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AN APPLICATION OF SEPARATE CONVERGENCE FOR CONTINUED FRACTIONS TO ORTHOGONAL POLYNOMIALS

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ABSTRACT. It is known that the n-th denominators $Q_n(\alpha,\beta,z)$ of a real J-fraction

$$\frac{a_1}{z+a_2} - \frac{a_2a_3}{z+(a_3+a_4)} - \frac{a_4a_5}{z+(a_5+a_6)} - \cdots,$$

where

$$a_1 := 1, \quad a_{2n} := \frac{-(\beta-\alpha+n-\frac{1}{2})(\alpha+n-1)}{(\beta+2n-2)(\beta+2n-1)},$$
$$a_{2n+1} := \frac{-(\beta-\alpha+n)(\alpha+n-\frac{1}{2})}{(\beta+2n-1)(\beta+2n)}, \quad n \geq 1 \text{ for } \beta \geq \alpha > 0,$$

form an orthogonal polynomial sequence (OPS) with respect to a distribution function $\psi(t)$ on $\mathbb{R}$. We use separate convergence results for continued fractions to prove the asymptotic formula

$$\lim_{n\to\infty}\left(\frac{2z}{n}\right)^n Q_n\left(\alpha,\beta,\left(\frac{2z}{n}\right)^{-1}\right) = e^{-z},$$

the convergence being uniform on compact subsets of $z \in \mathbb{C}$.

1. **Introduction.** Considerable attention has been given recently to the study of separate convergence of continued fractions

$$b_0(z) + \mathop{\mathbf{K}}_{n=1}^{\infty}\left(\frac{a_n(z)}{b_n(z)}\right) = b_0(z) + \frac{a_1(z)}{b_1(z)} + \frac{a_2(z)}{b_2(z)} + \frac{a_3(z)}{b_3(z)} + \cdots, \tag{1.1}$$

[5], [7], [10], [11]. The term *separate convergence* is used when both of the sequences $\{A_n(z)/\Gamma_n(z)\}$ and $\{B_n(z)/\Gamma_n(z)\}$ converge, where $A_n(z)$ and $B_n(z)$ denote the n-th numerator and denominator, respectively, of the continued fraction (1.1) and $\{\Gamma_n(z)\}$ is an "easily described" sequence of functions. The restriction that the Γ_n can be "easily described" is essential since, otherwise, one could choose $\Gamma_n(z) = B_n(z)$ and the distinction between ordinary and separate convergence would become meaningless. Clearly separate convergence implies ordinary convergence. One reason for the importance of separate convergence is that it can provide useful information on asymptotic properties of orthogonal polynomial sequences which occur as denominators of certain continued fractions. It also yields a method for representing special functions.

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From [6, Theorem 6.1] we obtain the continued fraction representation

$$\mathop{\mathbf{K}}_{n=1}^{\infty}\left(\frac{a_n\zeta}{1}\right)=\zeta\frac{{}_2F_1(\alpha,\alpha+\frac{1}{2};\beta+1;\zeta)}{{}_2F_1(\alpha,\alpha-\frac{1}{2};\beta;\zeta)}=:g(\alpha,\beta,\zeta) \tag{1.2a}$$

for

$$\beta\in\mathbb{C}\setminus[0,-1,-2,\cdots], \tag{1.2b}$$

where, for $n\geq 1$,

$$a_1:=1,\quad a_{2n}:=\frac{-(\beta-\alpha+n-\frac{1}{2})(\alpha+n-1)}{(\beta+2n-2)(\beta+2n-1)},\quad a_{2n+1}:=\frac{-(\beta-\alpha+n)(\alpha+n-\frac{1}{2})}{(\beta+2n-1)(\beta+2n)}. \tag{1.2c}$$

The convergence in (1.2a) is uniform in ζ on every compact subset of $\mathbb{C}$ containing no pole of $g(\alpha,\beta,\zeta)$. The function $g(\alpha,\beta,\zeta)$ is analytic at $\zeta=0$ and $g(\alpha,\beta,0)=0$. The symbol ${}_2F_1$ denotes the confluent hypergeometric function

$${}_2F_1(a,b;c;z):=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}, \tag{1.3}$$

which converges for $|z|<1$ and $c\in\mathbb{C}\setminus[0,-1,-2,\ldots]$; $(a)_0:=1$, $(a)_n:=a(a+1)(a+2)\cdots(a+n-1)$, $n\geq 1$. The continued fraction (1.2a) provides the analytic continuation of $g(\alpha,\beta,\zeta)$ to a larger domain in $\mathbb{C}$. It is readily seen that $a_n=-\frac{1}{4}+O(\frac{1}{n^2})$ so that

$$a:=\lim_{n\to\infty}a_n=-\frac{1}{4}\quad\text{and}\quad\sum_{n=1}^{\infty}\left|a_n+\frac{1}{4}\right|<\infty. \tag{1.4}$$

We let $A_n(\alpha,\beta,\zeta)$ and $B_n(\alpha,\beta,\zeta)$ denote the n-th numerator and denominator, respectively, of the continued fraction in (1.2) and let

$$C_n(\alpha,\beta,\zeta):=\frac{2^{n+1}A_n(\alpha,\beta,\zeta)}{(1+\sqrt{1-\zeta})^{n+1}},\quad D_n(\alpha,\beta,\zeta):=\frac{2^{n+1}B_n(\alpha,\beta,\zeta)}{(1+\sqrt{1-\zeta})^{n+1}},\quad n\geq 0. \tag{1.5}$$

It follows then from (1.4) and a result on separate convergence (see, e.g. [10, Coro. 3.2]) that the limits

$$C(\alpha,\beta,\zeta):=\lim_{n\to\infty}C_n(\alpha,\beta,\zeta)\text{ and }D(\alpha,\beta,\zeta):=\lim_{n\to\infty}D_n(\alpha,\beta,\zeta) \tag{1.6}$$

both exist and are analytic functions of ζ for

$$\zeta\in S:=\mathbb{C}\setminus[1,\infty). \tag{1.7}$$

The convergence is uniform on all compact subsets of S. In (1.5) and subsequently $\sqrt{}$ is chosen so that $\mathrm{Re}\sqrt{}>0$. Thus in this case the "easily described" function Γ_n is

$$\Gamma_n(z):=\left(\frac{1+\sqrt{1-z}}{2}\right)^{n+1},\quad n=0,1,2,\cdots. \tag{1.8}$$

Our interest here is primarily in a closely related continued fraction called a J-fraction. The even part [6, Sect. 2.4.2] of the regular C–fraction (1.2) is the associated continued fraction

$$(1.9) \qquad \frac{a_1\zeta}{1+a_2\zeta} - \frac{a_2a_3\zeta^2}{1+(a_3+a_4)\zeta} - \frac{a_4a_5\zeta^2}{1+(a_5+a_6)\zeta} - \cdots .$$

Applying an equivalence transformation [6, Sect. 2.3] to (1.9) and setting $\zeta = 1/z$, we obtain the J-fraction

$$(1.10) \qquad \frac{a_1z}{a_2+z} - \frac{a_2a_3}{(a_3+a_4)+z} - \frac{a_4a_5}{(a_5+a_6)+z} - \frac{a_6a_7}{(a_7+a_8)+z} - \cdots .$$

We let $P_n(\alpha,\beta,z)$ and $Q_n(\alpha,\beta,z)$ denote the n-th numerator and denominator, respectively, of (1.10). It can then be shown that

$$(1.11) \qquad z^nP_n(\alpha,\beta,z^{-1}) = A_{2n}(\alpha,\beta,z) \quad \text{and} \quad z^nQ_n(\alpha,\beta,z^{-1}) = B_{2n}(\alpha,\beta,z).$$

We prove (Theorems 2.8 and 2.10) that, for $z \in \mathbb{C}$ and $\beta \in \mathbb{C} \setminus [0,-1,-2,\cdots]$,

$$(1.12) \qquad \lim_{n\to\infty}\left(\frac{2z}{n}\right)^{n-1} P_n\left(\alpha,\beta,\left(\frac{2z}{n}\right)^{-1}\right) = \lim_{n\to\infty}\left(\frac{2z}{n}\right)^{n} Q_n\left(\alpha,\beta,\left(\frac{2z}{n}\right)^{-1}\right) = e^{-z}.$$

When the parameters α and β are real with $\beta \geq \alpha > 0$, $\{Q_n(\alpha,\beta,\zeta)\}$ is a sequence of polynomials orthogonal with respect to a distribution function $\psi(t)$ defined on $(0,\infty)$. Thus separate convergence for continued fractions yields asymptotic formulae for a class of orthogonal polynomial sequences. The case considered in this paper is related to Jacobi polynomials. It is precisely a special case of associated Jacobi polynomials discussed in the paper [4]. The parameters α, c, β in the paper [4] have been replaced by $\beta - 2\alpha + \frac{1}{2}$, $\alpha - \frac{1}{2}$ and $-\frac{1}{2}$, respectively.

2. **Separate Convergence.** In this section we let α and β be given (fixed) complex numbers satisfying (1.2b). The sequences $\{P_n(\alpha,\beta,z)\}$ and $\{Q_n(\alpha,\beta,z)\}$ are defined recursively by the difference equations

$$(2.1a) \qquad P_1(\alpha,\beta,z) := a_1z, \quad P_2(\alpha,\beta,z) := a_1z(z+a_3+a_4),$$

$$(2.1b) \qquad Q_1(\alpha,\beta,z) := a_2+z, \quad Q_2(\alpha,\beta,z) := z^2+(a_2+a_3+a_4)z+a_2a_4,$$

(2.1c)

$$\begin{pmatrix} P_n(\alpha,\beta,z) \\ Q_n(\alpha,\beta,z) \end{pmatrix} = (z+a_{2n-1}+a_{2n})\begin{pmatrix} P_{n-1}(\alpha,\beta,z) \\ Q_{n-1}(\alpha,\beta,z) \end{pmatrix} - a_{2n-2}a_{2n-1}\begin{pmatrix} P_{n-2}(\alpha,\beta,z) \\ Q_{n-2}(\alpha,\beta,z) \end{pmatrix}, \qquad n = 3,4,5,\cdots,$$

where the a_n are defined by (1.2c). Let $q_{n,j}$ denote coefficients defined by

$$(2.2) \qquad \sum_{j=0}^{n} q_{n,j}z^j := z^nQ_n(\alpha,\beta,z^{-1}), \quad n = 0,1,2,\ldots,$$

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THEOREM 2.1. *With the a_n defined by (1.2c), we have that*

$$a_n = -\frac{1}{4} + O\left(\frac{1}{n^2}\right). \tag{2.3a}$$

Then for $n = 1, 2, 3, \cdots$, and $j = 1, 2, 3, \cdots$,

$$q_{n,j} = \frac{1}{j!}\left(-\frac{n}{2}\right)^j + O(n^{j-1}). \tag{2.3b}$$

Our proof of Theorem 2.1 is based on several lemmas, the first of which is

LEMMA 2.2. *For each $n \geq 1$, the polynomials $P_n(\alpha, \beta, z)$ and $Q_n(\alpha, \beta, z)$ can be expressed in the form*

$$P_n(\alpha, \beta, z) = \sum_{j=0}^{n} r_{n,j}^{(1)} z^{n-j} \quad \text{and} \quad Q_n(\alpha, \beta, z) = \sum_{j=0}^{n} r_{n,j}^{(2)} z^{n-j} \tag{2.4a}$$

where

$$r_{n,j}^{(\ell)} := \sum_{i_j=\ell}^{2n-2j+2} a_{i_j} \sum_{i_{j-1}=i_j+2}^{2n-2j+4} a_{i_{j-1}} \cdots \sum_{i_2=i_3+2}^{2n-2} a_{i_2} \sum_{i_1=i_2+2}^{2n} a_{i_1}, \ \ell = 1, 2 \tag{2.4b}$$

and $p_{n,j} = r_{n,j}^{(1)}$ and $q_{n,j} = r_{n,j}^{(2)}$.

Lemma 2.2 can be proved by applying the Euler-Minding Formulae [8, p.9] or directly by an induction argument using the difference equations (2.1).

To aid in the proof of Theorem 2.1, we introduce to notation

$$\widehat{q_{n,j}} := \sum_{i_j=i_{j+1}+2}^{2n-2j+2} a_{i_j} \sum_{i_{j-1}=i_j+2}^{2n-2j+4} a_{i_{j-1}} \cdots \sum_{i_2=i_3+2}^{2n-2} a_{i_2} \sum_{i_1=i_2+2}^{2n} a_{i_1}. \tag{2.5}$$

Note that if $i_{j+1} = 0$, then $\widehat{q_{n,j}} = q_{n,j}$. Using an induction argument and equation (2.3a), we obtain

LEMMA 2.3. *For each $j = 1, 2, 3, \cdots$,*

(2.6)

$$\widehat{q_{n,j}} = \left(\frac{-1}{4}\right)^j \Bigg\{(2n)^j - (2n)^{j-1} i_{j+1} - (2n)^{j-2} \sum_{i_j=i_{j+1}+2}^{2n-2j+2} i_j - (2n)^{j-3} \sum_{i_j=i_{j+1}+2}^{2n-2j+2} \sum_{i_{j-1}=i_j+2}^{2n-2j+4} i_{j-1} -$$

$$\cdots - \sum_{i_j=i_{j+1}+2}^{2n-2j+2} \sum_{i_{j-1}=i_j+2}^{2n-2j+4} \cdots \sum_{i_2=i_3+2}^{2n-2} i_2 \Bigg\} + O(n^{j-1}).$$

An application of the binomial theorem yields

LEMMA 2.4. *For each* $j = 2, 3, 4, \cdots$,

$$\left\{\sum_{k=2}^{j} \frac{(k-1)(-1)^{j-k}}{k!(j-k+1)!}\right\} + \frac{(-1)^{j-1}}{(j+1)!} = \frac{j}{(j+1)!}. \tag{2.7}$$

LEMMA 2.5. *For each* $j = 2, 3, 4, \cdots$, *and* $m = 1, 2, 3, \cdots, (j-1)$,

$$\underbrace{\sum_{i_j=i_{j+1}+2}^{2n-2j+2} \sum_{i_{j-1}=i_j+2}^{2n-2j+4} \cdots \sum_{i_{j-m+1}=i_{j-m+2}+2}^{2n-2j+2m}}_{m \text{ sums}} i_{j-m+1} \tag{2.8}$$

$$= \frac{m}{(m+1)!}(2n)^{m+1} + \left\{\sum_{k=2}^{m}(-1)^{m-k+1}\frac{(k-1)(2n)^k}{k!(m-k+1)!}i_{j+1}^{m-k+1}\right\} + \frac{(-1)^m i_{j+1}^{m+1}}{(m+1)!} + O(n^m).$$

PROOF. By induction and Lemma 2.4.

PROOF OF THEOREM 2.1. Applying Lemmas 2.3 and 2.5, and setting $i_{j+1} = 0$, we have

$$\begin{aligned} q_{n,j} &= \left(\frac{-1}{4}\right)\left\{(2n)^j - (2n)^{j-2}\frac{(2n)^2}{2!} - (2n)^{j-3}(2n)^3\frac{2}{3!} - \cdots - (2n)^j\frac{j-1}{j!}\right\} + O(n^{j-1}) \\ &= \left(\frac{-n}{2}\right)^j\left\{1 - \frac{1}{2!} - \frac{2}{3!} - \cdots - \frac{j-1}{j!}\right\} + O(n^{j-1}) \\ &= \frac{1}{j!}\left(\frac{-n}{2}\right)^j + O(n^{j-1}). \end{aligned}$$ ∎

It follows from Theorem 2.1 that, for $n \geq 1$ and $j \geq 1$,

$$q_{n,j} = \frac{1}{j!}\left(\frac{-n}{2}\right)^j \gamma(\alpha, \beta, n, j), \tag{2.9a}$$

where

$$\gamma(\alpha, \beta, n, j) = 1 + O\left(\frac{1}{n}\right) \text{ as } n \longrightarrow \infty. \tag{2.9b}$$

For the special case of Theorem 2.1 when $\alpha = \beta$ it follows from the difference equations (2.1) that

THEOREM 2.6. *For* $\alpha = \beta$, $n = 1, 2, 3, \cdots$ *and* $j = 1, 2, 3, \cdots$,

$$\begin{aligned} q_{n,0} = 1, \quad \textit{and} \quad q_{n,j} &= \left(\frac{-1}{2}\right)^j \binom{n}{j} \frac{(2n-1)(2n-3)\cdots(2n-2j+1)}{(\alpha+2n-1)(\alpha+2n-2)\cdots(\alpha+2n-j)} \\ &= \frac{1}{j!}\left(\frac{-n}{2}\right)^j \gamma(\alpha, \alpha, n, j), \end{aligned} \tag{2.10}$$

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where

$$\gamma(\alpha,\alpha,n,j) := \frac{\left[\prod_{k=1}^{j}(1-\frac{2k-1}{2n})\right]\left[\prod_{k=1}^{j}(1-\frac{k-1}{n})\right]}{\left[\prod_{k=1}^{j}(1-\frac{k-\alpha}{2n})\right]} \tag{2.11}$$
$$= 1 + O\Big(\frac{1}{n}\Big) \text{ as } n \longrightarrow \infty.$$

From (2.2) and (2.3b) we obtain

$$z^n Q_n(\alpha,\beta,z^{-1}) = \sum_{j=0}^{n} \frac{1}{j!}\Big(\frac{-nz}{2}\Big)^j \gamma(\alpha,\beta,n,j), \quad n = 1,2,3,\cdots.$$

This suggests replacing z by $\dfrac{2z}{n}$ and considering

$$\Big(\frac{2z}{n}\Big)^n Q_n\Big(\alpha,\beta,\Big(\frac{2z}{n}\Big)^{-1}\Big) = \sum_{j=0}^{n} \frac{(-z)^j}{j!}\gamma(\alpha,\beta,n,j), \quad n = 1,2,3,\cdots. \tag{2.12}$$

Combining this with (2.9b), one can prove the second equality in (1.12).

LEMMA 2.7. *If $D_n(\alpha,\beta,z)$ is defined by (1.5), then*

$$\lim_{n\longrightarrow\infty} D_n\Big(\alpha,\beta,\frac{2z}{n}\Big) = D(\alpha,\beta,0), \text{ for all } z \in \mathbb{C}, \tag{2.13}$$

where the convergence is uniform in z on every compact subset of $\mathbb{C}$.

PROOF. Let K_1 denote an arbitrary compact subset of $\mathbb{C}$ and let $K_2 := [z \in \mathbb{C} : |z| \leq \frac{1}{2}]$. Then there exists an N_1 such that $\dfrac{2z}{n} \in K_2$ for all $z \in K_1$ and $n \geq N_1$. Let $\epsilon > 0$ be given. Since by (1.6) $D(\alpha,\beta,z)$ is a continuous function of z on K_2, there exists an N_2 such that

$$\Big|D(\alpha,\beta,0) - D\Big(\alpha,\beta,\frac{2z}{n}\Big)\Big| < \frac{\epsilon}{2} \text{ for all } z \in K_1 \text{ and } n \geq N_2. \tag{2.14}$$

Since $\{D_n(\alpha,\beta,w)\}$ converges uniformly on K_2 to $D(\alpha,\beta,w)$, there exists an N_3 such that

$$|D(\alpha,\beta,w) - D_n(\alpha,\beta,w)| < \frac{\epsilon}{2} \text{ for all } w \in K_2 \text{ and } n \geq N_3. \tag{2.15}$$

Combining (2.14) and (2.15) yields

$$\begin{aligned}\Big|D(\alpha,\beta,0) - D_n\Big(\alpha,\beta,\frac{2z}{n}\Big)\Big| &\leq \Big|D(\alpha,\beta,0) - D\Big(\alpha,\beta,\frac{2z}{n}\Big)\Big| \\ &\quad + \Big|D\Big(\alpha,\beta,\frac{2z}{n}\Big) - D_n\Big(\alpha,\beta,\frac{2z}{n}\Big)\Big| \\ &< \epsilon,\end{aligned}$$

for all $z \in K_1$ and $n \geq \max(N_1,N_2,N_3)$. ∎

We can now prove

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THEOREM 2.8. *For each $\beta \in \mathbb{C} \setminus [0, -1, -2, \cdots]$ and $z \in \mathbb{C}$,*

$$\lim_{n\to\infty} \left(\frac{2z}{n}\right)^n Q_n\left(\alpha, \beta, \left(\frac{2z}{n}\right)^{-1}\right) = e^{-z}. \tag{2.16}$$

PROOF. It is easily shown that

$$\lim_{n\to\infty} \left(\frac{1+\sqrt{1-\frac{2z}{n}}}{2}\right)^{2n+1} = e^{-z}, \quad z \in \mathbb{C}. \tag{2.17}$$

By (1.5), (1.6), Lemma 2.7, and the fact that $B_n(\alpha, \beta, 0) = 1$ for $n \geq 1$, we obtain
(2.18)

$$1 = D(\alpha, \beta, 0) = \lim_{n\to\infty} D_{2n}\left(\alpha, \beta, \frac{2z}{n}\right) = \lim_{n\to\infty} \frac{B_{2n}(\alpha, \beta, \frac{2z}{n})}{\left(\frac{1+\sqrt{1-(\frac{2z}{n})}}{2}\right)^{2n+1}} = \frac{\lim_{n\to\infty} B_{2n}\left(\alpha, \beta, \frac{2z}{n}\right)}{e^{-z}}$$

The assertion (2.16) follows from (2.17) and (2.18). ∎

LEMMA 2.9. *Let $h_n(\alpha, \beta, \zeta) := \dfrac{g_n(\alpha, \beta, \zeta)}{\zeta}$, where $g_n(\alpha, \beta, \zeta) = \dfrac{A_n(\alpha, \beta, \zeta)}{B_n(\alpha, \beta, \zeta)}$ denotes the n-th approximant of the continued fraction (1.2). If $h(\alpha, \beta, \zeta) := \lim_{n\to\infty} h_n(\alpha, \beta, \zeta)$, then*

$$\lim_{n\to\infty} h_n\left(\alpha, \beta, \frac{2z}{n}\right) =: h(\alpha, \beta, 0) = 1 \tag{2.19}$$

where the convergence is uniform on every compact subset of $\mathbb{C}$.

PROOF. Let K_1 be an arbitrary compact subset of $\mathbb{C}$ and let $K_2 := [z : |z| \leq \delta]$ where $\delta > 0$ is chosen small enough so that $h(\alpha, \beta, \zeta)$ is analytic on K_2. Then there exists an N_1 such that $\frac{2z}{n} \in K_2$ for all $z \in K_1$ and $n \geq N_1$. Let $\epsilon > 0$ be given. Since $h(\alpha, \beta, \zeta)$ is analytic on K_2, there exists an N_2 such that

$$\left|h(\alpha, \beta, 0) - h\left(\alpha, \beta, \frac{2z}{n}\right)\right| < \frac{\epsilon}{2} \text{ for all } z \in K_1 \text{ and } n \geq N_2. \tag{2.20}$$

Since $\{h_n(\alpha, \beta, w)\}$ converges uniformly on K_2 to $h(\alpha, \beta, w)$, there exists an N_3 such that

$$|h(\alpha, \beta, w) - h_n(\alpha, \beta, w)| < \frac{\epsilon}{2} \text{ for all } w \in K_2 \text{ and } n \geq N_3. \tag{2.21}$$

Combining (2.20) and (2.21) yields

$$\begin{aligned}\left|h(\alpha, \beta, 0) - h_n\left(\alpha, \beta, \frac{2z}{n}\right)\right| &\leq \left|h(\alpha, \beta, 0) - h\left(\alpha, \beta, \frac{2z}{n}\right)\right| \\ &\quad + \left|h\left(\alpha, \beta, \frac{2z}{n}\right) - h_n\left(\alpha, \beta, \frac{2z}{n}\right)\right| \\ &< \epsilon,\end{aligned}$$

for all $z \in K_1$ and $n \geq \max(N_1, N_2, N_3)$. ∎

THEOREM 2.10. *For all* $\beta \in \mathbb{C} \setminus [0, -1, -2, \cdots]$ *and* $z \in \mathbb{C}$,

$$\lim_{n\to\infty} \left(\frac{2z}{n}\right)^{n-1} P_n\left(\alpha, \beta, \left(\frac{2z}{n}\right)^{-1}\right) = e^{-z}. \tag{2.22}$$

PROOF. By (1.2), (1.6) and Lemma 2.9,

$$1 = h(\alpha, \beta, 0) = \lim_{n\to\infty} h_{2n}\left(\alpha, \beta, \frac{2z}{n}\right) = \lim_{n\to\infty} \left(\frac{2z}{n}\right)^{-1} \frac{A_{2n}(\alpha, \beta, \frac{2z}{n})}{B_{2n}(\alpha, \beta, \frac{2z}{n})}$$

$$= \frac{\lim_{n\to\infty} \left(\frac{2z}{n}\right)^{-1} A_{2n}\left(\alpha, \beta, \frac{2z}{n}\right)}{e^{-z}} = \frac{\lim_{n\to\infty} \left(\frac{2z}{n}\right)^{n-1} P_n\left(\alpha, \beta, \left(\frac{2z}{n}\right)^{-1}\right)}{e^{-z}}.$$

The assertion (2.22) follows. ∎

3. **Orthogonal Polynomial Sequences.** By a well known result attributed to Favard (see, e.g., [2, Theorem 4.4] and [6, p. 254]) if $\{V_n(z)\}$ is a sequence of polynomials satisfying a system of three-term recurrence relations of the form

$$V_{-1}(z) := 0, \qquad V_0(z) := 1, \tag{3.1a}$$

$$V_n(z) = (\ell_n + z)V_{n-1}(z) - k_n V_{n-2}(z), \quad n = 1, 2, 3, \ldots, \tag{3.1b}$$

where

$$\ell_n \in \mathbb{R} \quad \text{and} \quad k_n > 0, \quad n = 1, 2, 3, \ldots, \tag{3.1c}$$

then there exists a distribution function $\psi(t)$ on some (a, b), where $-\infty \leq a < b \leq +\infty$, such that $\{V_n(z)\}$ is an orthogonal polynomial sequence with respect to $\psi(t)$. Moreover, the real J-fraction

$$\frac{k_1}{\ell_1 + z} - \frac{k_2}{\ell_2 + z} - \frac{k_3}{\ell_3 + z} - \cdots \tag{3.2}$$

corresponds at $z = \infty$ to the formal power series (fps)

$$L^* = \frac{c_1}{z} + \frac{c_2}{z^2} + \frac{c_3}{z^3} + \cdots, \tag{3.3}$$

where

$$c_n := \int_a^b t^{n-1} d\psi(t), \qquad n = 1, 2, 3, \ldots. \tag{3.4}$$

In this section we consider a sequence $\{Q_n(\alpha, \beta, z)\}$ defined by the recurrence relations (2.1) and assume that

$$\beta \geq \alpha > 0. \tag{3.5}$$

Hence $a_n < 0$ for $n \geq 2$, where a_n is defined by (1.2c) and therefore $\{Q_n(\alpha, \beta, z)\}$ satisfies recurrence relations of the form (3.1) with $\ell_n = a_{2n-1} + a_{2n} \in \mathbb{R}$, $k_n = a_{2n-2}a_{2n-1} > 0$, $n \geq 1$. Since the S-fraction (1.2a) is convergent, it follows that the Stieltjes moment problem for the sequence $\{c_n\}$ is determinate. Therefore the distribution function $\psi(t)$ generating the moments (3.4) is uniquely determined and is constant outside the interval $(0, \infty)$ [6, Theorem 9.8(A)]. Hence we have proved

THEOREM 3.1. *Let $\beta \geq \alpha > 0$. Then: (A) There exists a distribution function $\psi(t)$ on $(0, \infty)$ such that $\{Q_n(\alpha, \beta, z)\}$ is an orthogonal polynomial sequence with respect to ψ.*

(B) $\{Q_n(\alpha, \beta, z)\}$ satisfies the asymptotic property

$$\lim_{n \to \infty} \left(\frac{2z}{n}\right)^n Q_n\left(\alpha, \beta, \left(\frac{2z}{n}\right)^{-1}\right) = e^{-z}, \quad z \in \mathbb{C}. \tag{3.6}$$

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