Canad. Math. Bull. Vol. 27 (4), 1984

LOCALLY UNIFORMLY ROTUND RENORMING AND INJECTIONS INTO $c_0(\Gamma)$

BY

G. GODEFROY, S. TROYANSKI, J. WHITFIELD* AND V. ZIZLER†

ABSTRACT. A norm $|\cdot|$ on a Banach space X is locally uniformly rotund (LUR) if $\lim |x_n - x| = 0$ for every x_n, $x \in X$ for which $\lim 2|x|^2 + 2|x_n|^2 - |x + x_n|^2 = 0$. It is shown that a Banach space X admits an equivalent LUR norm provided there is a bounded linear operator T of X into $c_0(\Gamma)$ such that $T^* c_0^*(\Gamma)$ is norm dense in X^*. This is the case e.g. if X^* is weakly compactly generated (WCG).

It is a well known result of J. Clarkson that a Banach space X admits an equivalent strictly convex norm if there is a bounded linear one-to-one operator T of X into some strictly convex Banach space Y (see [1] or [2]). For locally uniformly rotund norms the analogical result is no longer true, since e.g. $l_\infty(N)$, obviously possessing a bounded linear one-to-one operator into $l_2(N)$ still admits no equivalent LUR norm (see [2]). So, naturally, the following question arises: what additional property of a bounded linear one-to-one operator T of given Banach space X into, say, $c_0(\Gamma)$ would ensure that X admits an equivalent LUR norm? The space $c_0(\Gamma)$ can be chosen above since it is known to have an equivalent LUR norm (see [5] or [2]). One answer to this question is provided by Theorem 1. A good indication as to the uses of the methodology presented in this paper can be found in Theorem 2.

The main source of this paper was a more detailed study of the geometry in the Day's construction of a LUR norm on $c_0(\Gamma)$ ([5]) and its variant for the spaces with long Schauder basis ([7]). The paper originated in discussions made by the authors at the Winter School of Abstract Analysis in Czechoslovakia, January 1983 and was finished when the last named author was a member of Sonderforschungsbereich 72 der Universität Bonn.

We will work in real Banach spaces for which we will keep the standard notations. The letters i, j, k, l, m, n, p, s will be reserved to denote positive integers. The set of all positive integers will be denoted by N.

Received by the editors November 7, 1983.

* This author's research supported in part by NSERC (Canada) Grant A7535

† This author's research supported by Sonderforschungsbereich 72 der Universität Bonn (West Germany)

A. M. S. Classification number 46 B 20

Key words and phrases: renorming, local uniform rotundity, differentiability of the norm

https://doi.org/10.4153/CMB-1984-079-8 Published online by Cambridge University Press

DEFINITION 1. A norm $|\cdot|$ of a Banach space X is called locally uniformly rotund (LUR), resp. weakly locally uniformly rotund (WLUR), resp. weakly star locally uniformly rotund (W*LUR) (in the case of $X = Y^*$), if $\lim x_n = x$ in the norm, resp. weak, resp. weak star topology, for every x_n, $x \in X$ for which $\lim 2|x|^2 + 2|x_n|^2 - |x + x_n|^2 = 0$.

The key lemma of the paper is

LEMMA 1. *Suppose that the norm $|\cdot|$ of a Banach space X has the following two properties*

(i) *$|\cdot|$ is WLUR and*

(ii) *$|\cdot|^*$ – the dual norm of $|\cdot|$ on X^* – is LUR.*

Then X admits an equivalent LUR norm.

Proof. First, since $|\cdot|^*$ is LUR, there is a transfinite sequence Q_α of bounded linear projections $Q_\alpha : X^* \to X^*$, $0 \le \alpha \le \tau$, such that $Q_0 = 0$, $Q_\alpha \ne 0$ for $\alpha \ne 0$, $Q_\tau =$ Identity operator on X^*, $Q_\alpha Q_\beta = Q_\beta Q_\alpha = Q_\beta$ if $\beta \le \alpha$, and for all $x^* \in X^*$ and α, $Q_\alpha x^* \in \overline{\{Q_{\beta+1} x^*\}}$ and $(Q_{\alpha+1} - Q_\alpha)X^*$ is separable for all $0 \le \alpha < \tau$. These projections have the following properties:

(i) for all $x^* \in X^*$ and $\varepsilon > 0$, $\Lambda(x^*, \varepsilon) = \{\alpha < \tau, \ |Q_{\alpha+1} - Q_\alpha)x^*| \ge \varepsilon(|Q_\alpha| + |Q_{\alpha+1}|)\}$ is finite, and

(ii) for all $x^* \in X^*$,

$$x^* \in \operatorname{sp}\{(Q_{\alpha+1} - Q_\alpha)X^*, \alpha \in \Lambda(x^*)\}, \text{ where } \Lambda(x^*) = \bigcup \{\Lambda(x^*, \varepsilon), \varepsilon > 0\}.$$

This is a variant of a result of D. Amir and J. Lindenstrauss and was shown in [4]. Let us denote, for $0 \le \alpha < \tau$ and $f \in S_1^*$-the unit sphere of $(X, |\cdot|)^*$ by

$$h_\alpha(f) = |(Q_{\alpha+1} - Q_\alpha)f| / (|Q_{\alpha+1}| + |Q_\alpha|). \tag{1}$$

Furthermore, if K is a finite set of indexes α, $0 \le \alpha < \tau$, let $\{g_i^K\}_{i=1}^\infty$ be a sequence which is dense in the unit sphere of the space $\operatorname{sp}\{(Q_{\alpha+1} - Q_\alpha)X^*, \alpha \in K\}$, and for each such g_i^K, let $\{y_{i,j}^K\}_{j=1}^\infty$ be a sequence of the points of the unit sphere S_1 of X such that $\lim_j g_i^K(y_{i,j}^K) = 1$. Now we shall define a function which assigns to each four-terms sequence (f, n, p, l), $f \in S_1^*$, $n, p, l \in N$, a pseudonorm $E_{f,n,p,l}$ on X as follows:

First let $f \to Af = (\alpha_1, \alpha_2, \ldots)$ be a function which assigns to each $f \in S_1^* \subset (X, |\cdot|)^*$ a finite or infinite but countable sequence $(\alpha_1, \alpha_2, \ldots)$ $\alpha_j \in \Lambda(f)$ such that $h_{\alpha_{j+1}}(f) \ge h_{\alpha_j}(f)$, $j = 1, 2 \ldots$, and $\{h_{\alpha_j}(f)\}$ excerpts the whole set $\{h_\alpha(f), \alpha \in \Lambda(f)\}$. Now, if $j \in N$, let $M_{f,j}$ be the set (unordered) of the first j members of the sequence Af, if $j \le \operatorname{card} Af$; otherwise, for $j > \operatorname{card} Af$ put $M_{f,j} = M_{f,\operatorname{card} Af}$. Furthermore put

$$D_{f,n,p} = \operatorname{sp}\{y_{j,k}^K, K \subset M_{f,n}, j, k \le p\}$$

and let the desired pseudonorm $E_{f,n,p,l}$ be

$$E_{f,n,p,l}(x)=\left(f^2(x)+\frac{1}{l}\rho^2(x, D_{f,n,p})\right)^{1/2}, \quad \text{for} \quad x \in X,$$

where $\rho(x, D_{f,n,p})$ means the distance function to the subspace $D_{f,n,p}$. Now, if n, p, l are given positive integers, put

$$G_{n,p,l}(x)=\sup\{E_{f,n,p,l}(x), f \in S_1^* \subset (X, |\cdot|)^*\} \quad \text{for} \quad x \in X.$$

Finally, define the following norm on X:

$$\|x\|=\left(|x|^2+\sum \frac{1}{2^{n+p+l}} G^2_{n,p,l}(x)\right)^{1/2}, \quad \text{for} \quad x \in X.$$

Evidently, $\|\cdot\|$ is an equivalent norm on X. We shall now show that it is LUR. For it assume that x_j, $x \in X$ are so that

$$\lim 2\|x\|^2+2\|x_j\|^2-\|x+x_j\|^2=0 \tag{2}$$

and suppose without loss of generality that $|x|=1$. Then

$$2|x|^2+2|x_j|^2-|x+x_j|^2 \geq 2|x|^2+2|x_j|^2-(|x|+|x_j|)^2=(|x_j|-|x|)^2$$

and thus $\{x_j\}$ is bounded and by another simple convexity argument,

$$\lim 2|x|^2+2|x_j|^2-|x+x_j|^2=0. \tag{3}$$

Therefore, by WLUR of the norm $|\cdot|$ of X, we have that $\lim x_j=x$ in the weak topology of X. Thus to prove that $\lim|x_j-x|=0$, it suffices to show that x_j is precompact in the norm topology. Therefore, take an $\varepsilon>0$ and look for a finite ε-net for $\{x_j\}$. To find one, let first $f \in S_1^*$ be a unique element for which $f(x)=1$ (observe that $|\cdot|$ is Fréchet differentiable—see [1] or [2]). We show that

$$x \in \text{wcl}\{D_{f,n,p}, n, p \in N\},$$

where wcl$\{\cdot\}$ denotes the weak closure of $\{\cdot\}$. To see this, first observe that there are $g_{i_q}^{M_{f,s_q}}$, $q=1,2,\ldots$ (for definition of these see (1)), such that $\lim_q g_{i_q}^{M_{f,s_q}}=f$ (use the property (ii) of the projections Q_α). So, if we choose $y_{i_q,j_q}^{M_{f,s_q}}$ (for definition see again (1)), so that $g_{i_q}^{M_{f,s_q}}(y_{i_q,j_q}^{M_{f,s_q}})>1-\frac{1}{2}$, then we have that

$$|y_{i_q,j_q}^{M_{f,s_q}}+x| \geq g_{i_q}^{M_{f,s_q}}(y_{i_q,j_q}^{M_{f,s_q}}+x) \underset{q}{\to} 2.$$

Therefore, by WLUR of $|\cdot|$, we have that $\lim_q y_{i_q,j_q}^{M_{f,s_q}}=x$ in the weak topology of X. So, we can find a convex combination of some of these points which is no farther from x than $\varepsilon/4$. Thus there is a couple n, $p \in N$ such that

$$\rho(x, D_{f,n,p})<\varepsilon/4. \tag{4}$$

Suppose here without loss of generality that $D_{f,n,p}$ arises from $M_{f,n}$ which is formed by the first n members of the sequence $Af=(\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots)$ and that $h_{\alpha_n}(f) > \sup\{h_\alpha(f),\ \alpha \neq \alpha_i,\ i = 1, 2, \ldots, n\}$. Then, using the uniform equicontinuity of $\{h_\alpha(f)\}$, it is easy to see that there is a $\delta_1 > 0$ such that if $f_1 \in S_1^*$, $\rho(f_1, \{f, -f\}) < \delta_1$ then

$$\min\{h_\alpha(f_1), \alpha \in \{\alpha_1, \ldots, \alpha_n\}\} > \max\{h_\alpha(f_1), \alpha \notin \{\alpha_1, \ldots, \alpha_n\}\}$$

and thus

$$Af_1 = (\alpha_1', \alpha_2', \ldots, \alpha_n', \ldots), \text{ where } (\alpha_1', \alpha_2', \ldots, \alpha_n')$$

is a permutation of $(\alpha_1, \alpha_2, \ldots, \alpha_n)$. Therefore

$$M_{f,n} = M_{f_1,n} \quad \text{and thus} \quad D_{f,n,p} = D_{f_1,n,p}.$$

Then, by the use of Fréchet differentiability of $|\cdot|$, choose $\delta > 0$ so that if $h \in S_1^*$, $h^2(x) \geq 1 - \delta$, then $\rho(h, \{f, -f\}) < \delta_1$ and thus

$$D_{h,n,p} = D_{f,n,p}. \tag{5}$$

Finally chosen $l \in N$ so that $l > 4/\delta$. So, we have chosen $f \in S_1^*$, n, p, $l \in N$. We shall fix them by the end of our proof. From (2) we have that

$$\lim a_j = 0, \quad \text{where} \tag{6}$$

$a_j = 2G^2_{n,p,l}(x) + 2G^2_{n,p,l}(x_j) - G^2_{n,p,l}(x + x_j)$. Let $f_j \in S_1^*$ be such that

$$0 \leq c_j = \sup_{f \in S_1^*}\left(f^2(x + x_j) + \frac{1}{l}\rho^2(x + x_j, D_{f,n,p})\right) - f_j^2(x + x_j) - \frac{1}{l}\rho^2(x + x_j, D_{f_j,n,p}) \underset{j}{\rightarrow} 0. \tag{7}$$

Then we have

$$a_j \geq 2\left(f_j^2(x) + \frac{1}{l}\rho^2(x, D_{f_j,n,p})\right) + 2\left(f_j^2(x_j) + \frac{1}{l}\rho^2(x_j, D_{f_j,n,p})\right) - \left(f_j^2(x + x_j) + \frac{1}{l}\rho^2(x + x_j, D_{f_j,n,p})\right) - c_j = b_j - c_j, \text{ for some } b_j.$$

Here, by a simple convexity argument, $b_j \geq 0$ and since $b_j \leq a_j + c_j$, $\lim a_j + c_j = 0$ (see (6), (7)), we have that $\lim b_j = 0$. It follows from this and from a convexity argument that

$$\lim(f_j(x) - f_j(x_j)) = 0 \quad \text{and} \quad \lim(\rho(x_j, D_{f_j,n,p}) - \rho(x, D_{f_j,n,p})) = 0 \tag{8}$$

We now show that beginning with some j_0', each $f_j^2(x) \geq 1 - \delta$. Suppose the contrary is true, i.e. that there is a subsequence j_k, and elements $h_k \in S_1^*$ such that $h_k^2(x) > f_{j_k}^2(x) + \delta$. Then we would have by convexity arguments

$$a_{j_k} \geq 2\left(h_k^2(x) + \frac{1}{l}\rho^2(x, D_{h_k,n,p})\right) + 2\left(f_{j_k}^2(x_{j_k}) + \frac{1}{l}\rho^2(x_{j_k}, D_{f_{j_k},n,p})\right)$$

https://doi.org/10.4153/CMB-1984-079-8 Published online by Cambridge University Press

$$-\left(f_{j_k}^2(x+x_{j_k})+\frac{1}{l}\rho^2(x+x_{j_k}, D_{f_{j_k},n,p})\right)-c_{j_k}=$$

$$=2(h_k^2(x)-f_{j_k}^2(x))+\frac{2}{l}(\rho^2(x, D_{h_k,n,p})-\rho^2(x, D_{f_{j_k},n,p}))$$

$$+2\left(f_{j_k}^2(x)+\frac{1}{l}\rho^2(x, D_{f_{j_k},n,p})\right)+2\left(f_{j_k}^2(x_{j_k})+\frac{1}{l}\rho^2(x_{j_k}, D_{f_{j_k},n,p})\right)$$

$$-\left(f_{j_k}^2(x+x_{j_k})+\frac{1}{l}\rho^2(x+x_{j_k}, D_{f_{j_k},n,p})\right)-c_{j_k}$$

$$\geq 2(h_k^2(x)-f_{j_k}^2(x))+\frac{2}{l}(\rho^2(x, D_{h_k,n,p})-\rho^2(x, D_{f_{j_k},n,p}))-c_{j_k}$$

$$\geq 2\delta-\frac{4}{l}-c_{j_k}\geq \delta-c_{j_k}.$$

Since $\lim c_{j_k}=0$, we have arrived to the contradiction with the fact that $\lim a_j=0$. Thus, beginning with some j_0', $f_j^2(x)\geq 1-\delta$ and therefore $D_{f_j,n,p}=D_{f,n,p}$. So, by combining (8) with (4), we have that starting with some index j_0, it must be that

$$\rho(x_j, D_{f,n,p})<\varepsilon/2.$$

Since $\dim D_{f,n,p}<\infty$ and $\{x_j\}$ is bounded, one can easily find a finite $\varepsilon-$net for $\{x_j\}$. This completes the proof of Lemma 1.

THEOREM 1. *Let the norm $|\cdot|$ of a Banach space Y have the following two properties*

(i) *$|\cdot|$ is WLUR and*

(ii) *$|\cdot|^*$-the dual norm of $|\cdot|$ on X^* – is LUR.*

Let X be a Banach space which admits a bounded linear operator T of X into Y such that T^*Y^* is norm dense in X^*. Then X admits an equivalent LUR norm the dual of which is also LUR.

Proof. By a result in [3], X admits an equivalent norm, the dual of which is LUR. So, having in mind the Asplund's averaging technique (see [1] or [2]), to finish our proof by applying Lemma 1, it suffices to show that X admits an equivalent WLUR norm. This is easy to see, one can just construct the norm

$$\|x\|=(|x|^2+|Tx|^2)^{1/2}, \quad \text{for} \quad x\in X:$$

Then if we assume that x_n, $x\in X$ are such that $\lim 2\|x\|^2+2\|x_n\|^2-\|x+x_n\|^2=0$, then by the convexity argument, we have that $\{x_n\}$ is bounded and that $\lim Tx_n=Tx$ in the weak topology of Y, because of WLUR of $|\cdot|$ of Y. Thus, if $f\in Y^*$, then $\lim T^*f(x_n)=\lim f(Tx_n)=f(Tx)=T^*f(x)$ and since T^*Y^* is norm dense in X^* and $\{x_n\}$ is bounded, we have that $\lim x_n=x$ in the weak topology of X. This shows that $\|\cdot\|$ is WLUR. The proof of Theorem 1 is completed.

Remark 1. Theorem 1 applies e.g. if X^* is WCG. To see this use the well known fact that X^* then admits a relatively weakly compact Markuševič basis, i.e. biorthogonal system $\{e_\alpha, f_\alpha\}$, $\alpha \in \Gamma$, $e_\alpha \in X^*$, $f_\alpha \in X^{**}$ such that $\overline{\text{sp}}\{e_\alpha\} = X^*$, $\{f_\alpha\}$ is total on X^* and such that $\{e_\alpha\}$ is relatively weakly compact (see e.g. [3]). Then it is easy to see that the map T of X into $c_0(\Gamma)$ defined by $Tx(\alpha) = e_\alpha(x)$ can be used for Theorem 1, since $c_0(\Gamma)$ actually admits an equivalent LUR norm the dual of which is also LUR (see e.g. [2]). Since there are subspaces of WCG spaces, which are not themselves WCG ([6]), the following Theorem generalizes the remark above.

THEOREM 2. *Let X be a Banach space such that X^* is a subspace of a WCG Banach space Y. Then X admits an equivalent LUR norm.*

Proof. By the result in [3], X admits an equivalent norm the dual of which is LUR. So, to apply Lemma 1 we need only to show that X admits an equivalent WLUR norm. We show in fact that X^{**} admits an equivalent W*LUR norm. For it first take a bounded linear w^*-w continuous one-to-one map T of Y^* into $c_0(\Gamma)$ for some Γ (see Remark 1) and $|\cdot|$ be an equivalent LUR norm on $c_0(\Gamma)$. Then it is easy to check that the norm

$$\|f\| = (|f|^2 + |Tf|^2)^{1/2}, \quad \text{for} \quad f \in Y^*,$$

is a W*LUR dual norm on Y^*: Assuming that $\lim 2\|f_n\|^2 + 2\|f\|^2 - \|f + f_n\|^2 = 0$, we have by the LUR property of $|\cdot|$, that $\lim Tf_n = Tf$, and since T is a w^*-w homeomorphism on the balls of Y, we have that $\lim f_n = f$ pointwise on Y. Now, to see that X^{**} has an equivalent W*LUR norm, obviously it suffices to show the following simple fact: If Z_2 is a Banach space such that Z_2^* is W*LUR and Z_1 is a subspace of Z_2, then Z_1^* is also W*LUR. To see this, take $f, f_n \in Z_1^*$ such that $\lim 2\,|f|^2 + 2\,|f_n|^2 - |f + f_n|^2 = 0$. Let $\hat{f}$, $\hat{f}_n$ be the corresponding classes to f, f_n in $Z_2^*/Z_1^\perp$ and take f', $f'_n \in Z_2^*$ such that $f' \in \hat{f}$, $|f'| = |\hat{f}|$, $f'_n \in \hat{f}_n$, $|f'_n| = |\hat{f}_n|$. Then we have $0 \leq 2\,|f'|^2 + 2\,|f'_n|^2 - |f' + f'_n|^2 \leq 2\,|\hat{f}|^2 + |\hat{f}_n|^2 - |\hat{f} + \hat{f}_n|^2 = 2\,|f|^2 + 2\,|f_n|^2 - |f + f_n|^2 \to 0$, and thus $\lim f'_n = f'$ in the w^* topology of Z_2^*, since Z_2^* is W*LUR. Therefore $\lim f_n = f$ in the pointwise topology of Z_1^*. This completes the proof of Theorem 2.

ACKNOWLEDGEMENT. The last named author is deeply indebted to the Sonderforschungsbereich 72 der Universität Bonn for the support during the preparation of the paper. He also thanks the University of Dortmund for providing him with excellent working conditions during his stay here.

REFERENCES

1. M. M. Day, *Normed linear spaces*, 3rd. ed. Springer-Verlag, New York, 1973,

2. J. Diestel, *Geometry of Banach spaces.* Selected topics, Lecture Notes in Math. Vol. 485, Springer-Verlag, New York, 1975,

3. G. Godefroy, S. Troyanski, J. Whitfield and V. Zizler, *Smoothness in weakly compactly generated Banach spaces*, J. Funct. Anal. **52** (1983), 344–352.

4. K. John and V. Zizler, *Markuševič bases in some dual spaces*, Proc. Amer. Math. Soc. **50** (1975), 293–296,

5. J. Rainwater, *Local uniform convexity of Day's norm on* $c_0(\Gamma)$, Proc. Amer. Math. Soc. **22** (1969), 335–339,

6. H. P. Rosenthal, *The heredity problem for weakly compactly generated Banach spaces*, Compositio Math. **28** (1974), 83–111,

7. S. Troyanski, *On locally uniformly convex and differentiable norms in certain nonseparable Banach spaces*, Studia Math. **37** (1971), 173–180.

ÉQUIPE D'ANALYSE,
UNIVERSITÉ PARIS VI,
PLACE JUSSIEU, F-75230, PARIS,
CEDEX 05, FRANCE,
DEPARTMENT OF MATHEMATICS,
UNIVERSITY OF SOFIA,
BOUL. A. IVANOV 5,
SOFIA, BULGARIA,
DEPARTMENT OF MATHEMATICAL SCIENCES,
LAKEHEAD UNIVERSITY, THUNDER BAY,
ONTARIO, CANADA P7B5E1,
INSTITUT FÜR ANGEWANDTE MATHEMATIK DER UNIVERSITÄT,
WEGELERSTR. 6, D-5300 BONN,
WEST GERMANY
and
DEPARTMENT OF MATHEMATICS,
UNIVERSITY OF ALBERTA,
EDMONTON T6G261
ALBERTA, CANADA