EMBEDDINGS OF TOPOLOGICAL PRODUCTS OF CIRCULARLY CHAINABLE CONTINUA

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1. Introduction. In a recent paper **(5)**, the author has established the Euclidean spaces of least dimension in which the topological products of finite collections of k-cell-like continua can be embedded. Specifically, it was shown that, for each pair of positive integers k and n, the topological product of any collection of n k-cell-like continua can be embedded in Euclidean space of dimension $k(n + 1)$. This result includes a theorem of Bennett **(1)** that the topological product of any finite collection of n snakelike continua can be embedded in Euclidean space of dimension $n + 1$.

The purpose of the present paper is to extend the investigation of embeddings of topological products of continua that are normally defined in terms of cofinal sequences of open coverings whose nerves are members of characteristic classes of complexes. Such continua have been discussed by Mardesic and Segal **(7)** and shown to be topologically equivalent to inverse limits of inverse systems whose co-ordinate spaces are members of corresponding classes of polyhedra, a result that will be used in this study. The principal theorem of this paper is the following solution of the embedding problem for topological products of circularly chainable continua.

THEOREM. *The topological product of any finite collection of n circularly chainable continua can be embedded in Euclidean space of dimension $n + 2$.*

This theorem gives an affirmative answer to a question raised by Bing in a research seminar at the University of Wisconsin, 1964. Furthermore, in a subsequent paper to be presented by the author it will be shown that for each positive integer n this principal theorem is the best possible result.

In **(2)**, Bing has considered the problem of embedding circularly chainable continua in Euclidean spaces and has established a characterization of circularly chainable continua that can be embedded in the plane. A difficulty arises in embedding circularly chainable continua in the plane, which does not occur in embedding snakelike continua; it concerns the circling number of refinements in circular chains associated with the continuum in the sense of **(4)**. In the present paper, the embedding of topological products of circularly chainable continua in Euclidean spaces also involves problems that are more difficult than those encountered in the corresponding study for cell-like

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continua. Thus, we shall use somewhat more complicated techniques than were used in **(5)**.

2. Definitions and notation. The more standard terms used in this paper are defined in **(3** and **9)**. In addition we give the following definitions of terms that are either special terms or terms that are frequently given different although equivalent definitions in the literature.

A continuum K is said to be *circularly chainable* if K is homeomorphic with the inverse limit of an inverse system $\{S_i, f_i\}$ in which each co-ordinate space S_i is a simple closed curve and each bonding function f_i is a mapping of S_{i+1} onto S_i. We note that, from the results of Mardesic and Segal **(7)**, this definition of circular chainability is equivalent to that given by Bing in **(2)**.

A transformation f of a topological product space $S_1 \times S_2 \times \ldots \times S_n$ onto itself will be said to be a *monovariant function* (with respect to the topological product $S_1 \times S_2 \times \ldots \times S_n$) if there is an integer k, $1 \leqslant k \leqslant n$, such that f has the form

$$f(x_1, x_2, \ldots, x_n) = (x_1, x_2, \ldots, x_{k-1}, f_k(x_k), x_{k+1}, x_{k+2}, \ldots, x_n),$$

where x_i denotes a representative point of S_i, $i = 1, 2, \ldots, n$, and f_k is a continuous transformation of S_k onto S_k.

A homeomorphism h of a compact subset C of a space S onto a subset $h(C)$ of S is defined to be an *extensible homeomorphism* with respect to S if there is an open set U of S containing C and an extension of h to a homeomorphism of U into S.

We define a homeomorphism h of a space M_1 with metric d_1 into a space M_2 with metric d_2 to be *distance decreasing* if, for each pair of distinct points x and y in E_1,

$$d_2(h(x), h(y)) < d_1(x, y).$$

In this development it will be important to distinguish between different metric functions for topologically equivalent spaces. The metric function d for a space M that is the topological product of a finite collection of spaces $M_1, M_2, \ldots, M_n$ with metric functions $d_1, d_2, \ldots, d_n$, respectively, will be the standard product metric function

$$d((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)) = \left(\sum_{i=1}^{n} d_i(x_i, y_i)^2\right)^{\frac{1}{2}}.$$

In referring to the distance between functions it will be assumed that the functions have a common domain and that the usual function-space metric applies.

The notation E^n will be used to denote Euclidean space of dimension n and, if a and b are points of a particular Euclidean space, the notation $[a, b]$ will be used to denote the line segment with end points a and b.

3. Preliminary results. Before proceeding to give a proof of the principal theorem of this paper, a number of preliminary results are needed. First, we state two lemmas that will facilitate the construction of mappings from inverse limit spaces whose co-ordinate spaces are topological products onto inverse limit spaces whose co-ordinate spaces are subsets of an embedding space. It will be seen that both of these lemmas can be readily established using standard methods.

LEMMA 1. *If $S_1, S_2, \ldots, S_n$ is a finite collection of unit circles and M is a subset of E^k $(k \geqslant 1)$, then there is a distance-decreasing homeomorphism h embedding the topological product $S_1 \times S_2 \times \ldots \times S_n \times M$ in E^{n+k}. Furthermore, if M is an open subset of E^k, then $h(S_1 \times S_2 \times \ldots \times S_n \times M)$ is an open subset of E^{n+k}.*

LEMMA 2. *If S is a unit circle in E^3 and f is a mapping of S onto itself, then, for each positive number ϵ, there is an extensible homeomorphism h with respect to E^3 such that the distance from f to h is less than ϵ.*

Next, we state a result of McCord **(8**, §4, Theorem 8**)**.

LEMMA 3. *Let S be a compact metric space, let $\{S_i, f_i\}$ be an inverse system in which each co-ordinate space S_i is a compact subset of S, and suppose, for each positive number ϵ and each positive integer i, that there is an extensible homeomorphism $h_{\epsilon i}$ with respect to S having distance less than ϵ from the bonding mapping f_i. Then the inverse limit of $\{S_i, f_i\}$ can be embedded in S.*

In this development we shall be concerned with inverse limit systems of the form $\{S_i, f_i\}$, where, for each positive integer i, S_i is compact, $S_i = S_{i+1}$, and S_i is contained in an embedding space S that is a subset of a particular Euclidean space. It is observed that with these conditions the requirement in Lemma 3 that S be compact can be omitted.

4. Embedding topological products of circularly chainable continua. The purpose of this section is to establish the principal theorem of this paper that the topological product of any collection of n circularly chainable continua can be embedded in E^{n+2}. The spaces of the inverse systems considered in this theorem will be simple closed curves and generalized tori and the spaces of any given inverse system will have identical point sets. However, for the purposes of describing the transformations developed in the proof of the theorem, it will be convenient to index otherwise identical spaces according to their positions in a given inverse limit sequence.

THEOREM. *If $C_1, C_2, \ldots, C_n$ is a collection of circularly chainable continua, then the topological product $C_1 \times C_2 \times \ldots \times C_n$ can be embedded in Euclidean space of dimension $n + 2$.*

Proof. The proof will be presented in three sections. In the first section we shall show that the topological product $C_1 \times C_2 \times \ldots \times C_n$ can be expressed as the inverse limit of a system $\{P_i, f_i\}$ in which each co-ordinate space P_i is the topological product of n unit circles and each bonding mapping is a monovariant function. Next, modifications will be made to $\{P_i, f_i\}$ to produce a second inverse system $\{Q_i, g_i\}$ such that the inverse limits of $\{P_i, f_i\}$ and $\{Q_i, g_i\}$ are homeomorphic and $\{Q_i, g_i\}$ has a form that will facilitate the construction of a corresponding inverse system with co-ordinate spaces in E^{n+2}. Finally, in the third section of the proof, we shall develop a further inverse system whose co-ordinate spaces are embeddings of the co-ordinate spaces of $\{Q_i, g_i\}$ in E^{n+2} and whose bonding mappings are those induced by the corresponding bonding mappings of $\{Q_i, g_i\}$. It will be shown that the inverse limit of this third inverse system can be embedded in E^{n+2}.

We now consider the first section of the proof. It may be assumed that each circularly chainable continuum C_t, $t = 1, 2, \ldots, n$, is the inverse limit of a system $\{S_{rt}, b_{rt}\}$, $r = 1, 2, 3, \ldots$, in which each co-ordinate space S_{rt} is the unit circle and each bonding function b_{rt} is a mapping of $S_{r+1,t}$ onto S_{rt}. With these inverse limit systems we construct a new inverse system whose co-ordinate spaces are topological products of n unit circles.

First we define topological products

$$\{P_{rt}: r = 1, 2, 3, \ldots ; t = 1, 2, \ldots, n\}$$

in the following manner:

$$P_{rt} = S_{r+1,1} \times S_{r+1,2} \times \ldots \times S_{r+1,t-1} \times S_{rt} \times S_{r,t+1} \times \ldots \times S_{rn}.$$

In the case that $t = 1$ the defining equation for P_{rt} is to be interpreted as indicating that $P_{rt} = S_{r1} \times S_{r2} \times \ldots \times S_{rn}$. Next, the members of the collection $\{P_{rt}\}$ are ordered lexicographically with respect to their subscripts. To complete the definition of this inverse limit system we denote by x_{rt} a representative element of S_{rt} and define the bonding mapping f_{rt} of the successor of P_{rt} onto P_{rt} by the equation

$$f_{rt}(x_{r+1,1}, x_{r+1,2}, \ldots, x_{r+1,t}, x_{r,t+1}, x_{r,t+2}, \ldots, x_{rn})$$
$$= (x_{r+1,1}, x_{r+1,2}, \ldots, b_{rt}(x_{r+1,t}), x_{r,t+1}, x_{r,t+2}, \ldots, x_{rn}).$$

We denote the resulting lexicographically ordered inverse system by $\{P_i, f_i\}$ and indicate the inverse limit space of this system by L. The first section of the proof will be established by showing that L is homeomorphic with the topological product $C_1 \times C_2 \times \ldots \times C_n$.

To do this, note, since L is the inverse limit space of $\{P_i, f_i\}$, that each element of L has the form

$$u = ((x_{11}, x_{12}, \ldots, x_{1n}), (x_{21}, x_{12}, x_{13}, \ldots, x_{1n}), \ldots,$$
$$(x_{r+1,1}, x_{r+1,2}, \ldots, x_{r+1,t-1}, x_{rt}, x_{r,t+1}, \ldots, x_{rn}),$$
$$(x_{r+1,1}, x_{r+1,2}, \ldots, x_{r+1,t}, x_{r,t+1}, x_{r,t+2}, \ldots, x_{rn}), \ldots).$$

We shall refer to the expressions enclosed in inner parentheses in this equation for u as "co-ordinate terms" of u and use the projection mapping notation to indicate the position of a particular co-ordinate term in the sequence of co-ordinate terms. Thus $\pi_1(u)$ would denote the first co-ordinate term $(x_{11}, x_{12}, \ldots, x_{1n})$. The individual entries, ignoring inner parentheses, of the right-hand side of the equation for u will be referred to as "elementary terms." Thus x_{11} would be the first elementary term of u.

Now, in the expression for u,

$$b_{rt}(x_{r+1,t}) = x_{rt} \qquad \text{for } r = 1, 2, 3, \ldots; t = 1, 2, \ldots, n.$$

Furthermore the expression

$$v = ((x_{11}, x_{21}, x_{31}, \ldots), (x_{12}, x_{22}, x_{32}, \ldots), \ldots, (x_{1n}, x_{2n}, x_{3n}, \ldots)),$$

where

$$x_{rt} = b_{rt}(x_{r+1,t}) \qquad \text{for } r = 1, 2, 3, \ldots; t = 1, 2, \ldots, n,$$

represents a point of the topological product $C_1 \times C_2 \times \ldots \times C_n$. We define a transformation h of L onto $C_1 \times C_2 \times \ldots \times C_n$ by setting

$$h(u) = v.$$

It is easily verified that h is a one-to-one transformation of L onto

$$C_1 \times C_2 \times \ldots \times C_n.$$

To see that h is also a continuous transformation we note, since the co-ordinate spaces of $\{P_i, f_i\}$ are uniformly bounded metric spaces, that the product-space metric described in (**6**, Theorem 14, pp. 122–123) induces a topology for L equivalent to the usual Tychonoff topology. Hence, if $u_1, u_2, u_3, \ldots$ is a sequence of points of L converging to a point u of L and i is a positive integer, then $\pi_i(u_i), \pi_i(u_2), \pi_i(u_3), \ldots$ converges to $\pi_i(u)$. Furthermore, if $(x_{rt})_i$ is an elementary term of u_i, $i = 1, 2, 3, \ldots,$ then $(x_{rt})_1, (x_{rt})_2, (x_{rt})_3, \ldots$ converges to the elementary term x_{rt} of u. It follows that $h(u_1), h(u_2), h(u_3), \ldots$ converges to $h(u)$. Thus h is a continuous transformation, and we conclude that the inverse limit space L is homeomorphic with the topological product $C_1 \times C_2 \times \ldots \times C_n$.

The second section of the proof involves the modification of the inverse system $\{P_i, f_i\}$ to produce a second inverse system $\{Q_i, g_i\}$ having the properties described in the first paragraph of the proof. Let S denote the unit circle in E^3 with cylindrical co-ordinate representation $\{(r, \theta, z) : r = 1, z = 0\}$, let T denote the solid open torus in E^3 with cylindrical co-ordinate representation $\{(r, \theta, z) : (1 - r)^2 + z^2 < \frac{1}{2}\}$, and let D denote the open disk that is obtained by intersecting T with the half-plane $\{(r, \theta, z) : \theta = 0\}$. Then, if p is a point of S and c is a point of D, the ordered pair (p, c) can be considered as identifying a point of T whose θ co-ordinate is determined by p and whose r, z co-ordinates are determined by c. Thus, if i and j are integers, $1 \leqslant i, j \leqslant n$, $S_1, S_2, \ldots, S_n$

are copies of the unit circle S and T_i, T_j are copies of the open solid torus T such that T_i and T_j are associated with S_i and S_j, respectively, then there is a homeomorphism w_{ij} of

$$S_1 \times S_2 \times \ldots \times S_{i-1} \times T_i \times S_{i+1} \times S_{i+2} \times \ldots \times S_n$$

onto

$$S_1 \times S_2 \times \ldots \times S_{j-1} \times T_j \times S_{j+1} \times S_{j+2} \times \ldots \times S_n$$

such that the restriction of w_{ij} to $S_1 \times S_2 \times \ldots \times S_n$ is the identity mapping. In particular, we may choose w_{ij} to have the form

$$w_{ij}(p_1, p_2, \ldots, p_{i-1}, (p_i, c), p_{i+1}, p_{i+2}, \ldots, p_n)$$
$$= (p_1, p_2, \ldots, p_{j-1}, (p_j, c), p_{j+1}, p_{j+2}, \ldots, p_n),$$

where p_t is a representative point of S_t, $t = 1, 2, \ldots, n$, and c is a representative element of D. The homeomorphisms

$$\{w_{ij} : i = 1, 2, \ldots, n; j = 1, 2, \ldots, n\}$$

are now used in the construction of the inverse system $\{Q_i, g_i\}$ in the following manner:

(1) Each co-ordinate space of $\{Q_i, g_i\}$ is defined to be the topological product of n unit circles.

(2) The bonding mappings with odd subscripts of $\{Q_i, g_i\}$ are defined by the relationship $f_i = g_{2i-1}$, $i = 1, 2, 3, \ldots$.

(3) To define the bonding mappings with even subscripts of $\{Q_i, g_i\}$ we express each of the functions f_i, $i = 1, 2, 3, \ldots$, in the alternative double-subscript form $f_{r_i t_i}$ described in the development of the inverse system $\{P_i, f_i\}$. Then if $f_{r_i t_i}$ and $f_{r_i +1, t_i +1}$ are successive bonding mappings of the system $\{P_i, f_i\}$, we define $g_{2i} = w_{t_i +1, t_i}$, $i = 1, 2, 3, \ldots$.

It is observed that the co-ordinate spaces of $\{Q_i, g_i\}$ are identical with those of $\{P_i, f_i\}$, the bonding mappings of $\{Q_i, g_i\}$ form a sequence

$$f_{11}, w_{21}, f_{12}, w_{32}, f_{13}, \ldots, f_{1n}, w_{1n}, f_{21}, \ldots$$

in which the bonding mappings of $\{P_i, f_i\}$ alternate with homeomorphisms of the collection

$$\{w_{ij} : i = 1, 2, \ldots, n; j = 1, 2, \ldots, n\},$$

and the members of this collection are identity mappings on the co-ordinate spaces of $\{Q_i, g_i\}$. Thus it follows that the inverse limits of $\{P_i, f_i\}$ and $\{Q_i, g_i\}$ are topologically equivalent. The reasons for the particular choice of the homeomorphisms

$$\{w_{ij} : i = 1, 2, \ldots, n; j = 1, 2, \ldots, n\}$$

will become apparent in the next section of the proof.

We now consider the final section of the proof. In this section each co-ordinate space Q_i of the system $\{Q_i, g_i\}$ will be assumed to be represented as the topological product $S_1 \times S_2 \times \ldots \times S_n$ where each space S_t, $t = 1, 2, \ldots, n$, is a copy of the unit circle S. Now, if i is an odd integer, g_i is a monovariant function, so that we may choose a corresponding integer k_i with the property that g_i can be expressed in the form

$$g_i(x_1, x_2, \ldots, x_n) = (x_1, x_2, \ldots, x_{k_i-1}, g_{ik_i}(x_{k_i}), x_{k_i+1}, x_{k_i+2}, \ldots, x_n)$$

where x_t is a representative element of S_t, $t = 1, 2, \ldots, n$, and g_{ik_i} is a continuous transformation of S_{k_i} onto S_{k_i}. For each even integer i, we define a corresponding integer k_i by the relationship $k_i = k_{i-1}$, $i = 2, 4, 6, \ldots$. Let

$$B_t = S_1 \times S_2 \times \ldots \times S_{t-1} \times T_t \times S_{t+1} \times S_{t+2} \times \ldots \times S_n, \qquad t = 1, 2, \ldots, n,$$

where T_t is a copy of the solid open torus T such that T_t is associated with S_t. Then, by Lemma 1, there are distance-reducing homeomorphisms

$$\{e_t\colon t = 1, 2, \ldots, n\}$$

such that $e_t(B_t)$ is an open subset of E^{n+2}. With the homeomorphisms $\{e_t : t = 1, 2, \ldots, n\}$ and integers $\{k_i : i = 1, 2, 3, \ldots\}$, we construct a mapping of the inverse limit of the system $\{Q_i, g_i\}$ onto an inverse limit space whose co-ordinate spaces are contained in E^{n+2}.

We define the required mapping by means of the following inverse-limit diagram.

$$\begin{array}{ccccccc}
Q_1 & \xleftarrow{g_1} & Q_2 & \xleftarrow{g_2} & Q_3 & \xleftarrow{g_3} & Q_4 \leftarrow \ldots \\
\downarrow e_{k_1} & & \downarrow e_{k_2} & & \downarrow e_{k_3} & & \downarrow e_{k_4} \\
e_{k_1}(Q_1) & \xleftarrow[e_{k_1} g_1 e_{k_2}^{-1}]{} & e_{k_2}(Q_2) & \xleftarrow[e_{k_2} g_2 e_{k_3}^{-1}]{} & e_{k_3}(Q_3) & \xleftarrow[e_{k_3} g_3 e_{k_4}^{-1}]{} & e_{k_4}(Q_4) \leftarrow \ldots
\end{array}$$

Then, from the normal interpretation of this diagram, it follows that the inverse limit space of the system

$$\{e_{k_i}(Q_i), e_{k_i}\, g_i\, e_{k_{i+1}}^{-1}\}$$

is homeomorphic with the inverse limit space of the system $\{Q_i, g_i\}$. Furthermore, each co-ordinate space $e_{k_i}(Q_i)$ of the former system is a compact subset of E^{n+2}. In addition, since each embedding homeomorphism e_{k_i}, $i = 1, 2, 3, \ldots$, is characterized by the requirement that it be a homeomorphism of B_{k_i} into E^{n+2}, we may assume without loss in generality that, for each pair of positive integers i and j, $e_{k_i} = e_{k_j} w_{k_i k_j}$. Thus it will be supposed that

$$e_{k_1}(Q_1) = e_{k_2}(Q_2) = e_{k_3}(Q_3) = \ldots.$$

The proof of the theorem will be completed by showing that the inverse limit of the system

$$\{e_{k_i}(Q_i), e_{k_i}\, g_i\, e_{k_{i+1}}^{-1}\}$$

can be embedded in E^{n+2}.

We choose ϵ to be a positive number and consider two cases.

Case I: i is an odd integer. In this case g_i is a monovariant function with the form previously described, where g_{ik_i} is a continuous transformation of the unit circle S_{k_i} onto itself. Then, by Lemma 2, there is a homeomorphism H_{ik_i} of an open set G_{ik_i} of E^3 into E^3 such that G_{ik_i} contains S_{k_i} and the restriction h_{ik_i} of H_{ik_i} to S_{k_i} has distance less than ϵ from g_{ik_i}. It may be assumed that the domain and range of H_{ik_i} are each subsets of the solid open torus T_{k_i}. We now define homeomorphisms h_i and H_i related to h_{ik_i} and H_{ik_i}, respectively, in the following manner:

The function h_i has domain $S_1 \times S_2 \times \ldots \times S_n$ and satisfies the equation

$$h_i(x_1, x_2, \ldots, x_n) = (x_1, x_2, \ldots, x_{k_i-1}, h_{ik_i}(x_{k_i}), x_{k_i+1}, x_{k_i+2}, \ldots, x_n).$$

The function H_i has domain

$$S_1 \times S_2 \times \ldots \times S_{k_i-1} \times G_{ik_i} \times S_{k_i+1} \times S_{k_i+2} \times \ldots \times S_n$$

and satisfies the equation

$$H_i(x_1, x_2, \ldots, x_n) = (x_1, x_2, \ldots, x_{k_i-1}, H_{ik_i}(x_k), x_{k_i+1}, x_{k_i+2}, \ldots, x_n).$$

It is observed that h_i and H_i are homeomorphisms, H_i is an extension of h_i, and g_i has distance less than ϵ from h_i. Now, by the second assertion of Lemma 1 and the fact that G_{ik_i} is open with respect to E^3, it follows that the image under e_{k_i} of the domain of H_i is an open subset of E^{n+2}. Furthermore, since $e_{k_i} = e_{k_{i+1}}$ and the domain of H_i contains Q_{i+1}, the image under $e_{k_{i+1}}$ of the domain of H_i is an open subset of E^{n+2} which contains $e_{k_{i+1}}(Q_{i+1})$. In addition, the image under e_{k_i} of the range of H_i is an open subset of E^{n+2}. Hence, the composite function $e_{k_i}\, h_i\, e_{k_{i+1}}^{-1}$ with domain $e_{k_{i+1}}(Q_{i+1})$ is an extensible homeomorphism with respect to E^{n+2}. Finally, the functions $e_{k_i}\, g_i\, e_{k_{i+1}}^{-1}$ and $e_{k_i}\, h_i\, e_{k_{i+1}}^{-1}$ with domain $e_{k_{i+1}}(Q_{i+1})$ have distance apart equal to the distance from $e_{k_i} g_i$ to $e_{k_i} h_i$. Therefore, since e_{k_i} is a distance-reducing homeomorphism and g_i has distance less than ϵ from h_i, we conclude that the extensible homeomorphism $e_{k_i}\, h_i e_{k_{i+1}}^{-1}$ with respect to E^{n+2} has distance less than ϵ from $e_{k_i} g_i\, e_{k_{i+1}}^{-1}$.

Case II: i is an even integer. In this case g_i has the form described in condition (3) of the definition of the inverse system $\{Q_i, g_i\}$. Now, let j be the integer such that $i = 2j$ and note, from condition (2) of the definition of the inverse system $\{Q_i, g_i\}$, that $f_j = g_{i-1}$ and $f_{j+1} = g_{i+1}$. Hence from the equations that define the form of the bonding mappings of the inverse systems $\{P_i, f_i\}$ and $\{Q_i, g_i\}$, it follows that $k_{i-1} = t_j$ and $k_{i+1} = t_{j+1}$. Furthermore, since i is an even integer, k_{i-1} is also equal to k_i. Thus, from these last three equalities and the fact that $g_i = w_{t_j+1 t_j}$, we obtain the result that the bonding mapping g_i is equivalent to the homeomorphism $w_{k_i+1 k_i}$. Now, by Lemma 1, $e_{k_i+1}(B_{k_i+1})$ is an open subset of E^{n+2} containing $e_{k_i+1}(Q_{i+1})$. In addition, since $w_{k_i+1 k_i}(B_{k_i+1}) = B_{k_i}$ and $e_{k_i}(B_{k_i})$ is an open subset of E^{n+2}, it follows that the image under the homeomorphism $e_{k_i} w_{k_i+1 k_i} e_{k_i+1}^{-1}$ of the set $e_{k_i+1}(B_{k_i+1})$ is contained in E^{n+2}. Therefore, the restriction of $e_{k_i} w_{k_i+1 k_i} e_{k_i+1}^{-1}$ to $e_{k_i+1}(Q_{i+1})$ is an extensible homeomorphism with respect to E^{n+2} having distance zero from the restriction of $e_{k_i} g_i e_{k_i+1}^{-1}$ to $e_{k_i+1}(Q_{i+1})$.

We conclude, by Lemma 3 together with the observation following Lemma 3, that the inverse limit of the system $\{e_{k_i}(Q_i), e_{k_i} g_i e_{k_i+1}^{-1}\}$ can be embedded in E^{n+2}. Therefore, the topological product $C_1 \times C_2 \times \ldots \times C_n$ can be embedded in E^{n+2}.

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