction of Theorem 11.3 is given in Figure 11.2. In the applications of this result to the existence of asymptotically simple spacetimes, it is important to control the location of the exterior region $\mathcal{E}_{r_*}$ and to ensure that $r_* \not\to \infty$ as one moves along a family of initial data sets tending, say, to data for the Minkowski spacetime. This possible degeneracy has been dealt with by imposing some reflexion symmetry properties

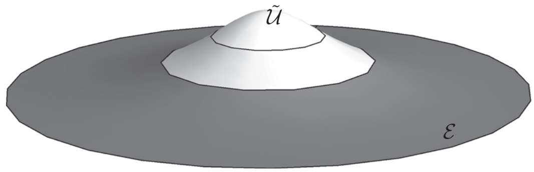

Figure 11.2 Schematic depiction of the exterior gluing construction given by Theorem 11.3. It contains an inner region $\tilde{\mathcal{U}}$ where the 3-metric has a fixed arbitrary value $\tilde{\boldsymbol{h}}$, an annular transition region between $\mathcal{E}_{r_*}$ and $\mathcal{E}_{2r_*}$ and an exterior region $\mathcal{E}$ where it is equal to data for a member of the Schwarzschild family of solutions.

on the metric $\tilde{\boldsymbol{h}}$; see Chruściel and Delay (2003). An alternative solution has been provided in Corvino (2007). This result makes use of symmetric $(0,2)$-tensors $\boldsymbol{k}$ satisfying the condition $\mathscr{R}_{-\boldsymbol{\delta}}(\boldsymbol{k}) = 0$. Making use of a York splitting the tensor $\boldsymbol{k}$ can be decomposed in a unique way into a traceless term with vanishing divergence, a trace part and a part which is the conformal Killing operator of a covector; see Chaljub-Simon (1982). The tensor $\boldsymbol{k}$ is said to be ***non-degenerate*** if its transverse-traceless part is non-zero. Using this terminology one has the following *stability result* (see Corvino (2007) for further details and its proof):

Theorem 11.4 (***stability of the exterior gluing construction***) *Let $\boldsymbol{k}$ be any smooth, compactly supported symmetric $(0,2)$-tensor on $\mathbb{R}^3$ with $\mathscr{R}_{-\boldsymbol{\delta}}(\boldsymbol{k}) = 0$. Moreover, for sufficiently small $\varepsilon > 0$ let*

$$\tilde{\boldsymbol{h}} = -\vartheta^4(\boldsymbol{\delta} + \varepsilon \boldsymbol{k})$$

be asymptotically flat and satisfy $r[\tilde{\boldsymbol{h}}] = 0$. If $\boldsymbol{k}$ is non-degenerate, there exists $r_* > 0$ so that for all ε small enough there is a metric $\bar{\boldsymbol{h}}$ with $r[\bar{\boldsymbol{h}}] = 0$ which agrees with $\tilde{\boldsymbol{h}}$ in the closed ball $\overline{\mathcal{B}}_{r_*}(0)$ and is exactly Schwarzschild on $\mathcal{E}_{2r_*}$. Consequently, the Riemannian manifold $(\mathbb{R}^3, \bar{\boldsymbol{h}})$ admits a smooth conformal point compactification in the sense of Definition 11.2.

This theorem guarantees the existence of time-symmetric solutions to the vacuum Einstein constraint equations which are both close to data for the Minkowski spacetime and exactly Schwarzschildean in a non-trivial exterior region; see Section 20.5.

Versions of the asymptotic exterior gluing construction for initial data sets with non-vanishing extrinsic curvature can be found in Chruściel and Delay (2003) and Corvino and Schoen (2006). There are adaptations of the exterior gluing method to the case of hyperboloidal initial data sets with constant scalar curvature; see Chruściel and Delay (2009).

11.9 Further reading

The best point of entry to the extensive literature on the Einstein constraint equations is through reviews such as those of Bartnik and Isenberg (2004) or Isenberg (2013). An older, classical review on the topic is given in Choquet-Bruhat and York (1980). An alternative review aimed at applications in numerical relativity is Cook (2000). A detailed account of the conformal method to solve the constraint equations, as seen by one of the main contributors of the topic, can be found in Choquet-Bruhat (2008) – this reference contains, in addition, a discussion of the basic aspects of weighted Sobolev spaces. Closely related to the latter is the reference Choquet-Bruhat et al. (2000). A discussion of basic aspects of the theory of elliptic differential equations and its application to the analysis of the Einstein constraints can be found in Rendall (2008). An alternative account of the basic aspects of the analysis of elliptic equations with a number of worked-out examples is Dain (2006). Finally, a detailed account of the conformal equations under the assumption of spherical symmetry is given in Guven and O'Murchadha (1995).

By contrast, the accounts on the conformal Einstein constraints are much more restricted in number. The original references are Friedrich (1983, 1984, 1986a, 1995, 2004); see also the discussion in Frauendiener (2004). A systematic analysis of hyperboloidal initial data sets can be found in Andersson et al. (1992) and Andersson and Chruściel (1993, 1994).

The notion of asymptotically Euclidean and regular manifolds can be traced back to the discussion in Geroch (1972b). These ideas have been further elaborated in Friedrich (1988, 1998c). Accounts of the use of *Dirac's deltas* to represent the points at infinity can be found in Beig and O'Murchadha (1991, 1994). A neat application of this approach to the construction of initial data sets with a conformal toroidal symmetry is given in Beig and Husa (1994). Applications of the method to the construction of initial data for the collision of Kerr-like black holes can be found in Dain (2001a,c). Finally, a detailed construction of initial data sets admitting expansions in powers of the geodesic distance is given in Dain and Friedrich (2001).

Appendix: some results of analysis

As in the main text of this chapter, let $(\mathcal{S}, \boldsymbol{h})$ denote a Riemannian manifold. Moreover, let $p \in \mathcal{S}$ denote a point and consider normal coordinates $\underline{x} = (x^\alpha)$ centred at p; that is, $x^\alpha(0) = 0$.

Order symbols. The behaviour of functions $f : \mathcal{S} \to \mathbb{R}$ near p can be conveniently described by means of the ***big*** O and ***small*** o notations. More precisely, given $f, g : \mathcal{S} \to \mathbb{R}$, if for some $\underline{x} = (x^\alpha)$ sufficiently close to 0 there exists a positive constant M such that

$$|f(\underline{x})| \leq M|g(\underline{x})|,$$

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one writes $f(\underline{x}) = O(g(\underline{x}))$, and one says that f *is at most of the order of* g. If, in addition, one has that

$$\partial_\alpha f(\underline{x}) = O(\partial_\alpha g(\underline{x})), \qquad \cdots \qquad \partial_{\alpha_1} \ldots \partial_{\alpha_k} f(\underline{x}) = O(\partial_{\alpha_1} \cdots \partial_{\alpha_k} g(\underline{x})),$$

for some integer k one writes $f(\underline{x}) = O_k(g(\underline{x}))$.

If given f, g one has $f(\underline{x})/g(\underline{x}) \to 0$ as $x^\alpha \to 0$, then one writes

$$f(\underline{x}) = o(g(\underline{x})),$$

and one says that *the order of* f *is bigger than that of* g. Again, if

$$\partial_\alpha f(\underline{x}) = o(\partial_\alpha g(\underline{x})), \qquad \ldots \qquad \partial_{\alpha_1} \cdots \partial_{\alpha_k} f(\underline{x}) = o(\partial_{\alpha_1} \cdots \partial_{\alpha_k} g(\underline{x})),$$

one writes $f(\underline{x}) = o_k(g(\underline{x}))$. For further discussion, see, for example, Courant and John (1989).

Taylor expansions. If a function $f : \mathbb{R}^n \to \mathbb{R}$ is of class C^k on the open ball $\mathcal{B}_a(0) \subset \mathbb{R}^n$ one has that

$$\begin{aligned} f(\underline{x}) = f(0) + \partial_\alpha f(0) x^\alpha + \frac{1}{2!} \partial_{\alpha_1} \partial_{\alpha_2} f(0) x^{\alpha_1} x^{\alpha_2} \\ + \cdots + \frac{1}{(k-1)!} \partial_{\alpha_1} \cdots \partial_{\alpha_{k-1}} f(0) x^{\alpha_1} \cdots x^{\alpha_{k-1}} + O(|x|^k). \end{aligned}$$

For further discussion, see, for example, Courant and John (1989).

Supremum norm. Given $\mathcal{U} \subset \mathbb{R}^n$ and $f \in C^k(\mathcal{U})$, one defines the supremum norm as

$$||f||_{C^k(\mathcal{U})} = \sum_{0 \le l \le k} \sup\{|\partial_{\alpha_1} \cdots \partial_{\alpha_l} f(\underline{x})| \, , \, \underline{x} \in \overline{\mathcal{U}}\}$$

where $\overline{\mathcal{U}}$ denotes the closure of $\mathcal{U}$. For further discussion on this and other related norms, see, for example, Ambrosetti and Prodi (1995).

Extension of smooth functions. Let $\mathcal{U} \subset \mathcal{S}$ denote a closed subset and $f : \mathcal{U} \to \mathbb{R}^k$ a smooth function. There exists a smooth function $\tilde{f} : \mathcal{S} \to \mathbb{R}^k$ such that $\tilde{f}|_{\mathcal{U}} = f$ and whose support is contained in $\mathcal{S} \setminus \mathcal{U}$; in other words, $\tilde{f}$ is non-vanishing in $\mathcal{S} \setminus \mathcal{U}$. In a slight abuse of notation $\tilde{f}$ will be denoted, again, by f. For more details on this result, see Lee (2002).

Dirac's delta. Let now $\mathcal{S}$ denote a compact manifold and $p \in \mathcal{S}$ a fixed point within. The Dirac's delta $\delta(p)$ with support on p is the *distribution* (i.e. a linear functional $C^0(\mathcal{S}) \to \mathbb{R}$) satisfying

$$\int_{\mathcal{S}} f(\underline{x}) \delta(p) \mathrm{d}\mu_{\boldsymbol{h}} = f(p), \qquad \text{for all } f \in C^0(\mathcal{S}).$$

In particular, one has that

$$\int_{\mathcal{S}} \delta(p) \, \mathrm{d}\mu_{\boldsymbol{h}} = 1.$$

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If $f(p) = 0$, one has the *distributional equality*

$$f(\underline{x})\delta(p) = 0.$$

For further details, the reader is referred to Appel (2007).

Divergence theorem. Given $(\mathcal{M}, \boldsymbol{g})$ a manifold with metric (Riemannian or Lorentzian) and, within, $\mathcal{U} \subset \mathcal{M}$ a compact subset and a smooth covector $\boldsymbol{\omega}$, one has

$$\int_{\mathcal{U}} \mathbf{div}\boldsymbol{\omega}\, \mathrm{d}\mu_{\boldsymbol{h}} = \int_{\partial\mathcal{U}} \langle \boldsymbol{\omega}, \boldsymbol{\nu} \rangle\, \mathrm{d}S_{\boldsymbol{h}},$$

with $\boldsymbol{\nu}$ the outward pointing unit normal to $\partial\mathcal{U}$; see, for example, Frankel (2003) for further details.

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