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GREEN'S FUNCTIONS OF FREE PRODUCTS OF OPERATORS, WITH APPLICATIONS TO GRAPH SPECTRA AND TO RANDOM WALKS

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Abstract. We systematically develop an algebraic technique of free products of operators and their Green's functions. We apply this framework to obtain, in a simple and uniform fashion, several results on the spectra of graph Laplaceans and random walks.

Introduction

Let X, Y be discrete groups, and let $X * Y$ be their free product. Let A, B be bounded operators on $\ell^2(X), \ell^2(Y)$ respectively. Their *free product*, $A * B$, is an operator on $\ell^2(X * Y)$. The pairing $A, B \mapsto A * B$ is a natural operation. If A and B are convolution operators, then $A*B$ is a convolution operator. Let S and T be generating sets of the groups X and Y, and let A and B be the corresponding *incidence operators*. Then $A*B$ is the incidence operator on $X * Y$ corresponding to the generating set $S \cup T$.

The Green's functions of A, B and $A * B$ satisfy a system of algebraic equations. If A and B are selfadjoint convolution operators, the system yields considerable information about the spectrum of $A * B$ [7]. For instance, $A * B$ has no singular continuous spectrum. If $\sigma_p(A)$ and $\sigma_p(B)$ are the point spectra of A and B, then $\sigma_p(A * B) \subset \sigma_p(A) + \sigma_p(B)$. The (absolutely) continuous spectrum of $A * B$ is a union of a finite number of intervals.

The present work is rooted in the observation that the group structure is largely irrelevant for these results. Let X and Y be arbitrary countable sets, each with a marked element, the *root*. We say that X, Y are *rooted sets.* Their *free product*, $X * Y$, is a rooted set, as well. Let A and B be operators on $\ell^2(X)$ and $\ell^2(Y)$, respectively. We define the free product operator, $A * B$, on $\ell^2(X * Y)$. If X, Y are groups, and the roots are the

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respective identity elements, then $X * Y$ is the free product of groups, and $A * B$ coincides with the free product operator in the group sense.

Our construction extends to an arbitrary number of factors, yielding the free products $\mathcal{A} = *_{i=1}^{n} A_i$. Let $G_i, 1 \le i \le n$, and $\mathcal{G}$ be the Green's functions of $A_i, 1 \le i \le n$, and $\mathcal{A}$, respectively. We obtain a system of $n+1$ algebraic equations on G_i, and $\mathcal{G}$ (see Theorem 1 and Corollaries 1–3, especially Corollary 3). This system is a source of information about $\mathcal{G}$. If the operators A_i are selfadjoint and *invariant* with respect to groups Γ_i, transitively acting on $X_i, 1 \le i \le n$, then $\mathcal{G}$ determines the spectrum of $\mathcal{A}$.

The system of equations of Corollary 3 can be explicitly solved only in very special cases. In Section 3 we investigate two such cases. In Section 3.1 we consider the free products $A * B$ where each of the operators A, B has two distinct eigenvalues (with arbitrary multiplicities). By Proposition 1, the Green's function of $A * B$ is single–valued on a double covering of $\mathbf{C}$, given by $\mathcal{R} = \{(t, w) : w^2 = R(t)\}$, where $R(t)$ is a quartic polynomial. In particular, $\mathcal{R}$ is an elliptic curve. In Section 3.2 we obtain the Green's function of any $*^n A$, if A has two eigenvalues (Proposition 5).

Explicit expressions for the Green's functions yield the spectra of the corresponding free product operators (Proposition 2, Theorems 2 and 3). The absolutely continuous spectrum consists of one or two intervals, whose endpoints are algebraic functions of the parameters involved (Theorems 2 and 3). The point spectrum is more involved. We completely determine the point spectra of $*^n A$ (Theorem 3), and get a good bound on the point spectra of $A * B$ (Proposition 2). We will report more comprehensively on this subject elsewhere [5].

There has been much interest in the spectra of graph Laplaceans, and, more generally, in the random walks on graphs [12]. In particular, there is considerable literature on random walks on free group products, and the spectra of related graphs (see the references in [12]). Several seemingly unrelated techniques have been used to calculate explicitly these spectra in special cases [1, 2, 3, 6, 7, 8, 10, 11].

The approach developed here allows to obtain these results in a simple and uniform way. Let K_n be the complete graph on $n+1$ vertices. In Section 4.1 we calculate the spectra of $K_m * K_n$ and $*^n K_r$ for arbitrary m, n, r (Theorems 4 and 5). In Section 4.2 we determine the spectra of the product random walks on $K_m * K_n$, and of the simple random walks on $*^n K_r$, for arbitrary m, n, r (Theorems 6 and 7). We use the results of

Section 3, and exploit the elementary fact that the spectrum of a complete graph consists of two numbers.

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§1. Preliminaries

A *rooted set*, (X, e), is a countable (at most) set with a distinguished element, the root. We denote by $\ell^2(X)$ the Hilbert space of square summable functions on X. There is a natural correspondence between operators and *kernels*:

$$(Af)(x) = \sum_{y\in X} A(x,y)f(y).$$

If $A(x,y) \neq 0$, we write $x \rightsquigarrow y$. If $x \rightsquigarrow y$ and $y \rightsquigarrow x$, we say that x, y are neighbors, and write $x \sim y$.

CONDITION 1. i) For all $x \in X$, we have $|\{y : x \sim y\}| = q(x) \leq q(A) = q < \infty$; ii) $\max_{x,y} |A(x,y)| = p(A) = p < \infty$.

EXAMPLE. The set X is a regular graph of degree q (i.e., q neighbors of a vertex), and A is the *incidence operator*: $A(x,y) = A(y,x) = 1$ if x and y are connected by an edge, and $A(x,y) = 0$ otherwise.

In what follows we assume Condition 1 (unless stated otherwise). Then A is bounded, $||A|| \leq pq$ (by Schwarz' inequality). Set

$$A^n(x,y) = \sum_{v_1,\dots,v_{n-1}} A(x,v_{n-1})\cdots A(v_1,y),\ n>0,\ A^0(x,y) = \delta_{x,y},$$

and define

$$F_{x,y}(t) = \sum_{n\geq 0} A^n(x,y)t^n. \tag{1}$$

By Condition 1, the series converges for sufficiently small $|t|$, and $F_{x,y}(t) = (1-tA)^{-1}(x,y)$. We set $F_x(t) = F_{x,x}(t)$, and call it the *return function* for x. The *first return function*, $f_x(t)$, is defined by

$$f_x(t) = \sum_{n\geq 1} t^n \left(\sum_{v_\imath \neq x} A(x,v_1)\cdots A(v_{n-1},x) \right), \tag{2}$$

and we have

$$F_x(t) = (1 - f_x(t))^{-1}. \tag{3}$$

We denote by $G_{x,y}(t)$ the kernel of the *resolvent*, $R(t) = (t - A)^{-1}$. For $|t|$ sufficiently large

$$G_{x,y}(t) = \frac{1}{t} F_{x,y}(t^{-1}). \tag{4}$$

We set $G_x(t) = G_{x,x}(t)$. In what follows we assume that $A(x,y) = A(y,x) \in \mathbf{R}$. Then A is a bounded selfadjoint operator. The formulas above provide a connection between the return functions and the *spectrum* of A.

EXAMPLE. For $|X| = n < \infty$, let $\lambda_1 \leq \cdots \leq \lambda_n$ be the eigenvalues of A, and let $\phi_i(x)$ be the corresponding orthonormal (real) eigenfunctions. Then

$$G_{x,y}(t) = \sum_i \frac{\phi_i(x)\phi_i(y)}{t - \lambda_i} \tag{5}$$

is rational, with the simple poles at the eigenvalues of A.

If A is the incidence operator of a graph X, the return functions have a geometric interpretation: $A^n(x,y)$ is the number of walks on X of length n starting at y and ending at x. Thus $F_x(t)$ is the counting function for the closed walks starting at x, and $f_x(t)$ counts the closed walks that do not come back prematurely.

DEFINITION 1. *Let A be an operator on $\ell^2(X)$. We say that A is* invariant *if there is a group, Γ, acting transitively on X, and $A(gx, gy) = A(x,y)$ for $g \in \Gamma$. If X is a graph, and its incidence operator is invariant, we say that the graph is* symmetric.

If A is invariant, then $F_x(t)$, $f_x(t)$, and $G_x(t)$ do not depend on x. Let $|X| < \infty$, and let A be an invariant operator. In the preceding example, for any x

$$G_x(t) = \sum_i \frac{\phi_i^2(x)}{t - \lambda_i}.$$

Denote by Λ the *set of eigenvalues* of A, and let $m(\lambda) \geq 1$ be the multiplicity, $\sum_\lambda m(\lambda) = |X|$. Since $\sum_x \phi_i^2(x) = 1$ for any i, we have

$$G_x(t) = \frac{1}{|X|} \sum_{\lambda \in \Lambda} \frac{m(\lambda)}{t - \lambda} = G(t), \quad F_x(t) = \frac{1}{|X|} \sum_{\lambda \in \Lambda} \frac{m(\lambda)}{1 - \lambda t} = F(t). \tag{6}$$

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Let the notation be as above. If $G_x(t)$ does not depend on $x \in X$, or if x is a distinguished element (see Section 2 below), we say that $G_x(t) = G(t)$ is the *Green's function*.

§2. Free Products of Operators and Green's Functions

Let $(X_i, e_i), 1 \leq i \leq n, n \geq 2$, be arbitrary rooted sets. Identifying the roots in the union, $\cup_i X_i$, we obtain the *bouquet*, $(\#_i X_i, e)$, of the sets X_i, where the root e is the image of the points e_i. The notation $(\#_{i \in I} X_i, e)$, where $I \subset \{1, \dots, n\}$, is self-explanatory. We will now define the *free product*, $*_{i=1}^n (X_i, e_i) = (\mathcal{X}, e)$.

Set $\tilde{X}_i = X_i \backslash \{e_i\}$, and $\mathcal{X}_1 = \#_{i=1}^n X_i$. Then $\mathcal{X}_1 = \{e\} \cup \tilde{X}_1 \cup \cdots \cup \tilde{X}_n$, a disjoint union. Let $x \in \mathcal{X}_1,\ x \neq e$. There is a unique $i \in \{1, \dots, N\}$, such that $x \in \tilde{X}_i$. We identify x with the root of the bouquet $\#_{j \neq i} X_j$. After having done this for all $x \in \mathcal{X}_1$, we obtain a new rooted set, $\mathcal{X}_2$, where $e \in \mathcal{X}_1 \subset \mathcal{X}_2$. Continuing this process indefinitely, we obtain the increasing tower $\{e\} \subset \mathcal{X}_1 \subset \mathcal{X}_2 \subset \cdots \subset \mathcal{X}_m \subset \cdots$. The union, $\mathcal{X} = \cup_{k=0}^{\infty} \mathcal{X}_k$, is the free product of $X_i, 1 \leq i \leq n$. For any point $(x) \in \mathcal{X}, (x) \neq e$, there is a unique sequence, $i_1, \dots, i_m$ of indices $i_k \in \{1, \dots, n\},\ i_k \neq i_{k-1}$ for all k, and for each i_k there is a unique $x_{i_k} \in \tilde{X}_{i_k}$ so that (x) is coded by the sequence $(x_{i_1}, \dots, x_{i_m})$. The correspondence between $\mathcal{X}$ and the set of such sequences is one-to-one. We think of elements $(x) \in \mathcal{X}$ as words, and of $m = |(x)|$ as the "length" of (x). Then $\mathcal{X}_m = \{(x) : |(x)| \leq m\}$, and $|(x)| = 0$ if and only if $(x) = e$.

The construction of $*_{i=1}^n X_i$ does require the sets X_i to be rooted, but nothing else. If each X_i is the vertex set of a graph with a distinguished vertex, e_i, then $*_{i=1}^n X_i$ is the *free product of the graphs* $X_i, 1 \leq i \leq n$. If each X_i is a group, and $e_i \in X_i$ is the identity, then $*_{i=1}^n X_i$ is the *free product of the groups* $X_i, 1 \leq i \leq n$, and e is the identity element.

Let the notation be as above, and let $A_i(x, y)$ be operators on $\ell^2(X_i)$, $1 \leq i \leq n$. We will define operators $\mathcal{A}_i$ on $\ell^2(\mathcal{X})$ via their kernels $\mathcal{A}_i((x), (y))$. Let, for simplicity of notation, $i = 1$. Let $(x) = x_{i_1} \cdots x_{i_k}$ and $(y) = y_{j_1} \cdots y_{j_l}$ be in $\mathcal{X}$. If $k = 0$ then $(x) = e$, and we set $\mathcal{A}_1(e, (y)) = A_1(e_1, y_1)$, if $l = 1$ and $j_1 = 1$, and $\mathcal{A}_1(e, (y)) = 0$ otherwise. For $k > 0$ we set $\mathcal{A}_1((x), (y)) = A_1(x_{i_k}, y_{j_k})$, if $l = k,\ x_{i_1} \cdots x_{i_{k-1}} = y_{j_1} \cdots y_{j_{k-1}}$, and $i_k = j_k = 1$. Otherwise $\mathcal{A}_1((x), (y)) = 0$. If all of A_i satisfy Condition 1, then $\mathcal{A}_i$ satisfy it as well, and $p(\mathcal{A}_i) = p(A_i), q(\mathcal{A}_i) = q(A_i)$.

Definition 2. *Let the notation be as above. The operator* $\sum_{i=1}^n \mathcal{A}_i$ *on*

$\ell^2(*_{i=1}^n X_i)$ *is the* free product *of the operators* $A_1, \dots, A_n$. *We denote it by* $\mathcal{A} = *_{i=1}^n A_i$.

EXAMPLES. 1. Let $X_i, 1 \le i \le n$, be rooted graphs, and let A_i be the incidence operator of X_i. Then $*_{i=1}^n A_i$ is the incidence operator of the graph $*_{i=1}^n X_i$. 2. Let A_i be the transition operator for a random walk on $X_i, 1 \le i \le n$. Let $\mu_i \ge 0, \sum_{i=1}^n \mu_i = 1$. The operator $*_{i=1}^n \mu_i A_i$ is the transition operator for a random walk on $*_{i=1}^n X_i$.

For $i \in \{1, \dots, n\}$ we denote by $\mathcal{X}^{(i)} \subset \mathcal{X}$ the set of words $(x) = x_{i_1} \cdots x_{i_k}$ such that $i_1 = i$, and, by convention, set $e \in \mathcal{X}^{(i)}$. Then $\mathcal{X}^{(i)}$ are rooted sets, and $\mathcal{X} = \#_{i=1}^n \mathcal{X}^{(i)}$. Set $\mathcal{Y}^{(i)} = \#_{j \ne i} \mathcal{X}^{(j)}$.

In what follows we consider various rooted sets. The root will always be clear from the context, and we suppress it from notation. For $1 \le i \le n$ we denote by $(f_i(z))$ $F_i(z), G_i(z)$ the (first) return function and the Green's function for A_i, and by $(\phi_i(z))$ $\Phi_i(z)$ the (first) return function for $\mathcal{A}_i$ on $\mathcal{X}^{(i)}$. In the formula below the "hat" above a symbol means the symbol is deleted.

THEOREM 1. *Let the setting be as above. Then, for* $1 \le i \le n$,

$$F_i\left(\frac{z}{1-\phi_1(z)-\cdots-\hat{\phi}_i-\cdots-\phi_n(z)}\right) = \frac{1-\phi_1(z)-\cdots-\hat{\phi}_i-\cdots-\phi_n(z)}{1-\sum_{k=1}^n \phi_k(z)}. \tag{7}$$

Proof. We will use the language of graphs in the argument, linking the vertices x, y whenever $C(x, y) \ne 0$, for a suitable kernel. Set, for simplicity of notation, $i = 1$, and compute the series $\phi_1(z)$. Let Γ be any loop contributing to $\phi_1(z)$. The first move of Γ is from e to some $x_1 \in X_1$. At x_1 there is a subset of $\mathcal{X}$, isomorphic to $\mathcal{Y}^{(1)}$, "attached" by its root to x_1. From x_1 the "walk" goes into $\mathcal{Y}^{(1)}$, and makes a loop, β_1, in $\mathcal{Y}^{(1)}$, which ends upon return to x_1 for the last time. After that Γ moves to another vertex, $x_2 \in X_1$. Repeating this construction, we obtain a unique decomposition:

$$\Gamma: \ e \to x_1, \beta_1, x_1 \to x_2, \beta_2, \dots, x_{k-1}, \beta_{k-1}, x_{k-1} \to e.$$

The points $x_1, \dots, x_{k-1}$ are in X_1, and are all different from e. Thus $e \to x_1 \to \cdots \to x_{k-1} \to e$ is a first return loop, γ, in X_1, and $\beta_1, \dots, \beta_{k-1}$ are (rooted) closed walks in $\mathcal{Y}^{(1)}$. The contribution of Γ to $\phi_1(z)$ factorizes in an obvious way. To write it down, we use the general notation $h(z|\alpha)$ for the contribution of α to a counting function, $h(z)$. Denote by $(\psi_i(z))$ $\Psi_i(z)$ the (first) return function for the operator $\mathcal{A}$ restricted to $\mathcal{Y}^{(i)}$. Then

$$\phi_1(z|\Gamma) = f_1(z|\gamma)\prod_{j=1}^{k-1}\Psi_1(z|\beta_j).$$

The correspondence $\Gamma = (\gamma; \beta_1 \ldots, \beta_{k-1})$ is one-to-one. Fixing γ and summing up over all $\beta_1 \ldots, \beta_{k-1}$, we obtain an expression for the contribution to $\phi_1(z)$ from all loops Γ corresponding to the same $\gamma = \bar{\Gamma}$, $|\gamma| = k > 0$:

$$\text{(8)} \qquad \sum_{\bar{\Gamma}=\gamma}\phi_1(z|\Gamma) = f_1(z|\gamma)\Psi_1(z)^{k-1} = f_1(z\Psi_1(z)|\gamma)/\Psi_1(z).$$

Since $\mathcal{Y}^{(1)} = \#_{i\neq 1}\mathcal{X}^{(i)}$, we have $\Psi_1(z) = \Big(1 - \hat{\phi}_1(z) - \phi_2(z)\cdots - \phi_n(z)\Big)^{-1}$. Substituting this into eq. (8), summing up over all γ, and replacing '1' with an arbitrary i:

$$\text{(9)} \quad \phi_i(z)/[1 - \sum_{j\neq i}\phi_j(z)] = f_i\left(\frac{z}{1 - \phi_1(z) - \cdots - \hat{\phi}_i - \cdots - \phi_n(z)}\right).$$

Expressing f_i in terms of F_i, by eq. (3), we obtain the claim. $\square$

Remark. The argument above does not use all of our assumptions on X_i and A_i, only that $q_i(x)$ is finite for any $x \in X_i$.

We denote by $\mathcal{G}(z)$ and $\mathcal{F}(z)$ the Green's function and the return function for the operator $\mathcal{A}$ on $\ell^2(\mathcal{X})$.

COROLLARY 1. (see ([7, 8] in the group case) *Let the notation be as above, and set*

$$s_i(w) = w(\phi_1(w^{-1}) + \cdots + \hat{\phi}_i(w^{-1}) + \cdots + \phi_n(w^{-1})).$$

Then, for $1 \leq i \leq n$,

$$\mathcal{G}(w) = G_i(w - s_i(w)).$$

Proof. By the proof of Theorem 1

$$\text{(10)} \qquad \mathcal{F}(z) = (1 - \phi_1(z) - \cdots - \phi_n(z))^{-1}.$$

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Substituting this into eq. (7), and using eq. (4), we obtain

$$\mathcal{G}(\frac{1}{z}) = G_i\left(\frac{1-\phi_1(z)-\cdots-\hat{\phi}_i-\cdots-\phi_n(z)}{z}\right). \tag{11}$$

With $z^{-1}=w$, this implies the claim. ∎

Remark. Specializing Corollary 1 to the case when A_i are convolution operators on the groups X_i, and $n=2$, we obtain a known result ([7], Theorem 5.1). If A_i are the incidence operators of Cayley graphs (here n is arbitrary), then Corollary 1 (equivalently, Theorem 1) is also in the literature ([8], Theorem 4.9).

COROLLARY 2. *Let the notation be as above, and set, for $1 \le i \le n$*

$$\xi_i(z) = \frac{1-\phi_1(z)-\cdots-\hat{\phi}_i-\cdots-\phi_n(z)}{z}. \tag{12}$$

Then eq. (7) is equivalent to the system ($1 \le i \le n$)

$$G_i(\xi_i(z)) = \frac{n-1}{\sum_{i=1}^n \xi_i(z) \;-\; z^{-1}}. \tag{13}$$

The Green's function of $(\mathcal{X}, e)$ satisfies

$$\mathcal{G}(z) = \frac{n-1}{\sum_{i=1}^n \xi_i(z^{-1}) \;-\; z}. \tag{14}$$

Proof. Straightforward computation, using eq. (11). ∎

We put Corollary 2 in a form more suitable for our applications.

COROLLARY 3. *Let $(X_i, e_i), 1 \le i \le n$, be arbitrary rooted sets, let A_i be operators on $\ell^2(X_i)$, and let $G_i(z)$ be their Green's functions. Let $x_i = x_i(t)$ be the solutions of the system ($1 \le i \le n$)*

$$G_i(x_i) = \frac{n-1}{\sum_{i=1}^n x_i(t) - t}. \tag{15}$$

Then the Green's function of the free product operator, $\mathcal{A} = *_{i=1}^n A_i$, is given by

$$\mathcal{G}(t) = \frac{n-1}{\sum_{i=1}^n x_i(t) - t}. \tag{16}$$

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The following is known in the group case (compare with [7], Corollary 5.2).

COROLLARY 4. *Let (X_i, e_i) be finite rooted sets, and let A_i be (selfadjoint) operators on $\ell^2(X_i)$, $1 \le i \le n$. Then the Green's function of the free product, $*_{i=1}^n A_i$, is* algebraic.

Proof. The Green's functions G_i are rational, hence the solutions of the system (15) are algebraic functions. Eq. (16) implies the claim. ∎

§3. Explicit Green's Functions and Spectra

The system (15) can be explicitly solved only in special cases. We will do this for two classes of examples. As a benefit, we will completely determine the spectra of certain free product operators.

Let G be a Green's function of a selfadjoint operator L. That is, for $t \in \mathbf{H}$, the upper half-plane, $G(t) =< f|(t-L)^{-1}f >$, for a certain vector f. Note that our definition of the resolvent, $R(t) = (t-L)^{-1}$, agrees with the one in [4, 7], and differs by sign from the one in [9]. The spectral measure of L is determined by $\lim G(x+i\epsilon)$, as $\epsilon \to 0$. We say that a Green's function is *algebraic*, if its analytic continuation is an algebraic function on a Riemann surface, $\mathcal{R}$, which is a finite-sheeted branched covering of the Riemann sphere, $p : \mathcal{R} \to \mathbf{C}$. The following problem typically arises in this situation. Given $G(t)$ as a function on $\mathcal{R}$, and $r \in \mathcal{R}$, such that $p(r) \in \mathbf{R} \subset \overline{\mathbf{H}}$, determine whether r belongs to the 'physical sheet' of $\mathcal{R}$. We will use a well known fact, which we formulate as a lemma, for future reference.

LEMMA 1. *Let G be an algebraic Green's function, and let $p : \mathcal{R} \to \mathbf{C}$ be the corresponding branched covering. Let $t \in \mathbf{R}$ be a point, which is not in the branch locus of p, and let $r \in \mathcal{R}$ be a point above t.*

1. If G has a pole at r, and $\mathrm{Res}_r G < 0$, then r is not in the physical sheet.

2. Let $G(r)$ be finite. If $G(r) \in \mathbf{R}$ and $G'(r) > 0$, then r is not in the physical sheet.

3.1. Free products of two operators

Throughout this subsection we consider invariant selfadjoint operators A and B on $\ell^2(X)$ and $\ell^2(Y)$ respectively, where X and Y are countable (e. g., finite) sets. Then $A * B$ is an invariant selfadjoint operator on $\ell^2(X * Y)$. We are interested in the Green's function and the spectrum of $A * B$. The standing assumption will be that the spectrum of each of the operators A, B is pure point, with two distinct eigenvalues.

By the preceding material, the Green's functions satisfy

$$G_A(z) = \frac{u}{z-a} + \frac{v}{z-b},\ G_B(z) = \frac{r}{z-c} + \frac{s}{z-d} \tag{17}$$

where $a < b, c < d$, and $u, v, r, s > 0$, see eq. (6). Set

$$g = (2u-1)b + (2v-1)a,\ h = (2r-1)d + (2s-1)c,\ S = a+b+c+d,$$

$$\Sigma = g+h,\ \Delta = g-h,\ T = (a+b)(c+d) + 2(ab+cd),$$

and define the following polynomials:

$$M(t) = -2t + S, \tag{18}$$

$$N(t) = -\Sigma t^2 + S\Sigma t - \frac{1}{2}[\Sigma T + g(c-d)^2 + h(a-b)^2], \tag{19}$$

$$D(t) = (t-(a+c))(t-(a+d))(t-(b+c))(t-(b+d)), \tag{20}$$

$$E(t) = ght^2 - Sght + \frac{1}{4}[g^2(c-d)^2 + h^2(a-b)^2 + 2ghT]. \tag{21}$$

PROPOSITION 1. *Let A and B be operators, satisfying the standing assumptions. Let the notation be as above, and set $R(t) = D(t) + E(t)$. Then the Green's function of $A * B$ satisfies*

$$\mathcal{G}(t) = \frac{N(t) - M(t)\sqrt{R(t)}}{2D(t)}. \tag{22}$$

Proof. Set $x = x_1(t), y = x_2(t)$. By Corollary 3

$$\frac{u}{x-a} + \frac{v}{x-b} = \frac{r}{y-c} + \frac{s}{y-d} = \frac{1}{x+y-t}.$$

This is equivalent to the system (here we use $u+v = r+s = 1$, see eq. (6))

$$xy = (t + \frac{1}{2}(g-a-b))x + \frac{1}{2}(g+a+b)y + (-\frac{1}{2}((g+a+b)t + ab),$$

$$xy = \frac{1}{2}(h+c+d)x + (t + \frac{1}{2}(h-c-d))y + (-\frac{1}{2}((h+c+d)t + cd).$$

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We separate the variables to arrive to two quadratic equations on x, y with polynomial (in t) coefficients. Solving them, and using eq. (16), we obtain the Green's function of $A * B$:

$$\mathcal{G}(t) = \frac{2(t + \frac{1}{2}(\Delta - S))(-t + \frac{1}{2}(\Delta + S))}{N(t) + M(t)\sqrt{R(t)}}. \tag{23}$$

Multiplying the numerator and the denominator in eq. (23) by $N(t) - M(t) \cdot \sqrt{R(t)}$, we get in the denominator:

$$N^2(t) - M^2(t)[D(t) + E(t)] = [N^2(t) - M^2(t)E(t)] - M^2(t)D(t).$$

We will prove the identity

$$N^2(t) - M^2(t)E(t) = \Delta^2 D(t).$$

Set $N^2(t) - M^2(t)E(t) = X(t) = \sum_{i=0}^4 X_i t^i$, and $D(t) = \sum_{i=0}^4 D_i t^i$. The coefficients X_i, D_i are polynomials in a, b, c, d, g, h, and the identity above is equivalent to the system $(0 \leq i \leq 4)$

$$X_i(a, b, c, d, g, h) = \Delta^2 D_i(a, b, c, d). \tag{24}$$

These identities are verified directly from eqs. (18–21). For $i = 4, 3$ the verification is very simple. To sketch a proof of eq. (24) for $i = 2, 1, 0$, we will use a self-explanatory notation for the coefficients of the polynomials $N(t), M(t)$. Then, by eqs. (18–21)

$$X_2 = -2\Sigma N_0 - 4E_0 + S^2\Sigma^2 - 5S^2gh = -2\Sigma N_0 - 4E_0 - S^2gh + \Delta^2 S^2.$$

By eqs. (19,21)

$$-2\Sigma N_0 - 4E_0 - S^2 gh = \Delta^2 T,$$

hence $X_2 = \Delta^2(S^2 + T)$. By eq. (18), $D_2 = S^2 + T$, which proves eq. (24) for $i = 2$. The identity $X_1 = \Delta^2 D_1$ is equivalent to the one we have just proved (note that $D_1 = -ST$). For $i = 0$ eq. (24) becomes

$$N_0^2 - S^2 E_0 = \Delta^2(a + c)(a + d)(b + c)(b + d).$$

This is proved directly from eqs. (19,21), and we leave it to the reader.

Our identity and eq. (18) yield

$$N^2(t) - M^2(t)R(t) = 4(t + \frac{1}{2}(\Delta - S))(-t + \frac{1}{2}(\Delta + S))D(t) \tag{25}$$

which proves the claim.

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Proposition 1 means, in particular, that the Green's function of $A * B$ is algebraic on the Riemann surface $\mathcal{R} = \{(t, w) : w^2 = R(t)\}$, which is a 2-sheeted covering of the Riemann sphere, via $p(t, w) = t$. The sheets of $\mathcal{R}$ correspond to the two branches of $\sqrt{R(t)}$, where on the physical sheet we have, asymptotically, $\sqrt{R(t)} \sim t^2$, as $t \to \infty$. The involution $(t, w) \mapsto (t, -w)$ interchanges the physical and the nonphysical sheets. Note that $\mathcal{R}$ is an elliptic curve, since R is a quartic polynomial, and $p : \mathcal{R} \to \mathbf{C}$ is the canonical covering.

Let $\mathcal{R}_-$ (resp. $\mathcal{R}_+$) be the part of the physical (resp. nonphysical) sheet above $\mathbf{H}$. Since the branch locus of $p : \mathcal{R} \to \mathbf{C}$ consists of zeros of the quartic polynomial R, and since $\mathcal{R}_-$ and $\mathcal{R}_+$ are disjoint, we conclude that R has no roots in $\mathbf{H}$. Thus, all roots of R are real, which also follows directly from eqs. (20-21), see the proof of Theorem 2, below.

Let the notation be as above. If the inequalities

$$a + d \neq S/2, \tag{26}$$

$$E(a + c), E(b + d), E(a + d), E(b + c) \neq 0 \tag{27}$$

hold, we say that the operators A, B are *in general position.*

PROPOSITION 2. *Let the operators A and B satisfy the standing assumptions. If they are in general position then the point spectrum of $A * B$ consists of at most two eigenvalues; One of them is contained in $\{a+c, b+d\}$, and the other in $\{a + d, b + c\}$.*

Proof. By the preceding discussion and eq. (22), the poles of $\mathcal{G}$ on the Riemann surface $\mathcal{R} = \{(t, w) : w^2 - R(t) = 0\}$ are contained in the set $p^{-1}(\{a+c, b+d, a+d, b+c\})$, where $p : \mathcal{R} \to \mathbf{C}$ is the canonical projection. By eqs. (26-27), $|p^{-1}(\{a + c, b + d, a + d, b + c\})| = 8$. Let t_0 be any of the four numbers $a + c, a + d, b + c, b + d$, and let $p_0, r_0 \in \mathcal{R}$ be the two points above it. Note that if $p_0 = (t_0, w_0)$, then $r_0 = (t_0, -w_0)$. Since, by eq. (25)

$$(N(t_0) - M(t_0)w_0)(N(t_0) + M(t_0)w_0) = 0$$

and

$$N(t_0) + M(t_0)w_0 = 0, N(t_0) - M(t_0)w_0 = 2N(t_0) \neq 0,$$

the function $\mathcal{G}$ has a pole at p_0, and r_0 is a regular point. Thus, $\mathcal{G}$ has four poles in $\mathcal{R}$, one above each of the numbers $a+c, a+d, b+c, b+d$. Moreover, the poles are simple, and the residue of $\mathcal{G}$ at p_0 is $2N(t_0)/D'(t_0) \neq 0$.

By straightforward computations

$$N(a+c)=N(b+d)=-\frac{1}{2}(b-a+d-c)[g(d-c)+h(b-a)],$$

$$N(a+d)=N(b+c)=-\frac{1}{2}(b+c-a-d)[g(c-d)+h(b-a)]$$

and

$$0<D'(b+d)=[b-a+d-c](d-c)(b-a)=-D'(a+c),$$

$$D'(b+c)=-[b+c-a-d](d-c)(b-a)=-D'(a+d).$$

Thus

$$\frac{N(a+d)}{D'(a+d)}=\frac{1}{2}\left[\frac{g}{b-a}-\frac{h}{d-c}\right]=-\frac{N(b+c)}{D'(b+c)}, \tag{28}$$

$$\frac{N(a+c)}{D'(a+c)}=\frac{1}{2}\left[\frac{g}{b-a}+\frac{h}{d-c}\right]=-\frac{N(b+d)}{D'(b+d)}. \tag{29}$$

Next, we invoke Lemma 1. By eq. (28) (resp. eq. (29)), at most one of the two poles of $\mathcal{G}$ above $a+d, b+c$ (resp. above $a+c, b+d$) is in the physical sheet. ∎

A complete analysis of the point spectrum of $A*B$ requires detailed calculations. We will return to this and related questions elsewhere [5]. Next we turn to the continuous spectrum of $A*B$.

PROPOSITION 3. *Let the operators A and B satisfy the standing assumptions (we don't assume they are in general position), and let the notation be as above. Then $A*B$ has no singular continuous spectrum: $\sigma_c(A*B)=\sigma_{ac}(A*B)$. If $t_1<t_2<t_3<t_4$ are the roots of the quartic polynomial $R=D+E$, then $\sigma_c(A*B)=[t_1,t_2]\cup[t_3,t_4]$.*

Proof. Set $L=A*B, Z=X*Y$. Then L is an invariant selfadjoint operator on $\ell^2(Z)$. Let $\mathcal{G}_z(t), z\in Z, t\in \mathbf{H}$ be the Green's function of L, corresponding to $\delta_x\in\ell^2(Z)$ [4]. By invariance of L, we have $\mathcal{G}_z(t)=\mathcal{G}_e(t)$, where e is the root of Z. Let μ be the spectral measure, corresponding to δ_e, and let μ_f be the one corresponding to $f\in\ell^2(Z)$ [4]. Then, by invariance of L, the measure μ_f is absolutely continuous with respect to μ [7], hence μ is the spectral measure of L.

Let $\mu=\mu_{ac}+\mu_{sc}+\mu_{pp}$ be the standard decomposition of μ. Using that on the physical sheet $\mathcal{G}_e=\mathcal{G}$, which is algebraic, by eq. (22), and the standard

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characterization of the singular continuous spectrum, [9] Section 1.1, we obtain that $\mu_{sc} = 0$. Thus $\mu = \mu_{ac} + \mu_{pp}$.

Let $t = x+i\epsilon$, and for any function on $\mathbf{H}$ set $f(x+i0) = \lim_{\epsilon\to 0} f(x+i\epsilon)$, if the limit exists. Then (see [9], Theorem 1.6)

$$\pi d\mu_{ac}(x) = |\Im\mathcal{G}(x+i0)|dx.$$

Eq. (22) implies the claim. □

Remark. The fact that $A * B$ has no singular continuous spectrum, probably, is quite general. For instance, $\mu_{sc}(A * B) = 0$ if A and B are arbitrary (right) convolution operators on discrete groups [7].

LEMMA 2. *Let A and B be a pair of operators, satisfying the standing assumptions, and let the notation be as above. Set*

$$\alpha = \frac{1}{4}(b+d-a-c)^2 + \frac{1}{4}(a+d-b-c)^2 - gh,$$

$$\gamma = (a-b)^2(c-d)^2 + g^2h^2 - g^2(c-d)^2 - h^2(a-b)^2.$$

Then $0 < \sqrt{\gamma} \le \alpha$. The equality $\alpha = \sqrt{\gamma}$ holds if and only if $b-a = d-c$ and $u = r, v = s$ (see eq. (17) for notation).

Proof. Substituting the expressions for g and h in terms of the spectral data of A, B into the formulas above, we obtain

$$2\alpha = [(a-b)-(c-d)]^2 + 2[(b-a)(d-c)[1-(u-v)(r-s)], \tag{30}$$

$$\gamma = (a-b)^2(c-d)^2[1-(u-v)^2][1-(r-s)^2]. \tag{31}$$

Since $|u-v|, |r-s| < 1$, we have

$$0 < [(a-b)-(c-d)]^2 < 2\alpha < [(a-b)+(c-d)]^2,$$

and $\gamma > 0$. The inequality $\alpha \ge \sqrt{\gamma}$ follows from

$$[1-(u-v)(r-s)]^2 \ge [1-(u-v)^2][1-(r-s)^2].$$

The equality $\alpha = \sqrt{\gamma}$ holds if and only if $(a-b)-(c-d) = 0$ and $u-v = r-s$. Since $u+v = r+s = 1$, the latter equation is equivalent to $u = r$. □

Recall that $S = a + b + c + d$, and set

$$\beta = \frac{1}{4}(g-h)[g(c-d)^2 - h(a-b)^2] + \frac{1}{16}(b+d-a-c)^2(a+d-b-c)^2.$$

LEMMA 3. *Let the setting be as above. Then the quartic polynomial R (see Proposition 1) satisfies*

$$R(w + S/2) = w^4 - \alpha w^2 + \beta. \tag{32}$$

Proof. By eqs. (20-21), $D(t)$ and $E(t)$ are invariant with respect to $t \to S - t$. Hence $R(w+S/2)$ is an even polynomial in w. The claim follows by a direct computation. □

DEFINITION 3. *Let A_i be bounded linear operators on Hilbert saces $\mathcal{H}_i$, $i = 1, 2$. They are* essentially unitarily equivalent *if there exist Hilbert spaces $\mathcal{M}_1, \mathcal{M}_2$ such that the operators $A_i \otimes Id_{\mathcal{M}_i}$ on $\mathcal{H}_i \otimes \mathcal{M}_i$, $i = 1, 2$ are unitarily equivalent.*

THEOREM 2. *Let the operators A and B satisfy the standing assumptions, and let the notation be as above.*
1) *Suppose that A and $B - \lambda Id$ are not essentially unitarily equivalent for any λ. Then the continuous spectrum of $A * B$ is a union of two disjoint intervals:*

$$\sigma_c(A*B) = \left[\frac{S}{2} - \sqrt{\frac{1}{2}(\alpha + \sqrt{\gamma})}, \frac{S}{2} - \sqrt{\frac{1}{2}(\alpha - \sqrt{\gamma})}\right] \cup \left[\frac{S}{2} + \sqrt{\frac{1}{2}(\alpha - \sqrt{\gamma})}, \frac{S}{2} + \sqrt{\frac{1}{2}(\alpha + \sqrt{\gamma})}\right].$$

2) *If there exists λ such that A and $B - \lambda Id$ are essentially unitarily equivalent, then the continuous spectrum of $A * B$ is a single interval:*

$$\sigma_c(A*B) = \left[\frac{S}{2} - \sqrt{\alpha}, \frac{S}{2} + \sqrt{\alpha}\right].$$

Proof. By Lemma 2 and Lemma 3, all roots of R are real. They are

$$\frac{S}{2} - \sqrt{\frac{1}{2}(\alpha + \sqrt{\gamma})} < \frac{S}{2} - \sqrt{\frac{1}{2}(\alpha - \sqrt{\gamma})} \le \frac{S}{2} + \sqrt{\frac{1}{2}(\alpha - \sqrt{\gamma})} < \frac{S}{2} + \sqrt{\frac{1}{2}(\alpha + \sqrt{\gamma})}.$$

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If $\alpha > \sqrt{\gamma}$, then the roots are distinct, and Proposition 3 applies. If $\alpha = \sqrt{\gamma}$, then

$$\sqrt{R(t)} = (t - S/2)\sqrt{(t - S/2 + \sqrt{\alpha})(t - S/2 - \sqrt{\alpha})}.$$

The argument of Proposition 3 applies and shows that $\sigma_c(A * B) = [S/2 - \sqrt{\alpha}, S/2 + \sqrt{\alpha}]$.

It remains to show that the dichotomy $\alpha > \sqrt{\gamma}$ versus $\alpha = \sqrt{\gamma}$ is equivalent to the dichotomy of the Theorem. By Lemma 2, $\alpha = \sqrt{\gamma}$ if and only if there exist $p, q > 0, p + q = 1$, and $\lambda \in \mathbf{R}$ such that

$$(33) \qquad G_A(z) = \frac{p}{z-a} + \frac{q}{z-b}, \ G_B(z) = \frac{p}{z-a-\lambda} + \frac{q}{z-b-\lambda}.$$

Equivalently, the eigenvalues of A (resp. B) are $a < b$ (resp. $a+\lambda < b+\lambda$), with the same relative multiplicities p and q. Let m, n be natural numbers such that $m|X| = n|Y| = h$. Setting $\mathcal{M} = \mathbf{C}^m, \mathcal{N} = \mathbf{C}^n$, we obtain that $A \otimes Id_{\mathcal{M}}$ and $(B - \lambda) \otimes Id_{\mathcal{N}}$ are selfadjoint operators, acting on the same space $\mathbf{C}^h$, with the same eigenvalues $a < b$, of the same multiplicities. Hence, they are unitarily equivalent. We leave the proof of the opposite implication to the reader. □

Using eqs. (30-31) we can directly express the intervals comprising the continuous spectrum of $A * B$ in terms of the spectral data of the operators A and B. Leaving the general case to the reader, we will restict ourselves to the case when $\sigma_c(A * B)$ is a single interval. The following is an immediate corollary of Theorem 2 and its proof.

COROLLARY 5. *Let the operators A and B satisfy the standing assumptions, and let the notation be as above. Then the set $\sigma_c(A * B)$ is connected if and only if A and B are essentially unitarily equivalent.*
Let this be the case, and let a, b, λ, p, q be as in eq. (33). *Then*

$$\sigma_c(A*B) = [a+b+\lambda-2(b-a)\sqrt{pq}, a+b+\lambda+2(b-a)\sqrt{pq}] \subset [2a+\lambda, 2b+\lambda].$$

The inclusion is strict (on both ends), unless $p = q = 1/2$.

3.2. Free powers of an operator

In this subsection we consider arbitrary 'free powers' of a single operator A on $\ell^2(X)$, where (X, e) is a rooted set. Let $G(t)$ be the Green's function of A, based at e. For $n > 1$ we denote by $*^n A$ the n-th free power of A, i.e., $*^n A = *_{i=1}^n A_i$, where $A_i = A$ for all i.

PROPOSITION 4. *Let $\mathcal{G}_n(t)$ be the Green's function of $*^n A, n > 1$. Then*

$$\mathcal{G}_n(t) = \frac{n-1}{nx-t} \tag{34}$$

where $x = x(t)$ satisfies

$$G(x) = \frac{n-1}{nx-t}. \tag{35}$$

Proof. This is a special case of Corollary 3, where we have, by symmetry, $x_i = x_j$ for all i, j. □

From now until the end of Section 3 the standing assumption will be that $|X| < \infty$ and that A is invariant and selfadjoint. In addition, we assume that A has two eigenvalues. Thus $G(z) = p/(z-a) + q/(z-b)$, where $a < b$ and $p, q > 0, p + q = 1$. Some of the propositions to follow remain valid, with obvious modifications, if $|X| = \infty$.

PROPOSITION 5. *Let A satisfy the standing assumptions, and let the notation be as above. Set*

$$P_n(t) = (n-2)t + n^2(pb+qa) - n(n-1)(a+b),$$

$$\begin{aligned} R_n(t) = t^2 &+ 2[(n-2)(pb+qa) - (n-1)(a+b)]t \\ &+ [n(pb+qa) - (n-1)(a+b)]^2 + 4(n-1)ab. \end{aligned}$$

Then the Green's function of $*^n A$ satisfies

$$\frac{1}{2}\mathcal{G}_n(t) = \frac{n\sqrt{R_n(t)} - P_n(t)}{(t-na)(t-nb)}. \tag{36}$$

Proof. Specializing in eq. (35), we have

$$\frac{p}{x-a} + \frac{q}{x-b} = \frac{n-1}{nx-t}$$

which gives a quadratic equation on x. Substituting $x(t)$ into eq. (34) and getting rid of the radicals in the denominator, like in the proof of Proposition 1, we obtain the claim. □

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Proposition 5 means, in particular, that the Green's function $\mathcal{G}_n$ of $*^n A$ is algebraic on the Riemann surface $\mathcal{R}_n = \{(t, w) : w^2 = R_n(t)\}$, which is a 2-sheeted covering of the Riemann sphere, via $p_n(t, w) = t$. The sheets of $\mathcal{R}$ correspond to the two branches of $\sqrt{R_n(t)}$, where on the physical sheet we have, asymptotically, $\sqrt{R_n(t)} \sim t$, as $t \to \infty$. Our considerations involving $\mathcal{G}_n$ as a meromorphic function on the Riemann surface $\mathcal{R}_n$ are analogous to those of Section 3.1, concerning the Green's function of $A * B$. If anything, they are simpler in the present case. Note that $\mathcal{R}_n$ is the Riemann sphere, since R_n is a quadratic polynomial.

THEOREM 3. *Let A be an operator satisfying the standing assumptions, and let $G(z) = p/(z-a) + q/(z-b)$ be its Green's function. We consider the operators $*^n A, n > 1$, and assume that $np, nq \neq 1$ for any n.*
1. *The operator $*^n A$ has no singular continuous spectrum. Its absolutely continuous spectrum is a single interval:*

$$\begin{aligned}\sigma_{ac}(*^n A) &= \sigma_c(*^n A)\\ &= \Big[(1+p(n-2))a + (1+q(n-2))b - 2(b-a)\sqrt{(n-1)pq},\\ &\qquad (1+p(n-2))a + (1+q(n-2))b + 2(b-a)\sqrt{(n-1)pq}\Big].\end{aligned}$$

2. *Let $p < q$ (resp. $q < p$). Then the point spectrum of $*^n A$ consists of the single point nb (resp. na), as long as $n < 1/p$ (resp. $n < 1/q$). The point spectrum is empty for $n > 1/p$ (resp. $n > 1/q$).*

Proof. 1. The proof is analogous to the argument of Theorem 2 and Proposition 3, and we leave details to the reader. The endpoints of $\sigma_{ac}(*^n A)$ are the two roots of R_n.
2. The argument follows the proof of Proposition 2. By eq. (36), the only possible poles of $\mathcal{G}_n$ in $\mathcal{R}_n$ are above $na, nb \in \mathbf{R}$. Denote by t any of the two numbers. A direct computation gives

$$P_n(na) = n(nq-1)(a-b), \; P_n(nb) = n(np-1)(b-a). \tag{37}$$

Hence, t is not in the branch locus, and exactly one of the two points of $\mathcal{R}_n$ above t is a pole of $\mathcal{G}_n$. Let $p(t) \in \mathcal{R}_n$ be the pole, and let $r(t) \in \mathcal{R}_n$ be the regular point (we suppress the dependence on n, to simplify the notation). Both poles are simple, and from eqs. (36-37)

$$\mathrm{Res}_{p(a)}\mathcal{G}_n = 1 - nq, \; \mathrm{Res}_{p(b)}\mathcal{G}_n = 1 - np.$$

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A direct computation gives

$$\mathcal{G}'_n(p(a)) = c(a)(1 - nq), \ \mathcal{G}'_n(p(b)) = c(b)(1 - np),$$

where $c(a), c(b) > 0$. Hence the sign of $\mathcal{G}'$ at the regular point above a (resp. b) is the same as the sign of $1 - nq$ (resp. $1 - np$). Lemma 1 implies that $p(a)$ (resp. $p(b)$) is in the physical sheet if and only if $1 - nq > 0$ (resp. $1 - np > 0$). Since $np + nq = n \geq 2$, both $p(a)$ and $p(b)$ cannot be in the physical sheet, which implies the claim. ∎

§4. Applications

4.1. Graph spectra

By the *spectrum of a graph*, Γ, we mean the spectrum of its incidence operator, A_Γ. Another set, frequently associated with Γ, is the spectrum of the *graph Laplacean* (see, e. g, [2]). If Γ is *regular* (every vertex has the same number of edges), the two spectra are related by a translation. We will consider only symmetric (therefore regular) graphs.

If Γ_1, Γ_2 are two rooted graphs, we denote by $\Gamma_1 * \Gamma_2$ their *free product*. By the construction of Section 2, this is a special case of the free product of rooted sets. If A_1, A_2 are the incidence operators, then $A_1 * A_2$ is the incidence operator of $\Gamma_1 * \Gamma_2$. If Γ_1, Γ_2 are symmetric graphs, i.e. there are groups, H_1, H_2, acting transitively on Γ_1, Γ_2 by automorphisms, then $H_1 * H_2$ transitively acts on $\Gamma_1 * \Gamma_2$, thus $\Gamma_1 * \Gamma_2$ is symmetric. In particular, $\Gamma_1 * \Gamma_2$ does not depend on the choice of roots in Γ_1, Γ_2. Everything we said so far about the free product of two graphs extends to the free products of any number of graphs. We will use the self-explanatory notation: $*_{i=1}^n \Gamma_i$, $*^n \Gamma$.

We denote by $K_n, n \geq 1$, the *complete graph* on $n + 1$ vertices (any two vertices are neighbors). It is symmetric (under the natural action of S_{n+1}, the symmetric group).

THEOREM 4. *Let $1 \leq m < n$. The (absolutely) continuous spectrum of $K_m * K_n$ is the union of two disjoint intervals:*

$$I_{m,n} = \frac{1}{2}\Big[m + n - 2 - \sqrt{4(\sqrt{m} + \sqrt{n})^2 + (m - n)^2}, \\ m + n - 2 - \sqrt{4(\sqrt{m} - \sqrt{n})^2 + (m - n)^2}\Big]$$

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and

$$J_{m,n} = \frac{1}{2}\Big[m+n-2+\sqrt{4(\sqrt{m}-\sqrt{n})^2+(m-n)^2},$$
$$m+n-2+\sqrt{4(\sqrt{m}+\sqrt{n})^2+(m-n)^2}\Big].$$

The point spectrum of $K_m * K_n$ *is the set* $\{-2, m-1\}$.

Proof. The spectrum of K_n consists of two points: -1 and n. The multiplicities are n and 1 respectively. We denote by A, B the incidence operators of K_m, K_n, and use the notation of Section 3.1. Thus $a = 1, b = m, c = -1, d = n, u = m/(m+1), v = 1/(m+1), r = n/(n+1), s = 1/(n+1)$. Substituting this into eqs. (18-22), we obtain

$$N(t) = -(m+n-2)t^2 + (m+n-2)^2 t - [(m+n+2)(mn-7)+24], \tag{38}$$

$$R_{m,n}(t) = (t - \frac{m+n}{2}+1)^4 - \left[2(m+n)+\frac{(m-n)^2}{2}\right](t-\frac{m+n}{2}+1)^2 + \frac{1}{16}(m-n)^2[(m+n+4)^2-4mn], \tag{39}$$

and for the Green's function of the graph $K_m * K_n$:

$$\mathcal{G}_{m,n}(t) = \frac{N(t)+(2t-m-n+2)\sqrt{R_{m,n}(t)}}{2(t+2)(t+m+n)(t-m+1)(t-n+1)}. \tag{40}$$

We use Theorem 2 to find the absolutely continuous spectrum of $K_m * K_n$. By eq. (39), $\alpha > 0, \alpha^2 - 4\beta > 0$, and $\beta > 0$ (because $m < n$). Thus, by Theorem 2, the continuous spectrum of $K_m * K_n$ is a union of two intervals, whose endpoints are $\frac{1}{2}(m+n) - 1 \pm \sqrt{(\sqrt{m}\pm\sqrt{n})^2 + (\frac{1}{2}(m-n))^2}$.

The inequalities eqs. (26-27) hold (because $m < n$), thus A and B are in general position. We use Proposition 2 and explicit calculations, like in the proof of Theorem 3, to compute the point spectrum of $K_m * K_n$. □

EXAMPLES. 1 (see [7, 8]). Let $m = 1, n = 2$. From eq. (40) we obtain the Green's function of $K_1 * K_2$:

$$\mathcal{G}_{1,2}(t) = \frac{-t^2+t+1+(2t-1)\sqrt{t^4-2t^3-5t^2-6t+1}}{2(t+2)(t(t-1)(t+3)}$$

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By Theorem 4, the point spectrum of $K_1 * K_2$ is $\{-2, 0\}$, and the continuous spectrum:

$$\left[\frac{1}{2} - \frac{1}{2}\sqrt{13 + 8\sqrt{2}}, \frac{1}{2} - \frac{1}{2}\sqrt{13 - 8\sqrt{2}}\right] \cup \left[\frac{1}{2} + \frac{1}{2}\sqrt{13 - 8\sqrt{2}}, \frac{1}{2} + \frac{1}{2}\sqrt{13 + 8\sqrt{2}}\right].$$

2. Set $m = 1, n > 1$. By Theorem 4, the point spectrum is $\{-2, 0\}$, and $\sigma_c(K_1 * K_n) = I_n \cup J_n$, where

$$I_n = \left[\frac{n-1}{2} - \frac{1}{2}\sqrt{(n+1)^2 + 4 + 8\sqrt{n}}, \frac{n-1}{2} - \frac{1}{2}\sqrt{(n+1)^2 + 4 - 8\sqrt{n}}\right],$$

$$J_n = \left[\frac{n-1}{2} + \frac{1}{2}\sqrt{(n+1)^2 + 4 - 8\sqrt{n}}, \frac{n-1}{2} + \frac{1}{2}\sqrt{(n+1)^2 + 4 + 8\sqrt{n}}\right].$$

THEOREM 5. *Let $K_r, r \geq 1$, be the complete graph on $r + 1$ vertices, and let $\mathcal{G}_n^{(r)}$ be the Green's function of $*^n K_r, n > 1$. Then*

$$\mathcal{G}_n^{(r)} = \frac{(n-2)t + n(r-1) - n\sqrt{t^2 - 2(r-1)t + (r+1)^2 - 4nr}}{2(t+n)(t-nr)}.$$

For $r > 1$ the point spectrum of $*^n K_r$ is a single point, $\{-n\}$, if $2 \leq n \leq r$, and is empty if $n > r$. The graph $*^n K_1$ has no point spectrum. The (absolutely) continuous spectrum of $*^n K_r$:

$$\sigma_c(*^n K_r) = [r - 1 - 2\sqrt{(n-1)r}, r - 1 + 2\sqrt{(n-1)r}].$$

Proof. Straightforward from Proposition 5 and Theorem 3. □

EXAMPLES. 1. The graph $*^{n+1} K_1$ is the $n+1$-regular tree, T_n. Specializing to $r = 1$ in Theorem 5, we obtain its Green's function:

$$G_{T_n}(t) = \frac{(n-1)t - (n+1)\sqrt{t^2 - 4n}}{2(t+n+1)(t-n-1)},$$

and the absolutely continuous spectrum: $\sigma(T_n) = [-2\sqrt{n}, 2\sqrt{n}]$ (there is no point spectrum). These results are well known (see, e.g., [2]). 2. Setting $r = 2, n = 2$ in Theorem 5, we obtain: $\sigma_c(*^2 K_2) = [1 - 2\sqrt{2}, 1 + 2\sqrt{2}]$, and $\sigma_p(*^2 K_2) = \{-2\}$. We leave details to the reader.

4.2. Random walks on free products

A random walk on a graph, Γ, is given by the probabilities, $0 \leq p(e) \leq 1$, on the edges of Γ, so that $\sum_{e\sim x} p(e) = 1$ for any vertex $x \in \Gamma$. Equivalently, a random walk is determined by its *transition operator*, $(Pf)(x) = \sum_{y\sim x} P(x,y)f(y)$, and $P(x,y) = p(e)$, where e is the edge joining x with y. We say that a random walk on Γ is *invariant* if its transition operator is invariant (under a transitive group on Γ). By the spectrum (Green's function) of a random walk on Γ we mean the spectrum (Green's function) of its transition operator.

An invariant random walk on a rooted graph, (Γ, e), is determined by the probabilities of the edges of e. For the *simple random walk* these probabilities are equal to $1/d$, where d is the number of edges of e. Let $\Gamma_i, 1 \leq i \leq n$, be symmetric graphs, and let P_i be the transition operator for an invariant random walk on $\Gamma_i, 1 \leq i \leq n$. Any n-tuple, $p_i > 0, \sum_{i=1}^n p_i = 1$, defines a *product random walk* on $*_{i=1}^n \Gamma_i$, its transition operator is $*_{i=1}^n p_i P_i$. If the random walks on Γ_i are simple for $1 \leq i \leq n$, and $p_1 = \cdots = p_n = 1/n$, then the product random walk on $*_{i=1}^n \Gamma_i$ is also simple.

THEOREM 6. ([3]) *Let $r \geq 1, n > 1$. The (absolutely) continuous spectrum of the simple random walk on $*^n K_r$ is $\frac{1}{rn}[r-1-2\sqrt{(n-1)r}, r-1+2\sqrt{(n-1)r}]$. The point spectrum is nonempty if and only if $2 \leq n \leq r$. Then it consists of a single point, $\{-1/r\}$.*

Proof. The transition operator of the simple random walk on a k-regular graph, Γ, is $k^{-1}A_\Gamma$. In our case, $k = nr$, and the spectrum of A_Γ is given by Theorem 5. ∎

THEOREM 7. *Let $1 \leq m \leq n$, and $p, q > 0, p+q = 1$. We assume that $(p,m) \neq (q,n)$, and consider the product random walk on $K_m * K_n$ defined by these data. Its (absolutely) continuous spectrum is the union of two intervals, I, J (depending on p,q,m,n). Define $\ell_\pm(p,q,m,n) > 0$ by*

$$\ell_\pm^2 = \frac{1}{4}(p-q)^2 + \frac{1}{4}(\frac{p}{m} - \frac{q}{n})^2 + \frac{1}{2}[\frac{p}{m} + \frac{q}{n} \pm \frac{4pq}{\sqrt{mn}}].$$

Then

$$I = \frac{1}{2}[1 - (\frac{p}{m} + \frac{q}{n}) - 2\ell_+, 1 - (\frac{p}{m} + \frac{q}{n}) - 2\ell_-],$$

$$J = \frac{1}{2}[1 - (\frac{p}{m} + \frac{q}{n}) + 2\ell_-, 1 - (\frac{p}{m} + \frac{q}{n}) + 2\ell_+].$$

If $m < n$, the point spectrum is $\{-(p/m + q/n), p - q/n\}$. If $m = n > 1$, the point spectrum is $\{-1/m\}$. If $m = n = 1$, the point spectrum is empty.

Proof. The transition operator of this random walk is the free product $m^{-1}pA_m * n^{-1}qA_n$, where A_ℓ is the incidence matrix of the complete graph K_ℓ. In the notation of Section 3.1, $a = -p/m, b = p, c = -q/n, d = q, u = m/(m+1), v = n/(n+1), r = n/(n+1), s = 1/(n+1)$. We compute the parameters α and γ of Theorem 2, and obtain

$$m^2n^2[\alpha \pm \sqrt{\gamma}] = \frac{1}{2}m^2n^2(p-q)^2 + \frac{1}{2}(pn - qm)^2 + mn[pn + qm \pm 4pq\sqrt{mn}].$$

It is elementary to check that $pn + qm \pm 4pq\sqrt{mn} \geq 0$, hence the expression above is positive (we have ruled out $p = q, m = n$). Thus $\ell_\pm = \sqrt{(\alpha \pm \sqrt{\gamma})/2} > 0$, and Theorem 2 gives the continuous spectrum.

We use Proposition 2 and explicit calculations, like in the proof of Theorem 3, to determine the point spectrum. □

Remark. Theorems 6 and 7 together yield the spectra of all product random walks on $K_m * K_n$. Proposition 1 allows to find explicitly their Green's functions. We leave this to the reader.

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