

NON-TAME AUTOMORPHISMS OF A FREE GROUP OF RANK 3 IN $\mathfrak{A}_p\mathfrak{A}$

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TO PROFESSOR ROGER BRYANT ON HIS 60th BIRTHDAY

Abstract. We give a way of constructing non-tame automorphisms of a free group of rank 3 in the variety $\mathfrak{A}_p\mathfrak{A}$, with p prime.

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1. Introduction. For any group G , we write G' for the derived group of G . Let $\text{IA}(G)$ denote the kernel of the natural mapping from $\text{Aut}(G)$ into $\text{Aut}(G/G')$. The elements of $\text{IA}(G)$ are called IA-automorphisms of G . For a positive integer n , with $n \geq 2$, let F_n be a free group of rank n with a basis (in other words, a free generating set) $\{f_1, \dots, f_n\}$. For any variety of groups \mathfrak{V} , let $\mathfrak{V}(F_n)$ denote the verbal subgroup of F_n corresponding to \mathfrak{V} . Also, let $F_n(\mathfrak{V}) = F_n/\mathfrak{V}(F_n)$: thus $F_n(\mathfrak{V})$ is a relatively free group of rank n and it has a basis $\{x_1, \dots, x_n\}$, where $x_i = f_i\mathfrak{V}(F_n)$, $i = 1, \dots, n$. If ϕ is an automorphism of $F_n(\mathfrak{V})$ then $\{x_1\phi, \dots, x_n\phi\}$ is also a basis of $F_n(\mathfrak{V})$ and every basis of $F_n(\mathfrak{V})$ has this form. (For information concerning relatively free groups and varieties of groups see [14].) Since $\mathfrak{V}(F_n)$ is a characteristic subgroup of F_n , every automorphism φ of F_n induces an automorphism $\bar{\varphi}$ of $F_n(\mathfrak{V})$ in which $x_i\bar{\varphi} = (f_i\varphi)\mathfrak{V}(F_n)$ for $i = 1, \dots, n$. Thus we obtain a homomorphism of automorphism groups

$$\alpha : \text{Aut}(F_n) \longrightarrow \text{Aut}(F_n(\mathfrak{V})).$$

An automorphism of $F_n(\mathfrak{V})$ which belongs to the image of α is called *tame*. The image of α is denoted by $T_{\mathfrak{V}}$ (or, briefly, T if no confusion is likely to arise). An element $h \in F_n(\mathfrak{V})$ is called *primitive* if h is contained in a basis of $F_n(\mathfrak{V})$. We say that h is induced by a primitive element of F_n if there exists a primitive element g of F_n such that $g\mathfrak{V}(F_n) = h$. For a non-negative integer m , \mathfrak{A}_m denotes the variety of all abelian groups of exponent dividing m , interpreted in such a way that $\mathfrak{A}_0 = \mathfrak{A}$ is the variety of all abelian groups. Furthermore we write $\mathfrak{A}_m = \mathfrak{A}_m\mathfrak{A}$ for the variety of all extensions of groups in \mathfrak{A}_m by groups in \mathfrak{A} .

Let R be a commutative ring with identity and m be a positive integer. We write $\text{GL}_m(R)$ for the general linear group of degree m with entries in R and $\text{SL}_m(R)$ for the corresponding special linear group. Let $E_m(R)$ denote the subgroup of $\text{SL}_m(R)$ that is generated by the elementary matrices. We say a matrix $(a_{ij}) \in \text{SL}_m(R)$ is elementary if $a_{ii} = 1$ for $i = 1, \dots, m$ and there exists at most one ordered pair of subscripts (i, j) with $i \neq j$ such that $a_{ij} \neq 0$. Furthermore we write $\text{GE}_m(R)$ for the subgroup of $\text{GL}_m(R)$ generated by the invertible diagonal matrices and $E_m(R)$. A subset S of R is said to be *multiplicative closed* if $1 \in S$ and the product of any two

elements of S is an element of S . We write $\mathcal{L}_S(R)$ for the localization of R at S . Let $R[a_1, \dots, a_r]$ be the polynomial ring in indeterminates a_1, \dots, a_r with coefficients in R . Let S be the multiplicative monoid generated by the set $\{a_1, \dots, a_r\}$. Then $\mathcal{L}_S(R[a_1, \dots, a_r]) = R[a_1^{\pm 1}, \dots, a_r^{\pm 1}]$ is the Laurent polynomial ring in indeterminates a_1, \dots, a_r with coefficients in R . Let \mathbb{Z} denote the ring of integers. By a famous result of Suslin [20], $\mathrm{SL}_m(\mathbb{Z}[a_1^{\pm 1}, \dots, a_r^{\pm 1}]) = \mathrm{E}_m(\mathbb{Z}[a_1^{\pm 1}, \dots, a_r^{\pm 1}])$ for all integers $m \geq 3$ and $r \geq 1$. For $m = r = 2$, it is well-known that $\mathrm{SL}_2(\mathbb{Z}[a_1^{\pm 1}, a_2^{\pm 1}]) \neq \mathrm{E}_2(\mathbb{Z}[a_1^{\pm 1}, a_2^{\pm 1}])$ (see [4, 8]).

Chen [6] gave an example of a non-tame automorphism of $F_3(\mathfrak{V}_0)$. Bachmuth and Mochizuki [5] have shown that $\mathrm{Aut}(F_3(\mathfrak{V}_0))$ is not finitely generated. Hence $\mathrm{IA}(F_3(\mathfrak{V}_0))$ is not finitely generated as a group on which T acts by conjugation. Thus there exist infinitely many non-tame automorphisms of $F_3(\mathfrak{V}_0)$. Roman'kov [17] has shown that there exists a primitive element of $F_3(\mathfrak{V}_0)$ that is not induced by a primitive element of F_3 . Such a primitive element of $F_3(\mathfrak{V}_0)$ is called a non-induced primitive element. The existence of non-induced primitive element starts from the fact that $\mathrm{SL}_2(\mathbb{Z}[a_1^{\pm 1}, a_2^{\pm 1}]) \neq \mathrm{E}_2(\mathbb{Z}[a_1^{\pm 1}, a_2^{\pm 1}])$. Evans [8, Theorem C] has presented a method of constructing elements of $\mathrm{SL}_2(\mathbb{Z}[a_1^{\pm 1}, a_2^{\pm 1}])$ not in $\mathrm{E}_2(\mathbb{Z}[a_1^{\pm 1}, a_2^{\pm 1}])$. From the papers of Evans [8] and Roman'kov [17], it follows that there exists a way of constructing non-tame automorphisms of $F_3(\mathfrak{V}_0)$.

Our main purpose in this paper is to give a way of constructing non-tame automorphisms of $F_3(\mathfrak{V}_p)$ with p prime. In the next few lines we shall explain our method of how to construct non-tame automorphisms of $F_3(\mathfrak{V}_p)$: For each automorphism ϕ of $F_3(\mathfrak{V}_p)$ we define the Jacobian matrix J_ϕ over $\mathbb{F}_p A_3$, where \mathbb{F}_p denotes the finite field with p elements, and A_3 is the free abelian group F_3/F'_3 with a basis $\{s_1, s_2, s_3\}$, where $s_i = f_i F'_3$, $i = 1, 2, 3$. Let ζ be the Bachmuth representation of $\mathrm{IA}(F_3(\mathfrak{V}_p))$, that is, the group monomorphism $\zeta : \mathrm{IA}(F_3(\mathfrak{V}_p)) \rightarrow \mathrm{GL}_3(\mathbb{F}_p A_3)$ defined by $\phi\zeta = J_\phi$. Notice that the Bachmuth representation is essentially via Fox derivatives. Let S be the multiplicative monoid of $\mathbb{F}_p A_3$ generated by $s_1 - 1$, and let $\mathcal{L}_S(\mathbb{F}_p A_3)$ be the localization of $\mathbb{F}_p A_3$ at S . As in the paper of Bachmuth and Mochizuki [5], we conjugate $(\mathrm{IA}(F_3(\mathfrak{V}_p)))\zeta$ by a specific element (c_{ij}) of $\mathrm{GL}_3(\mathcal{L}_S(\mathbb{F}_p A_3))$ to obtain a group homomorphism η from the image of ζ into $\mathrm{GL}_2(\mathcal{L}_S(\mathbb{F}_p A_3))$. Let H be a finitely generated subgroup of $\mathrm{IA}(F_3(\mathfrak{V}_p))$ containing $T \cap \mathrm{IA}(F_3(\mathfrak{V}_p))$. Let ρ be the mapping from H into A_3 defined by $\phi\rho = \det J_\phi = s_1^{\mu_1} s_2^{\mu_2} s_3^{\mu_3}$ where $\mu_1, \mu_2, \mu_3 \in \mathbb{Z}$. Since $T \cap \mathrm{IA}(F_3(\mathfrak{V}_p)) \subseteq H$, it is easily verified that ρ is a group epimorphism. We write N for the kernel of ρ . Since H/N is finitely presented and H is finitely generated, we obtain from a result of Hall [10, page 421] N is finitely generated as a group on which H acts by conjugation. Let \mathcal{H} and \mathcal{N} be the images of H and N , respectively, via η . We show in Lemma 3.2 that $\mathcal{N} \subseteq \mathrm{E}_2(\mathcal{L}_P(\mathbb{F}_p A_3))$ for some suitable multiplicative monoid P in $\mathbb{F}_p A_3$. The most difficult part of our method is to show that $\mathrm{SL}_2(\mathcal{L}_P(\mathbb{F}_p A_3)) \neq \mathrm{E}_2(\mathcal{L}_P(\mathbb{F}_p A_3))$ (see Lemma 3.6). We note that the multiplicative monoid P depends upon \mathcal{H} . Taking H to be $T \cap \mathrm{IA}(F_3(\mathfrak{V}_p))$ and an explicitly given multiplicative monoid P , we construct infinitely many non-tame automorphisms of $F_3(\mathfrak{V}_p)$ (see Theorem 4.2). That is, using Lemmas 3.3, 3.5 and 3.6, a particular 2×2 matrix, Δ , is constructed which is not a product of elementary matrices, which is nonetheless in the image of the automorphism group of M_3 (as is seen by explicitly writing an appropriate 3×3 matrix and conjugating it) and in the kernel of the map ρ . Lemma 3.2 then allows one to conclude that no tame automorphism can produce Δ , since tame automorphisms in the kernel of ρ are products of elementary

matrices. The process of constructing these non-tame automorphisms is effective (see Examples 4.3).

2. Notation and preliminaries. We first fix some notation which is used throughout this paper. For any group G , we write G' for the derived group of G . Recall that $\text{IA}(G)$ denotes the group of IA-automorphisms of G . If a_1, \dots, a_c are elements of G then $[a_1, a_2] = a_1^{-1}a_2^{-1}a_1a_2$ and for $c \geq 3, [a_1, \dots, a_c] = [[a_1, \dots, a_{c-1}], a_c]$. For elements a and b of G, b^a denotes the conjugate $a^{-1}ba$. For a positive integer n , let F_n be a free group of rank n with a basis $\{f_1, \dots, f_n\}$. Let $A_n = F_n/F'_n$, the free abelian group of rank n . Thus $\{s_1, \dots, s_n\}$, with $s_i = f_iF'_n$ ($i = 1, \dots, n$), is a basis for A_n . Fix a prime integer p . The variety \mathfrak{V}_p is the class of all groups satisfying the laws $[[f_1, f_2], [f_3, f_4]]$ and $[f_1, f_2]^p$. Thus $\mathfrak{V}_p(F_3) = F''_3(F_3)^p$ and so, every element w of $\mathfrak{V}_p(F_3)$ is a product $w = w_1 \cdots w_k$, where for $i = 1, \dots, k$, either (i) $w_i \in F''_3$, or (ii) $w_i = [u, v]^p$ with $u, v \in F_3$. Let $M_3 = F_3(\mathfrak{V}_p)$ and let $x_i = f_i\mathfrak{V}_p(F_3), i = 1, 2, 3$. Thus $\{x_1, x_2, x_3\}$ is a basis for M_3 . Let \mathbb{Z} and \mathbb{F}_p be the ring of integers and the field of p elements, respectively. We write $\mathbb{Z}G$ (resp. \mathbb{F}_pG) for the integral group ring (resp. the group algebra over \mathbb{F}_p and G).

2.1. Fox derivatives. We use the partial derivatives introduced by Fox [9]. In our notation these are defined as follows : For $j = 1, 2, 3$, the (left) Fox derivative associated with f_j is the linear map $D_j : \mathbb{Z}F_3 \rightarrow \mathbb{Z}F_3$ satisfying the conditions

$$D_j(f_j) = 1, \quad D_j(f_i) = 0 \text{ for } i \neq j \tag{1}$$

and

$$D_j(uv) = D_j(u) + uD_j(v) \text{ for all } u, v \in F_3. \tag{2}$$

It follows that $D_j(1) = 0$ and $D_j(u^{-1}) = -u^{-1}D_j(u)$ for all $u \in F_3$. Let ε be the unit augmentation map $\varepsilon : \mathbb{Z}F_3 \rightarrow \mathbb{Z}$. It is well-known (see, for example, [7, page 5]) that the kernel of ε (i.e., the augmentation ideal of $\mathbb{Z}F_3$) is a free left $\mathbb{Z}F_3$ -module with basis $\{f_j - 1 : j = 1, 2, 3\}$. If $u \in \mathbb{Z}F_3$ then $u - u\varepsilon = \sum_{i=1}^3 u_i(f_i - 1)$, with $u_i \in \mathbb{Z}F_3, i = 1, 2, 3$. By applying D_j , we obtain $D_j(u) = u_j$ and so, we get the following Fox's fundamental formula

$$u - u\varepsilon = \sum_{i=1}^3 D_i(u)(f_i - 1) \tag{3}$$

for all $u \in \mathbb{Z}F_3$.

There is a natural group epimorphism $\kappa : F_3 \rightarrow A_3$ which extends to a ring epimorphism $\kappa : \mathbb{Z}F_3 \rightarrow \mathbb{Z}A_3$. Furthermore we write γ for the natural ring epimorphism from $\mathbb{Z}A_3$ into \mathbb{F}_pA_3 which agrees on \mathbb{Z} with the natural ring homomorphism from \mathbb{Z} onto \mathbb{F}_p . Set $\delta = \kappa \circ \gamma$ and let λ be the natural group epimorphism $\lambda : M_3 \rightarrow A_3$ which extends to a ring epimorphism $\lambda : \mathbb{F}_pM_3 \rightarrow \mathbb{F}_pA_3$. Note that, for all $f \in F_3$,

$$f\delta = (f\kappa)\gamma = (fF'_3)\gamma = fF'_3 = (f\mathfrak{V}_p(F_3))\lambda. \tag{4}$$

The equation (4) is really a statement about a rather natural commuting triangle. By an easy calculation, for all $u, v \in F_3$ and $j = 1, 2, 3$,

$$D_j([u, v]) = u^{-1}(v^{-1} - 1)D_j(u) + u^{-1}v^{-1}(u - 1)D_j(v). \tag{5}$$

Let $u = u_1u_2 \cdots u_k$, with $u_1, u_2, \dots, u_k \in F_3$ and $k \geq 2$. It follows from (2) and an inductive argument on k that

$$D_j(u) = D_j(u_1) + u_1D_j(u_2) + \cdots + u_1 \cdots u_{k-1}D_j(u_k) \tag{6}$$

for $j = 1, 2, 3$. We may deduce from (6) that

$$D_j([u, v]^p) = (1 + [u, v] + \cdots + [u, v]^{p-1})D_j([u, v]) \tag{7}$$

for all $u, v \in F_3$ and $j = 1, 2, 3$. Every element w of $\mathfrak{Y}_p(F_3)$ is a product $w = w_1 \cdots w_k$, where for $i = 1, \dots, k$, either (i) $w_i \in F_3''$, or (ii) $w_i = [u, v]^p$ with $u, v \in F_3$. It follows from (4)–(7) that

$$(D_j(w))\delta = 0 \tag{8}$$

for $j = 1, 2, 3$ and $w \in \mathfrak{Y}_p(F_3)$.

For j , with $j \in \{1, 2, 3\}$, we define

$$d_j(f\mathfrak{Y}_p(F_3)) = (D_j(f))\delta$$

for all $f \in F_3$. It is easily verified that d_j is well-defined. Since D_j is a linear map and δ is a ring homomorphism, we obtain d_j extends to a linear map from $\mathbb{F}_p M_3$ into $\mathbb{F}_p A_3$ for $j = 1, 2, 3$. From (1), we obtain

$$d_j(x_j) = 1, \quad d_j(x_i) = 0 \quad \text{for } i \neq j.$$

Furthermore $d_j(u^{-1}) = -(u\lambda)^{-1}d_j(u)$ for all $u \in M_3$. Let $u, v \in M_3$, with $u = f\mathfrak{Y}_p(F_3)$, $v = g\mathfrak{Y}_p(F_3)$ and $f, g \in F_3$. We may deduce from (2) and (4) that

$$d_j(uv) = d_j(u) + (u\lambda)d_j(v). \tag{9}$$

Note that if $u \in M_3'$ and $v \in M_3$ then (9) becomes

$$d_j(uv) = d_j(u) + d_j(v). \tag{10}$$

Furthermore, by (5), (9) and since δ is a ring homomorphism, we obtain

$$d_j([u, v]) = (u^{-1}\lambda)(v^{-1}\lambda - 1)d_j(u) + (u^{-1}\lambda)(v^{-1}\lambda)(u\lambda - 1)d_j(v). \tag{11}$$

Let ϕ be an automorphism of M_3 . The *Jacobian matrix* J_ϕ is defined to be the 3×3 matrix over $\mathbb{F}_p A_3$ whose (i, j) entry is $d_j(x_i\phi)$ for $i, j = 1, 2, 3$. Since δ is a ring homomorphism, it follows from (3) that

$$u\lambda - 1 = \sum_{i=1}^3 d_i(u)(s_i - 1) \tag{12}$$

for all $u \in M_3$. Since M_3' is a vector space over \mathbb{F}_p , it may be regarded as a right $\mathbb{F}_p(M_3/M_3')$ -module in the usual way, where the module action comes from conjugation

in M_3 . The group epimorphism $\lambda : M_3 \rightarrow A_3$ induces an isomorphism from M_3/M'_3 to A_3 . So, we may regard M'_3 as a right $\mathbb{F}_p A_3$ -module. For $w \in M'_3$ and $s \in \mathbb{F}_p A_3$, we write w^s to denote the image of w under the action of s . For $s \in \mathbb{F}_p A_3$ write $s = \sum_i m_i r_i$, where $m_i \in \mathbb{F}_p$ and $r_i \in A_3$ for each i and define

$$s^* = \sum_i m_i r_i^{-1}.$$

Thus $s \mapsto s^*$ is an involutory linear mapping from $\mathbb{F}_p A_3$ to $\mathbb{F}_p A_3$. For $w \in M'_3$ and $s \in \mathbb{F}_p A_3$, it is easily verified that

$$d_j(w^s) = s^* d_j(w). \tag{13}$$

The proof of the following result is elementary.

LEMMA 2.1. *Let M_3 be the free group of rank 3 in the variety \mathfrak{A}_p , with p prime.*

- (i) *For $u \in M'_3$, $[x_i, x_j]^u = [x_i, x_j]$ for all i, j .*
- (ii) *For all $u \in M_3$ such that $u \equiv u' \pmod{M'_3}$,*

$$[x_i, x_j]^u = [x_i, x_j]^{u'}.$$

- (iii) *For all $u, v \in M_3$,*

$$[x_i, x_j]^{uv} = [x_i, x_j]^{vu}.$$

- (iv) *$d_i(x_i[x_i, x_j]) = s_j^{-1}$, $d_j(x_i[x_i, x_j^{-1}]) = 1 - s_i$ and $d_j(x_i[x_i, x_j]) = s_j^{-1}(s_i - 1)$ for $i \neq j$.*
- (v) *Let $w \in M'_3$. Then we may write*

$$w = \prod_{\substack{i,j \\ 1 \leq i < j \leq 3}} [x_i, x_j]^{v_{ij}}$$

where $v_{ij} \in \mathbb{F}_p A_3$ for all i, j .

By Lemma 2.1, $[x_i, x_j]^u$ is really determined by the congruence classes of u modulo M'_3 .

2.2. Ihara's Theorem. If R is a unique factorization domain (UFD) and $S \subseteq R \setminus \{0\}$ is a multiplicative closed subset then $\mathcal{L}_S(R)$ is a UFD (see, for example, [1, Chapter 2]). Recall that $\mathbb{F}_p[s_1, s_2, s_3]$ is a UFD. Let C be the monoid generated by $\{s_1, s_2, s_3\}$. It is easily verified that $\mathbb{F}_p A_3 = \mathcal{L}_C(\mathbb{F}_p[s_1, s_2, s_3])$ and so $\mathbb{F}_p A_3$ is a UFD. Let Q denote the quotient field of $\mathbb{F}_p A_3$. The field Q has a discrete valuation determined by the powers of s_3 . More precisely, if $v \in \mathbb{F}_p A_3 \setminus \{0\}$, then v can be uniquely written as

$$v = \sum_{i=t}^r v_i s_3^i = s_3^t \sum_{i=0}^{r-t} v_{i+t} s_3^i, \quad t < r,$$

where $v_{i+t} \in \mathbb{F}_p A_2$, $i=0, \dots, r-t$. Define the s_3 -value of v to be $v v = t$. If $u, v \in \mathbb{F}_p A_3 \setminus \{0\}$, then we define $(u/v)v = uv - tv$. Let \mathcal{O} be the valuation ring of v i.e., \mathcal{O}

is the ring consisting of 0 and all $w \in Q \setminus \{0\}$ such that $wv \geq 0$. For the proof of the following result, we refer to [19, Corollary 1, page 79].

LEMMA 2.2. $SL_2(Q)$ is the free product of $SL_2(\mathcal{O})$ and $SL_2(\mathcal{O}) \begin{pmatrix} 1 & 0 \\ 0 & s_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & s_3 \end{pmatrix}^{-1} SL_2(\mathcal{O}) \begin{pmatrix} 1 & 0 \\ 0 & s_3 \end{pmatrix}$ with their intersection D amalgamated.

2.3. Irreducible polynomials over finite fields. We recall a general principle of obtaining new irreducible polynomials from known ones (see, for example, [12, Chapter 3]). Let $f \in \mathbb{F}_p[x]$ be a non-zero polynomial. If f has a non-zero constant term, then the least positive integer e for which $f(x)$ divides $x^e - 1$ is called the *order* of f and denoted by $\text{ord}(f)$. Let f be an irreducible polynomial in $\mathbb{F}_p[x]$ of degree m , with a non-zero constant term. Then $\text{ord}(f)$ is equal to the order of any root of f in the multiplicative group $\mathbb{F}_{p^m}^*$, where \mathbb{F}_{p^m} denotes the field with p^m elements. Let $f(x)$ be a monic irreducible polynomial in $\mathbb{F}_p[x]$ of degree m and order e , and let $t \geq 2$ be an integer whose prime factors divide e but not $(p^m - 1)/e$. Furthermore if $t \equiv 0 \pmod 4$ then $p^m \equiv 1 \pmod 4$. Then $f(x^t)$ is a monic irreducible polynomial in $\mathbb{F}_p[x]$ of degree mt and order et (see [12, Theorem 3.35]). The following lemma is probably well-known.

LEMMA 2.3. Let \mathbb{N} be the set of positive integers. There exists an injective mapping ω from \mathbb{N} into itself and a monic irreducible polynomial $\pi(x)$ in $\mathbb{F}_p[x]$ such that $\pi(x^{n\omega})$ is a monic irreducible polynomial in $\mathbb{F}_p[x]$ for all $n \geq 1$.

Proof. Let $p = 2$. Then $x^{2^{3n}} + x^{3^n} + 1$ is irreducible in $\mathbb{F}_2[x]$ for all $n \geq 1$ (see [12, Chapter 3, page 146]). Thus we may assume that p is an odd prime. Let q be an odd prime divisor of $p - 1$. Then there exists $a \in \{2, \dots, p - 1\}$ such that $a^{\frac{p-1}{q}} \neq 1$. Since $x^q - a$ has no root in \mathbb{F}_p , we obtain $x^{q^n} - a$ is irreducible in $\mathbb{F}_p[x]$ for all $n \geq 1$ (see [12, Theorem 3.75 and page 145]). Thus we may assume that p has the form $1 + 2^r$, with $r \geq 1$, and so p is a Fermat prime and $r = 2^\beta$, with $\beta \in \{0, 1, 2, \dots\}$. Let $\beta = 0$. Since $x^3 - x - 1$ is irreducible in $\mathbb{F}_3[x]$ and $\text{ord}(x^3 - x - 1) = 13$, we obtain $x^{3 \cdot 13^n} - x^{13^n} - 1$ is irreducible in $\mathbb{F}_3[x]$ for all $n \geq 1$. Finally, we assume that $\beta \geq 1$. Recall that $\left(\frac{3}{p}\right) = 1$ if and only if $p \equiv 1, -1 \pmod{12}$ (see [15, page 139]). Since $p \equiv 5 \pmod{12}$, we obtain $\left(\frac{3}{p}\right) = -1$ and so $x^2 - 3$ is irreducible in $\mathbb{F}_p[x]$. Since $\text{ord}(x^2 - 3) = 2^{r+1}$ and $\frac{p^2-1}{2^{r+1}} = 1 + 2^{r-1}$, we obtain $x^{2^n} - 3$ is irreducible in $\mathbb{F}_p[x]$ for all $n \geq 1$ (see, also, [12, Theorem 3.75]). Therefore there exists an irreducible polynomial $\pi(x)$ in $\mathbb{F}_p[x]$ and an injective mapping ω from \mathbb{N} into itself such that $\pi(x^{n\omega})$ is irreducible for all n .

REMARK 2.4. It is well-known that, for a prime power q , the number of monic irreducible polynomials of degree n over \mathbb{F}_q is given by $\frac{1}{n} \sum_{d|n} \mu\left(\frac{n}{d}\right) q^d$. Thus there are infinitely many monic irreducible polynomials over \mathbb{F}_q of different degree. The proof of Lemma 2.3 is needed in Section 4 for constructing non-tame automorphisms of $F_3(\mathfrak{A}_p)$.

It is elementary to show that if R is a principal ideal domain (PID), which is not a field, and $a \in R \setminus \{0\}$ is a non-unit of R then, $\mathcal{L}_S(R)$ is a PID, where $S = \{a^n : n \geq 0\}$.

LEMMA 2.5. Let π be a monic irreducible polynomial of a positive degree with a non-zero constant term in $\mathbb{F}_p[s_1]$, J the ideal of $\mathbb{F}_p[s_1]$ generated by π and I the ideal of $\mathbb{F}_p A_1$ generated by π . Then $\mathbb{F}_p[s_1]/J$ is isomorphic to $\mathbb{F}_p A_1/I$.

Proof. Let $\pi = s_1^n + c_{n-1}s_1^{n-1} + \dots + c_1s_1 + c_0$, with $n \geq 1$ and $c_0 \neq 0$, be a monic irreducible element in $\mathbb{F}_p[s]$. Let I be the ideal in \mathbb{F}_pA_1 generated by π . Let $E = I \cap \mathbb{F}_p[s]$. We claim that $E = J$. Since E is an ideal in $\mathbb{F}_p[s_1]$ and $\mathbb{F}_p[s_1]$ is a PID, E is generated by an element d , say. Since $\pi \in \mathbb{F}_p[s_1]$, $\pi \in E$ and so $J \subseteq E$. To show that $E \subseteq J$ it is enough to prove that $d \in J$. But $d \in I$ and so $d = \pi u$ for some $u \in \mathbb{F}_pA_1$. Write $u = s_1^m v$, where $v = a_0 + a_1s_1 + \dots + a_\mu s_1^\mu \in \mathbb{F}_p[s_1]$ and $a_0 \neq 0$. Suppose that $m < 0$. Since $c_0a_0 \neq 0$, we obtain a contradiction. Thus $m \geq 0$ and $d \in J$. Therefore $E = J$. Observe that π is irreducible in \mathbb{F}_pA_1 . Since both $\mathbb{F}_p[s_1]$ and \mathbb{F}_pA_1 are PID, we obtain $\mathbb{F}_p[s_1]/J$ and \mathbb{F}_pA_1/I are fields. Let δ be the natural ring homomorphism from $\mathbb{F}_p[s_1]$ into \mathbb{F}_pA_1/I defined by $v\delta = v + I$ for all $v \in \mathbb{F}_p[s_1]$. It is easily verified that $\ker \delta = J$ and so δ induces a ring monomorphism $\bar{\delta}$ from $\mathbb{F}_p[s_1]/J$ into \mathbb{F}_pA_1/I such that $(v + J)\bar{\delta} = v\delta$. We claim that $\bar{\delta}$ is surjective. Let $u = w + I$, where $w \in \mathbb{F}_pA_1$, and write $w = s_1^m w_1$ where $w_1 \in \mathbb{F}_p[s_1]$. If $m \geq 0$, we obtain $(w + J)\bar{\delta} = u$. Suppose that $m < 0$. Then, since $(\mathbb{F}_p[s_1] + I)/I$ is a field, there exists $x \in \mathbb{F}_p[s_1]$ such that $(s_1 + I)(x + I) = 1 + I$. Also, $(s_1 + I)(s_1^{-1} + I) = 1 + I$. Therefore $x + I = s_1^{-1} + I$ and so, $(x^{-m}w_1 + J)\bar{\delta} = x^{-m}w_1 + I = s_1^m w_1 + I = w + I = u$. Thus $\bar{\delta}$ is surjective and so, $\mathbb{F}_p[s_1]/J$ is isomorphic to \mathbb{F}_pA_1/I .

3. A method. We denote by Ω a free left \mathbb{F}_pA_3 -module with a basis $\{t_1, t_2, t_3\}$. The set $A_3 \times \Omega$ becomes a group by defining a multiplication

$$(\bar{u}, m_1)(\bar{v}, m_2) = (\bar{u}\bar{v}, m_1 + \bar{u}m_2) = (\overline{uv}, m_1 + \bar{u}m_2)$$

for all $\bar{u}, \bar{v} \in A_3$, where $\bar{u} = uF'_3$ and $\bar{v} = vF'_3$, with $u, v \in F_3$, and $m_1, m_2 \in \Omega$. Let χ be the mapping from F_3 into $A_3 \times \Omega$ defined by $f\chi = (\bar{f}, d_1(u)t_1 + d_2(u)t_2 + d_3(u)t_3)$, with $u = f\mathfrak{A}_p(F_3)$. It is easily verified that χ is a group homomorphism. But $\ker \chi = \mathfrak{A}_p(F_3)$ (see, for example, [11, Proposition 1]). Hence M_3 is embedded into $A_3 \times \Omega$ by χ satisfying the conditions $x_i\chi = (s_i, t_i)$, $i = 1, 2, 3$. The proof of the following result is elementary.

LEMMA 3.1. For $u \in M'_3$,

$$u\chi = (1, d_1(u)t_1 + d_2(u)t_2 + d_3(u)t_3).$$

Let ϕ be an IA-automorphism of M_3 satisfying the conditions $x_i\phi = x_iu_i$, where $u_i \in M'_3$, $i = 1, 2, 3$, and let $\hat{\phi} = \chi^{-1}\phi\chi$. It is easily verified that $\hat{\phi}$ is an IA-automorphism of $M_3\chi$. Thus, for $i \in \{1, 2, 3\}$,

$$\begin{aligned} (s_i, t_i)\hat{\phi} &= (x_i\phi)\chi = (x_i\chi)(u_i\chi) \\ \text{(Lemma 3.1)} &= (s_i, t_i)(1, d_1(u_i)t_1 + d_2(u_i)t_2 + d_3(u_i)t_3) \\ \text{(Equation (9))} &= (s_i, d_1(x_iu_i)t_1 + d_2(x_iu_i)t_2 + d_3(x_iu_i)t_3) \\ &= (s_i, a_{i1}t_1 + a_{i2}t_2 + a_{i3}t_3), \end{aligned} \tag{14}$$

where $a_{ij} = d_j(x_iu_i)$ for $i, j \in \{1, 2, 3\}$, and

$$a_{i1}(s_1 - 1) + a_{i2}(s_2 - 1) + a_{i3}(s_3 - 1) = s_i - 1 \tag{15}$$

for $i = 1, 2, 3$. The equation (15) is just a restatement of Fox's fundamental formula, as stated in equation (12). Notice that equations (14) give $J_\phi = (a_{ij})$. The mapping ζ from $\text{IA}(M_3)$ into $\text{GL}_3(\mathbb{F}_pA_3)$ given by $\phi \mapsto J_\phi = (a_{ij})$ is a faithful representation of $\text{IA}(M_3)$. (Indeed, write $\gamma_{ij} = d_j(x_i\phi\psi)$, $a_{ij} = d_j(x_i\phi)$ and $b_{ij} = d_j(x_i\psi)$. By Lemma 2.1

and equations (9), (10), (13) and (11), we get $\gamma_{ij} = \sum_{\kappa=1}^3 a_{i\kappa} b_{\kappa j}$ for $i, j = 1, 2, 3$ and so, ζ is a group homomorphism. If $x_i\phi = x_i u_i$, with $u_i \in M_3^i$, $i = 1, 2, 3$, and $d_j(x_i\phi) = d_j(x_i u_i) = \delta_{ij}$, where δ_{ij} is the Kronecker's delta, then $d_j(u_i) = 0$ for $i, j = 1, 2, 3$. Hence $u_i \in \mathfrak{V}_p(F_3)$ with $i = 1, 2, 3$ (see, for example, [11, Proposition 1]). Therefore ζ is a group monomorphism.) Suppose that $(a_{ij}) \in \text{Im}\zeta$. Thus the determinant of (a_{ij}) is a unit in $\mathbb{F}_p A_3$, and its rows satisfy equations (15). Since the units of $\mathbb{F}_p A_3$ are of the form $q a$ (see, for example, [18, Lemma 3.2, page 55]), where $q \in \mathbb{F}_p \setminus \{0\}$ and $a \in A_3$, we obtain

$$\det(a_{ij}) = q s_1^{\mu_1} s_2^{\mu_2} s_3^{\mu_3} \tag{16}$$

where $q \in \mathbb{F}_p \setminus \{0\}$ and $\mu_1, \mu_2, \mu_3 \in \mathbb{Z}$. For the next few lines, for each $u \in \mathbb{F}_p A_3$, we write \widehat{u} for the element of \mathbb{F}_p obtained from u substituting s_1, s_2, s_3 by 1. For $s_2 = s_3 = 1$, equations (15) give $a_{11}(s_1^{\pm 1})(s_1 - 1) = s_1 - 1$ and $a_{i1}(s_1^{\pm 1})(s_1 - 1) = 0$ with $i = 2, 3$. Since $\mathbb{F}_p A_3$ is an integral domain, we get $a_{11}(s_1^{\pm 1}) = 1$ and $a_{21}(s_1^{\pm 1}) = a_{31}(s_1^{\pm 1}) = 0$. Thus $\widehat{a}_{11} = 1$ and $\widehat{a}_{21} = \widehat{a}_{31} = 0$. Similarly, $\widehat{a}_{22} = \widehat{a}_{33} = 1$ and $\widehat{a}_{ij} = 0$ for $i \neq j$. Thus equation (16) (for $s_1 = s_2 = s_3 = 1$) gives $q = 1$. Therefore, for an element $(a_{ij}) \in \text{Im}\zeta$, its determinant is equal to $s_1^{\mu_1} s_2^{\mu_2} s_3^{\mu_3}$, with $\mu_1, \mu_2, \mu_3 \in \mathbb{Z}$, and its rows satisfy equations (15). For the converse, the proof of Lemma 1 in [2] carries over with minor changes apart from some obvious misprints. We note that the aforementioned equivalent statements are stated in [3, Proposition 2]. This is the *Bachmuth representation* of $\text{IA}(M_3)$.

We write \mathcal{A} for the image of $\text{IA}(M_3)$ via ζ . Let S be the multiplicative monoid of $\mathbb{F}_p A_3$ generated by $s_1 - 1$. Since $\mathbb{F}_p A_3$ is a UFD and $S \subseteq \mathbb{F}_p A_3 \setminus \{0\}$, we obtain $\mathcal{L}_S(\mathbb{F}_p A_3)$ is a UFD. We conjugate \mathcal{A} by the element

$$(c_{ij}) = \begin{pmatrix} s_1 - 1 & 0 & 0 \\ s_2 - 1 & (s_1 - 1)^{-1} & 0 \\ s_3 - 1 & 0 & 1 \end{pmatrix}.$$

Using equation (15) it is easy to verify that

$$(c_{ij})^{-1}(a_{ij})(c_{ij}) = \begin{pmatrix} 1 & a_{12}(s_1 - 1)^{-2} & a_{13}(s_1 - 1)^{-1} \\ 0 & b_{11} & b_{12} \\ 0 & b_{21} & b_{22} \end{pmatrix}, \tag{17}$$

where $b_{11} = [a_{22}(s_1 - 1) - a_{12}(s_2 - 1)](s_1 - 1)^{-1}$, $b_{12} = a_{23}(s_1 - 1) - a_{13}(s_2 - 1)$, $b_{21} = [a_{32}(s_1 - 1) - a_{12}(s_3 - 1)](s_1 - 1)^{-2}$ and $b_{22} = [a_{33}(s_1 - 1) - a_{13}(s_3 - 1)](s_1 - 1)^{-1}$. It is easily verified that the map η from \mathcal{A} into $\text{GL}_2(\mathcal{L}_S(\mathbb{F}_p A_3))$ defined by $(a_{ij})\eta = (b_{k\ell})$ is a group homomorphism.

Let H be a finitely generated subgroup of $\text{IA}(M_3)$ containing $T \cap \text{IA}(M_3)$. Let ρ be the mapping from H into A_3 defined by $\phi\rho = \det J_\phi = s_1^{\mu_1} s_2^{\mu_2} s_3^{\mu_3}$, where $\mu_1, \mu_2, \mu_3 \in \mathbb{Z}$. Since $T \cap \text{IA}(M_3) \subseteq H$, it is easily verified that ρ is an epimorphism. Thus we obtain the following short exact sequence

$$1 \rightarrow N \rightarrow H \xrightarrow{\rho} A_3 \rightarrow 1,$$

where N denotes the kernel of ρ . Since H/N is finitely presented and H is finitely generated, we obtain from a result of Hall [10, page 421] N is finitely generated as a group on which H acts by conjugation. The proof of the following result is based on some ideas given in the proof of Lemma 5 in [5].

LEMMA 3.2. Let \mathcal{H} and \mathcal{N} be the images of H and N , respectively, in $\text{GL}_2(\mathcal{L}_S(\mathbb{F}_p A_3))$ via the group homomorphism η . Let $(a_{ij1}), \dots, (a_{ijr})$ be a generating set of \mathcal{H} , and let $(b_{ij1}), \dots, (b_{ijs})$ be a generating set of \mathcal{N} as a group on which \mathcal{H} acts by conjugation. Then there exist irreducible elements $\alpha_1, \dots, \alpha_q \in \mathbb{F}_p A_3$ such that if P is the multiplicative monoid generated by $\mathbb{F}_p \setminus \{0\}$, $\{s_1^{\pm 1}, s_2^{\pm 1}, s_3^{\pm 1}\}$, $s_1 - 1$ and $\alpha_j, j = 1, \dots, q$, then $(d_{ij}) \in E_2(\mathcal{L}_P(\mathbb{F}_p A_3))$ for all $(d_{ij}) \in \mathcal{N}$.

Proof. From (17), $a_{12k}, b_{12\ell} \in \mathbb{F}_p A_3$ for $k = 1, \dots, r$ and $\ell = 1, \dots, s$. Let $\alpha_1, \dots, \alpha_q$ be the irreducible elements in $\mathbb{F}_p A_3$ which appear as a factor of a_{12k} or $b_{12\ell}, k = 1, \dots, r, \ell = 1, \dots, s$. Let P be the multiplicative monoid generated by $\mathbb{F}_p \setminus \{0\}, \{s_1^{\pm 1}, s_2^{\pm 1}, s_3^{\pm 1}\}, s_1 - 1$ and $\alpha_j, j = 1, \dots, q$. Since P is a multiplicative closed set not containing the zero element and $\mathbb{F}_p A_3$ is a UFD, we obtain $\mathcal{L}_P(\mathbb{F}_p A_3)$ is a UFD and

$$\mathbb{F}_p A_3 \subseteq \mathcal{L}_S(\mathbb{F}_p A_3) \subseteq \mathcal{L}_P(\mathbb{F}_p A_3) \subseteq \mathcal{Q}.$$

Let $(d_{ij}) \in \mathcal{H}$. We claim that $(d_{ij}) \in \text{GE}_2(\mathcal{L}_P(\mathbb{F}_p A_3))$. By (17), $a_{11k}a_{22k} - a_{12k}a_{21k}$ is a unit in $\mathcal{L}_P(\mathbb{F}_p A_3)$ for $k = 1, \dots, r$. Fix $k, k = 1, \dots, r$, and write e_{ij} for $a_{ijk}, i, j = 1, 2$. Let $e_{12} = 0$. Then

$$\begin{pmatrix} e_{11} & 0 \\ e_{21} & e_{22} \end{pmatrix} = \begin{pmatrix} e_{11} & 0 \\ 0 & e_{11}^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ e_{11}e_{21} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e_{11}e_{22} \end{pmatrix} \in \text{GE}_2(\mathcal{L}_P(\mathbb{F}_p A_3)).$$

Thus we may assume that $e_{12} \neq 0$. Since $e_{12} \in P$, we obtain $e_{21} = e_{12}^{-1}(e_{11}e_{22} - u)$, where $u = e_{11}e_{22} - e_{21}e_{12}$. Then

$$\begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix} = \begin{pmatrix} e_{12} & 0 \\ 0 & e_{12}^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ e_{12}e_{22} & 1 \end{pmatrix} \begin{pmatrix} u^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -e_{11}e_{12}^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & u \\ -u & 0 \end{pmatrix} \in \text{GE}_2(\mathcal{L}_P(\mathbb{F}_p A_3)).$$

Thus (d_{ij}) is a product of the (a_{ijk}) , whence $(d_{ij}) \in \text{GE}_2(\mathcal{L}_P(\mathbb{F}_p A_3))$. For the next few lines, we write $E(x)$ for the matrix $\begin{pmatrix} x & 1 \\ -1 & 0 \end{pmatrix}$ for $x \in \mathcal{L}_P(\mathbb{F}_p A_3)$. Note that, for invertible element x ,

$$\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} = E(0)^{-1}E(x^{-1})E(x)E(x^{-1})E(0)^{-1},$$

$$E(0) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and

$$E(x) = \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} E(0).$$

Thus $\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \in E_2(\mathcal{L}_S(\mathbb{F}_p A_3))$. Applying similar arguments as above, we obtain $(b_{ij\ell}) \in E_2(\mathcal{L}_P(\mathbb{F}_p A_3))$ for all $\ell = 1, \dots, s$. The group \mathcal{N} is generated as a group by the elements

$$(a_{ij})^{-1}(b_{ij\ell})(a_{ij}),$$

where $\ell = 1, \dots, s$, and $(a_{ij}) \in \mathcal{H}$. Since $E_2(\mathcal{L}_P(\mathbb{F}_p A_3))$ is a normal subgroup of $GE_2(\mathcal{L}_P(\mathbb{F}_p A_3))$, we obtain $(a_{ij})^{-1}(b_{ij\ell})(a_{ij}) \in E_2(\mathcal{L}_P(\mathbb{F}_p A_3))$. Thus $(d_{ij}) \in E_2(\mathcal{L}_P(\mathbb{F}_p A_3))$ if $(d_{ij}) \in \mathcal{N}$.

We need some notation and auxiliary lemmas before we prove a result (namely, Lemma 3.6) which is the key ingredient of our method of constructing non-tame automorphisms of $F_3(\mathfrak{A}_p)$.

First we recall some elementary facts about unique factorization domains (UFD) (see, for example, [1, Chapter 2]). Let R be a UFD. Two elements u and v in R are said to be associates if $u = cv$, where c is a unit. Define a relation \equiv on R as follows : $u \equiv v$ if u and v are associates. It is an equivalence relation on R . Denote by $[u]$ the equivalence class of u . An element $a \in R$ is irreducible if and only if it is prime. For a non-empty subset X of $R \setminus \{0\}$ we write $\text{Irr}(X)$ for the set of equivalence classes $[u]$, where u is an irreducible element of R which appears in the factorization of some element of X . Let $u, v \in R \setminus \{0\}$. If $v = ua$ for some $a \in R$ we say u divides v (written $u \mid v$); otherwise we write $u \nmid v$. Any set Y of nonzero elements of R has a greatest common divisor (gcd). Note that any two gcds of Y are associates. If 1 is a gcd of Y , then we say that the set Y is relatively prime.

Recall from the proof of Lemma 3.2 that P is the multiplicative monoid generated by $\mathbb{F}_p \setminus \{0\}$, $\{s_1^{\pm 1}, s_2^{\pm 1}, s_3^{\pm 1}\}$, $s_1 - 1$, and $\alpha_j, j = 1, \dots, q$. A typical element of $P \setminus \{1\}$ has the form

$$d a (s_1 - 1)^n \alpha_{j_1} \cdots \alpha_{j_\mu},$$

where $d \in \mathbb{F}_p \setminus \{0\}$, $a \in A_3$, n a non-negative integer, and $\alpha_{j_k} \in \{\alpha_1, \dots, \alpha_q\}$, $k = 1, \dots, \mu$. Each $\alpha_j, j = 1, \dots, q$, has a unique expression as an element in $\mathbb{F}_p A_3$

$$\alpha_j = s_3^{m_j} \left(\sum_{i_j=m_j}^{n_j} u_{i_j} s_3^{i_j - m_j} \right),$$

where $m_j \leq n_j$, $u_{i_j} \in \mathbb{F}_p A_2$, $i_j = m_j, \dots, n_j$, $u_{m_j} \neq 0$ and $u_{n_j} \neq 0$. Write $h_j = s_3^{-m_j} \alpha_j$ for $j = 1, \dots, q$. Let P_{s_3} be the submonoid of P generated by $\mathbb{F}_p \setminus \{0\}$, $\{s_1^{\pm 1}, s_2^{\pm 1}\}$, $s_1 - 1$ and h_1, \dots, h_q . Thus an element of $P_{s_3} \setminus \{1\}$ has the form

$$d h (s_1 - 1)^n h_{j_1} \cdots h_{j_\mu},$$

where $d \in \mathbb{F}_p \setminus \{0\}$, $h \in A_2$, n a non-negative integer, and $h_{j_k} \in \{h_1, \dots, h_q\}$, $k = 1, \dots, \mu$. Note that $P_{s_3} \subseteq \mathbb{F}_p A_2[s_3]$ and $\mathcal{L}_P(\mathbb{F}_p A_3) \cap \mathcal{O} = \mathcal{L}_{P_{s_3}}(\mathbb{F}_p A_2[s_3])$. Let Ψ be the ring epimorphism from $\mathbb{F}_p A_2[s_3]$ onto $\mathbb{F}_p A_2$ satisfying the conditions $u\Psi = u$ for all $u \in \mathbb{F}_p A_2$ and $s_3\Psi = 0$. Thus $P_{s_3}\Psi$ is the monoid generated by $\mathbb{F}_p \setminus \{0\}$, $\{s_1^{\pm 1}, s_2^{\pm 1}\}$, $s_1 - 1$ and u_{m_1}, \dots, u_{m_q} . An element of $P_{s_3}\Psi \setminus \{1\}$ is written as

$$d h (s_1 - 1)^n u_{j_1} \cdots u_{j_\mu},$$

where $d \in \mathbb{F}_p \setminus \{0\}$, $h \in A_2$, n is a non-negative integer, and $u_{j_1}, \dots, u_{j_\mu} \in \{u_{m_1}, \dots, u_{m_q}\}$. Hence $\text{Irr}(P_{s_3}\Psi)$ is finite. Since $0 \notin P_{s_3}\Psi$, the epimorphism Ψ induces a ring epimorphism $\tilde{\Psi}$ from $\mathcal{L}_{P_{s_3}}(\mathbb{F}_p A_2[s_3])$ onto $\mathcal{L}_{P_{s_3}\Psi}(\mathbb{F}_p A_2)$ such that $\frac{f}{t}\tilde{\Psi} = \frac{f\Psi}{t\Psi}$.

LEMMA 3.3. *Let $\pi(s_1)$ and ω be as in the statement of Lemma 2.3. Then, for infinitely many n , $\pi(s_1^{n\omega})$ is not invertible in $\mathcal{L}_{P_{s_3}\Psi}(\mathbb{F}_p A_2)$.*

Proof. Let $\pi(s_1)$ and ω be as in the statement of Lemma 2.3. Then $\pi(s_1^{n\omega})$ is an irreducible polynomial in $\mathbb{F}_p A_2$ for all $n \geq 1$. Suppose that $\pi(s_1^{n\omega})$ is invertible in $\mathcal{L}_{P_{s_3}\Psi}(\mathbb{F}_p A_2)$ for some n . Then there exists $u \in \mathcal{L}_{P_{s_3}\Psi}(\mathbb{F}_p A_2)$ such that $\pi(s_1^{n\omega})u = 1$. Write $u = \frac{v}{t}$ where $v \in \mathbb{F}_p A_2$ and $t \in P_{s_3}\Psi$. Thus $\pi(s_1^{n\omega})v = t$ in $\mathbb{F}_p A_2$. Since $\mathbb{F}_p A_2$ is a UFD, we obtain there exists t_1 an irreducible element in $\mathbb{F}_p A_2$ which appears in the factorization of t such that $\pi(s_1^{n\omega}) \in [t_1]$. Observe that if $\pi(s_1^{m\omega}) \in [t_1]$ then $\pi(s_1^{m\omega})$ does not belong to $[t_1]$ for $m \neq n$. Indeed, if $\pi(s_1^{m\omega}) \in [t_1]$ then $\pi(s_1^{m\omega}) = \pi(s_1^{n\omega})c$, where c is a unit in $\mathbb{F}_p A_2$. Since the only units in $\mathbb{F}_p A_2$ are the elements of $\mathbb{F}_p \setminus \{0\}$ and the elements of A_2 , we obtain a contradiction. Thus $\pi(s_1^{m\omega})$ does not belong to $[t_1]$ for $m \neq n$. Since $\text{Irr}(P_{s_3}\Psi)$ is finite whereas $\pi(s_1^{n\omega})$ is irreducible for all $n \geq 1$, we obtain $\pi(s_1^{n\omega})$ is not invertible in $\mathcal{L}_{P_{s_3}\Psi}(\mathbb{F}_p A_2)$ for infinitely many n .

REMARK 3.4. Let π be a monic irreducible polynomial in $\mathbb{F}_p[s_1]$ subject to π is not invertible in $\mathcal{L}_{P_{s_3}\Psi}(\mathbb{F}_p A_2)$. Then $\pi \nmid x$ for all $x \in P_{s_3}\Psi$. Indeed, suppose that there exists $x \in P_{s_3}\Psi$ such that $\pi \mid x$. Thus $x = \pi x'$ for some $x' \in \mathbb{F}_p A_2$. Since $x \in P_{s_3}\Psi$, we obtain x is invertible in $\mathcal{L}_{P_{s_3}\Psi}(\mathbb{F}_p A_2)$. Therefore π is invertible in $\mathcal{L}_{P_{s_3}\Psi}(\mathbb{F}_p A_2)$ which is a contradiction. By Remark 2.4, there are infinitely many irreducible polynomials of different degrees in $\mathbb{F}_p[s_1]$. Thus there are infinitely many irreducible polynomials in $\mathbb{F}_p A_2$. The arguments given in the proof of Lemma 3.3 guarantee that there are infinitely many irreducible elements in $\mathbb{F}_p A_2$ which are not invertible in $\mathcal{L}_{P_{s_3}\Psi}(\mathbb{F}_p A_2)$.

By the proof of Lemma 3.3 (and Remark 3.4), we may choose a monic irreducible polynomial π of degree m in $\mathbb{F}_p[s_1]$ subject to $\pi \nmid x$ for all $x \in P_{s_3}\Psi$, and there exists an odd prime divisor q of $p^m - 1$. Let I be the ideal of $\mathbb{F}_p A_1$ generated by π . By Lemma 2.5, $\mathbb{F}_p A_1/I$ is a field of p^m elements.

From now on, we fix π and write K for $\mathbb{F}_p A_1/I$. The natural mapping ϑ from $\mathbb{F}_p A_1$ onto K induces a ring epimorphism ϑ_1 from $\mathbb{F}_p A_2$ onto $K[s_2^{\pm 1}]$ in a natural way. Since $P_{s_3}\Psi$ is a multiplicative closed subset of $\mathbb{F}_p A_2$, we obtain $P_{s_3}\Psi\vartheta_1$ is a multiplicative closed subset of $K[s_2^{\pm 1}]$. Suppose that $0 \in P_{s_3}\Psi\vartheta_1$. Then there exists $v \in P_{s_3}\Psi$ such that $v\vartheta_1 = 0$. Since $v \in P_{s_3}\Psi$, we obtain v is invertible in $\mathcal{L}_{P_{s_3}\Psi}(\mathbb{F}_p A_2)$. Write $v = \sum v_\ell s_2^\ell$, with $v_\ell \in \mathbb{F}_p A_1$. By applying ϑ_1 , we obtain

$$v\vartheta_1 = \sum (v_\ell \vartheta) s_2^\ell = 0$$

and so, $v_\ell \in \ker \vartheta$ for all ℓ . Since $\ker \vartheta$ is the ideal in $\mathbb{F}_p A_1$ generated by π , we obtain π divides v_ℓ for all ℓ and so, π divides v in $\mathbb{F}_p A_2$ which is a contradiction by the choice of π . Therefore $0 \notin P_{s_3}\Psi\vartheta_1$ and so, $\mathcal{L}_{P_{s_3}\Psi\vartheta_1}(K[s_2^{\pm 1}]) \neq \{0\}$. Observe that $\text{Irr}(P_{s_3}\Psi\vartheta_1)$ is finite. The epimorphism ϑ_1 induces a ring epimorphism $\tilde{\vartheta}_1$ from $\mathcal{L}_{P_{s_3}\Psi}(\mathbb{F}_p A_2)$ onto $\mathcal{L}_{P_{s_3}\Psi\vartheta_1}(K[s_2^{\pm 1}])$ by defining $\frac{u}{t}\tilde{\vartheta}_1 = \frac{u\vartheta_1}{t\vartheta_1}$ for all $u \in \mathbb{F}_p A_2$ and $t \in P_{s_3}\Psi$.

Let b be an element of $K \setminus \{0\}$ such that $b^{\frac{p^m-1}{q}} \neq 1$. Since $s_2^q - b$ has no root in K , we obtain $s_2^{q^n} - b$ is irreducible in $K[s_2]$ for all $n \geq 1$ (see [12, Theorem 3.75 and page 145]). Since $\text{Irr}(P_{s_3}\Psi\vartheta_1)$ is finite whereas $s_2^{q^n} - b$ is irreducible in $K[s_2^{\pm 1}]$ for all $n \geq 1$, we obtain $s_2^{q^n} - b$ is not invertible in $\mathcal{L}_{P_{s_3}\Psi\vartheta_1}(K[s_2^{\pm 1}])$ for infinitely many n . Thus we obtain the following result.

LEMMA 3.5. *There exists $b \in K$ such that $s_2^{q^n} - b$ is irreducible in $K[s_2^{\pm 1}]$ for all $n \geq 1$. Furthermore, for infinitely many n , $s_2^{q^n} - b$ is not invertible in $\mathcal{L}_{P_{s_3}\Psi\vartheta_1}(K[s_2^{\pm 1}])$.*

Choose $s_2^{q^n} - b$ an irreducible element in $K[s_2]$ subject to $s_2^{q^n} - b$ is not invertible in $\mathcal{L}_{P_{s_3}\Psi}(K[s_2^{\pm 1}])$. Let c be an element of $\mathbb{F}_p A_1$ such that $c\vartheta = b$. Then $s_2^{q^n} - c$ is an irreducible element in $\mathbb{F}_p A_1[s_2]$. Hence $s_2^{q^n} - c$ is irreducible in $\mathbb{F}_p A_2$. It is easy to verify that $s_2^{q^n} - c$ is not invertible in $\mathcal{L}_{P_{s_3}\Psi}(\mathbb{F}_p A_2)$. Furthermore $s_2^{q^n} - c \nmid y$ for all $y \in P_{s_3}\Psi$, and $\pi, s_2^{q^n} - c$ are relatively prime elements in $\mathbb{F}_p A_2$.

Next we shall construct an element Δ of $SL_2(\mathcal{L}_P(\mathbb{F}_p A_3)) \setminus E_2(\mathcal{L}_P(\mathbb{F}_p A_3))$. The proof of the following result is based on some ideas given in the proof of Theorem C in [8].

LEMMA 3.6. *Let π be an irreducible element in $\mathbb{F}_p A_1$ subject to $\pi \nmid x$ for any element $x \in P_{s_3}\Psi$. Let $K = \mathbb{F}_p A_1/I$, where I is the ideal of $\mathbb{F}_p A_1$ generated by π . Let σ be an irreducible element in $\mathbb{F}_p A_2$ such that (i) π and σ are relatively prime in $\mathbb{F}_p A_2$, (ii) $\sigma \nmid x$ for any element $x \in P_{s_3}\Psi$ and (iii) $\sigma\vartheta_1$ is not invertible in $\mathcal{L}_{P_{s_3}\Psi}(K[s_2^{\pm 1}])$. Then, for $t \in \mathcal{L}_{P_{s_3}}(\mathbb{F}_p A_2[s_3])$ with $tv = 0$ and $t\tilde{\Psi} \neq 0$, the matrix*

$$\Delta = \begin{pmatrix} 1 + \sigma\pi t^2 s_3^{-1} & -\sigma^2 t^2 s_3^{-1} \\ \pi^2 t^2 s_3^{-1} & 1 - \sigma\pi t^2 s_3^{-1} \end{pmatrix}$$

is an element of $SL_2(\mathcal{L}_P(\mathbb{F}_p A_3)) \setminus E_2(\mathcal{L}_P(\mathbb{F}_p A_3))$.

Proof. Throughout the proof, we write X for $\begin{pmatrix} 1 & 0 \\ 0 & s_3 \end{pmatrix}$. By Lemma 2.2, $SL_2(Q) = SL_2(\mathcal{O}) *_D SL_2(\mathcal{O})^X$, where $D = SL_2(\mathcal{O}) \cap SL_2(\mathcal{O})^X$. Clearly $\Delta \in SL_2(Q)$. Now,

$$\Delta = \begin{pmatrix} 1 & \sigma/\pi \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \pi^2 t^2 s_3^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & -\sigma/\pi \\ 0 & 1 \end{pmatrix}.$$

It is easily verified that $\begin{pmatrix} 1 & \pm\sigma/\pi \\ 0 & 1 \end{pmatrix} \in SL_2(\mathcal{O}) \setminus D$ and $\begin{pmatrix} 1 & 0 \\ \pi^2 t^2 s_3^{-1} & 1 \end{pmatrix} \in SL_2(\mathcal{O})^X \setminus D$. The normal form theorem for the free products with amalgamation (see [13, Corollary 4.4.2]) implies that if $\Delta = g_1 g_2 \cdots g_r$, where the g_i are alternately in $SL_2(\mathcal{O}) \setminus D$ and $SL_2(\mathcal{O})^X \setminus D$, then $r = 3$, $g_1, g_3 \in SL_2(\mathcal{O}) \setminus D$, and $g_2 \in SL_2(\mathcal{O})^X \setminus D$. Note that π is not invertible in $\mathcal{L}_P(\mathbb{F}_p A_3)$. Indeed, let $w \in \mathcal{L}_P(\mathbb{F}_p A_3)$ such that $\pi w = 1$. Write $w = s_3^{uv} \tilde{v}$ for some $u \in \mathcal{L}_P(\mathbb{F}_p A_2[s_3])$ and $v \in P_{s_3}$. Since $\pi v = 0$, we obtain $wv = 0$. By applying $\tilde{\Psi}$, we obtain π is invertible in $\mathcal{L}_{P_{s_3}\Psi}(\mathbb{F}_p A_2)$ which is a contradiction by our hypothesis. Let $B = SL_2(\mathcal{L}_P(\mathbb{F}_p A_3)) \cap SL_2(\mathcal{O})$, $\Gamma = SL_2(\mathcal{L}_P(\mathbb{F}_p A_3)) \cap SL_2(\mathcal{O})^X$ and $G = \langle B, \Gamma \rangle$. We claim that $E_2(\mathcal{L}_P(\mathbb{F}_p A_3)) \leq G$. But

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in B,$$

and so

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -f & 1 \end{pmatrix},$$

for all $f \in \mathcal{L}_P(\mathbb{F}_p A_3)$. To show our claim, it is enough to prove that

$$\begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix} \in G$$

for all $f \in \mathcal{L}_p(\mathbb{F}_p A_3)$. Furthermore

$$\begin{pmatrix} 1 & 0 \\ s_3^{-1} & 1 \end{pmatrix} \in \Gamma \quad \text{and} \quad \begin{pmatrix} 1 & -s_3 \\ 0 & 1 \end{pmatrix} \in B,$$

and so

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ s_3^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & -s_3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ s_3^{-1} & 1 \end{pmatrix} = \begin{pmatrix} -s_3^{-1} & 0 \\ 0 & -s_3 \end{pmatrix} \in G.$$

Let $f \in \mathcal{L}_p(\mathbb{F}_p A_3)$ and let r be a positive integer such that $s_3^{2r}f \in \mathcal{L}_p(\mathbb{F}_p A_2[s_3])$. Since

$$\begin{pmatrix} 1 & s_3^{2r}f \\ 0 & 1 \end{pmatrix} \in G$$

we obtain

$$\begin{pmatrix} -s_3^{-r} & 0 \\ 0 & -s_3^r \end{pmatrix} \begin{pmatrix} 1 & s_3^{2r}f \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -s_3^r & 0 \\ 0 & -s_3^{-r} \end{pmatrix} = \begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix} \in G.$$

Thus $E_2(\mathcal{L}_p(\mathbb{F}_p A_3)) \leq G$. Suppose that $\Delta \in E_2(\mathcal{L}_p(\mathbb{F}_p A_3))$. Note that $B \cap D = \Gamma \cap D$. Since $E_2(\mathcal{L}_p(\mathbb{F}_p A_3)) \leq G$, we may write $\Delta = g_1 g_2 \cdots g_r$ where the g_i are alternately in B and Γ , and no g_i lies in D . Thus by the normal form theorem for free products with amalgamation, we may write

$$\Delta = \begin{pmatrix} d & e \\ f & g \end{pmatrix} \begin{pmatrix} h & is_3 \\ js_3^{-1} & k \end{pmatrix} \begin{pmatrix} \ell & m \\ n & q \end{pmatrix},$$

where $d, e, f, g, h, i, j, k, \ell, m, n, q \in \mathcal{L}_{p,3}(\mathbb{F}_p A_2[s_3])$. Making the calculations, we obtain

$$\Delta = \begin{pmatrix} dh\ell + ejs_3^{-1}\ell + is_3dn + ekn & dhm + ejs_3^{-1}m + is_3dq + ekq \\ fh\ell + gjs_3^{-1}\ell + fis_3n + gkn & fhm + gjs_3^{-1}m + fis_3q + gkq \end{pmatrix}.$$

Therefore

$$1 + \sigma\pi t^2 s_3^{-1} = dh\ell + ejs_3^{-1}\ell + is_3dn + ekn \tag{18}$$

and so, we obtain from (18)

$$\sigma\pi t^2 = (-1 + dh\ell + ekn)s_3 + ej\ell + is_3^2dn. \tag{19}$$

By applying $\tilde{\Psi}$ on (19), we obtain

$$\sigma\pi(t^2\tilde{\Psi}) = (e\tilde{\Psi})(j\tilde{\Psi})(\ell\tilde{\Psi}). \tag{20}$$

Similarly,

$$\pi^2(t^2\tilde{\Psi}) = (g\tilde{\Psi})(j\tilde{\Psi})(\ell\tilde{\Psi}). \tag{21}$$

Since $\mathcal{L}_{P_{s_3}\Psi}(\mathbb{F}_p A_2)$ is an integral domain, and by the choice of t , we obtain from (20) and (21)

$$\sigma(g\tilde{\Psi}) = \pi(e\tilde{\Psi}). \tag{22}$$

Write $g\tilde{\Psi} = \frac{u}{t_1}$ and $e\tilde{\Psi} = \frac{v}{t'_1}$, where $u, v \in \mathbb{F}_p A_2$ and $t_1, t'_1 \in P_{s_3}\Psi$. Thus (22) becomes

$$\sigma ut'_1 = vt_1\pi.$$

By our hypothesis, (i) and (ii), and since $\mathbb{F}_p A_2$ is a UFD, we obtain σ divides v and π divides u . Therefore $g\tilde{\Psi} = \pi e_1$ and $e\tilde{\Psi} = \sigma e_2$, where $e_1, e_2 \in \mathcal{L}_{P_{s_3}\Psi}(\mathbb{F}_p A_2)$. Since $dg - ef = 1$, we have

$$(d\tilde{\Psi})(g\tilde{\Psi}) - (e\tilde{\Psi})(f\tilde{\Psi}) = 1$$

and so

$$(d\tilde{\Psi})\pi e_1 - \sigma e_2(f\tilde{\Psi}) = 1. \tag{23}$$

By applying \tilde{v}_1 on (23), we obtain $\sigma\tilde{v}_1$ is invertible in $\mathcal{L}_{P_{s_3}\Psi\vartheta_1}(K[s_2^{\pm 1}])$ which is a contradiction by (iii). Therefore $\Delta \in \text{SL}_2(\mathcal{L}_P(\mathbb{F}_p A_3)) \setminus \text{E}_2(\mathcal{L}_P(\mathbb{F}_p A_3))$.

4. A construction of non-tame automorphisms. It is well-known (see, for instance, [13, Section 3.6, Theorem N4]) that $\text{IA}(F_3)$ is generated by the following automorphisms K_{ij} and K_{ijk} , where $i, j, k \in \{1, 2, 3\}$, satisfying the conditions

$$\begin{aligned} (f_i)K_{ij} &= f_j^{-1}f_j f_i && \text{for } i \neq j \\ (f_m)K_{ij} &= f_m && \text{if } m \neq i \end{aligned}$$

and

$$\begin{aligned} (f_i)K_{ijk} &= f_i[f_j, f_k] && \text{for } i \neq j < k \neq i \\ (f_m)K_{ijk} &= f_m && \text{if } m \neq i. \end{aligned}$$

The natural mapping from F_3 onto M_3 induces a group homomorphism, say α , from $\text{Aut}(F_3)$ into $\text{Aut}(M_3)$. We write τ for the restriction of α on $\text{IA}(F_3)$. It is easily verified that the image of τ is equal to $T \cap \text{IA}(M_3)$. It is generated by $\tau_{ij} = K_{ij}\tau$ for all $i \neq j$ and $\tau_{ijk} = K_{ijk}\tau$ for $i \neq j < k \neq i$. Thus $x_i\tau_{ij} = x_j^{-1}x_i x_j$ for $i \neq j$, $x_m\tau_{ij} = x_m$ if $m \neq i$, and $x_i\tau_{ijk} = x_i[x_j, x_k]$ for $i \neq j < k \neq i$ and $x_m\tau_{ijk} = x_m$ if $m \neq i$. Note that $\tau_{ijk}^{-1} = \tau_{ikj}$. Define $\mathcal{T} = \{\tau_{ij}, \tau_{ijk} : i \neq j < k \neq i\}$. Thus \mathcal{T} is a generating set of $T \cap \text{IA}(M_3)$. Recall that we have the following short exact sequence

$$1 \rightarrow \ker \rho_1 \rightarrow T \cap \text{IA}(M_3) \xrightarrow{\rho_1} A_3 \rightarrow 1,$$

where $\phi\rho_1 = \det J_\phi = s_1^{\mu_1} s_2^{\mu_2} s_3^{\mu_3}$, $\mu_1, \mu_2, \mu_3 \in \mathbb{Z}$. Note that $\tau_{31}^{-1}\tau_{21}, \tau_{32}^{-1}\tau_{12}, \tau_{23}^{-1}\tau_{13} \in \ker \rho_1$. Write $\mathcal{Q} = \{\tau_{123}, \tau_{213}, \tau_{312}, \tau_{31}^{-1}\tau_{21}, \tau_{32}^{-1}\tau_{12}, \tau_{23}^{-1}\tau_{13}, (\tau_{ij}, \tau_{\mu\nu}), (\tau_{\alpha\beta\gamma}, \tau_{\kappa\ell m}), (\tau_{\alpha\beta\gamma}, \tau_{ij}) : i \neq j, \mu \neq \nu, \alpha \neq \beta < \gamma \neq \alpha, \kappa \neq \ell < m \neq \kappa\}$.

LEMMA 4.1. *The kernel of ρ_1 is finitely generated by \mathcal{Q} as a group on which $T \cap \text{IA}(M_3)$ acts by conjugation.*

Proof. Let N_Q be the normal closure of Q in $T \cap \text{IA}(M_3)$, that is, the intersection of all normal subgroups of $T \cap \text{IA}(M_3)$ containing Q . It is easy to show that N_Q is generated by the set $\{\gamma^{-1}x\gamma : x \in Q, \gamma \in T \cap \text{IA}(M_3)\}$. We claim that $N_Q = \ker \rho_1$. Since $Q \subseteq \ker \rho_1$ and $\ker \rho_1$ is normal in $T \cap \text{IA}(M_3)$, it is enough to show that $\ker \rho_1 \subseteq N_Q$. For the next few lines, we set $E = T \cap \text{IA}(M_3)$. Since E/E' is finitely presented and E is finitely generated, we obtain E' is finitely generated as a group on which E acts by conjugation. In fact, E' is generated by the set $\{(\tau_{ij}, \tau_{\mu\nu}), (\tau_{\alpha\beta\gamma}, \tau_{\kappa\ell m}), (\tau_{\alpha\beta\gamma}, \tau_{ij}) : i \neq j, \mu \neq \nu, \alpha \neq \beta < \gamma \neq \alpha, \kappa \neq \ell < m \neq \kappa\}$ as a group on which E acts by conjugation. Thus $E' \subseteq N_Q$. Note that E/N_Q is an abelian group generated by 3 elements. Since

$$(E/N_Q)/(\ker \rho_1/N_Q) \cong E/\ker \rho_1$$

and $E/\ker \rho_1$ is a free abelian group of rank 3, we obtain $\ker \rho_1 \subseteq N_Q$. Therefore $\ker \rho_1 = N_Q$.

In the Appendix, we write down all $J_\phi = (a_{ij})$ for $\phi \in T \cup Q$ subject to $a_{13} \neq 0$ or $a_{23} \neq 0$. For simplicity, we write (J_ϕ, a_{13}, a_{23}) for $\phi \in T \cup Q$. Let P be the multiplicative monoid generated by $\mathbb{F}_p \setminus \{0\}, \{s_1^{\pm 1}, s_2^{\pm 1}, s_3^{\pm 1}\}, s_1 - 1, s_2 - 1, s_3 - 1$, and $\delta_1, \dots, \delta_5$ (see Appendix). Recall that for any element $u = \sum_i m_i r_i \in \mathbb{F}_p A_3$, with $m_i \in \mathbb{F}_p$ and $r_i \in A_3$, $u^* = \sum_i m_i r_i^{-1}$, and $(u^*)^* = u$. Furthermore, for $w \in M_3$ and $u \in \mathbb{F}_p A_3$, $d_j(w^u) = u^* d_j(w)$ for $j = 1, 2, 3$. Notice that $P_{s_3}\Psi$ is the multiplicative monoid generated by $\mathbb{F}_p \setminus \{0\}, \{s_1^{\pm 1}, s_2^{\pm 1}\}, s_1 - 1, s_2 - 1$.

THEOREM 4.2. *Let π be an irreducible element in $\mathbb{F}_p A_1$ subject to $\pi \nmid x$ for any element $x \in P_{s_3}\Psi$. Let $K = \mathbb{F}_p A_1/I$, where I is the ideal of $\mathbb{F}_p A_1$ generated by π . Let σ be an irreducible element in $\mathbb{F}_p A_2$ such that (i) π and σ are relatively prime in $\mathbb{F}_p A_2$, (ii) $\sigma \nmid x$ for any element $x \in P_{s_3}\Psi$ and (iii) $\sigma \tilde{\nu}_1$ is not invertible in $\mathcal{L}_{P_{s_3}\Psi \vartheta_1}(K[s_2^{\pm 1}])$. Then, for $t \in \mathcal{L}_{P_{s_3}}(\mathbb{F}_p A_2[s_3])$ with $tv = 0$ and $t\tilde{\Psi} \neq 0$, the automorphism ϕ of M_3 satisfying the conditions*

$$\begin{aligned} x_1\phi &= x_1 \\ x_2\phi &= x_2 [x_3, x_1]^{(s_1 s_2^{-1} \sigma^2)^*} [x_2, x_1]^{(-s_1 s_3^{-1} (s_1 - 1) \sigma \pi)^*} \\ x_3\phi &= x_3 [x_3, x_1]^{(s_1 s_3^{-1} (s_1 - 1) \sigma \pi)^*} [x_2, x_1]^{(-s_1 s_2 s_3^{-2} (s_1 - 1)^2 \pi^2)^*} \end{aligned}$$

is non-tame.

Proof. Since M_3 is a free group in the variety \mathfrak{A}_p with a free generating set $\{x_1, x_2, x_3\}$, ϕ extends uniquely to a group homomorphism of M_3 . Write $b_i = s_i - 1$ for $i = 1, 2, 3$. Using the equations (9), (10), (11) and (13), we calculate $d_j(x_i\phi)$, with $i, j \in \{1, 2, 3\}$, and so, the Jacobian matrix J_ϕ becomes

$$J_\phi = \begin{pmatrix} 1 & 0 & 0 \\ \sigma^2 b_3 s_3^{-1} - \sigma \pi b_1 b_2 s_3^{-1} & 1 + \sigma \pi b_1^2 s_3^{-1} & -\sigma^2 b_1 s_3^{-1} \\ -\pi^2 b_1^2 b_2 s_3^{-1} + \sigma \pi b_1 b_3 s_3^{-1} & \pi^2 b_1^3 s_3^{-1} & 1 - \sigma \pi b_1^2 s_3^{-1} \end{pmatrix}.$$

Since $\det J_\phi = 1$ and the rows of J_ϕ satisfy the conditions (15), we obtain $J_\phi \in \text{Im} \zeta$. Since ζ is a group monomorphism, we get $\phi \in \text{IA}(M_3)$. To get a contradiction, we assume that ϕ is tame. Since $\phi \in T \cap \text{IA}(M_3)$ and $\det J_\phi = 1$, we obtain $\phi \in \ker \rho_1$. To get its image in $\text{GL}_2(\mathcal{L}_S(\mathbb{F}_p A_3))$ we conjugate it by

$$(c_{ij}) = \begin{pmatrix} b_1 & 0 & 0 \\ b_2 & b_1^{-1} & 0 \\ b_3 & 0 & 1 \end{pmatrix}$$

which implies that

$$\Delta = \begin{pmatrix} 1 + \sigma\pi b_1^2 s_3^{-1} & -\sigma^2 b_1^2 s_3^{-1} \\ \pi^2 b_1^2 s_3^{-1} & 1 - \sigma\pi b_1^2 s_3^{-1} \end{pmatrix} \in (\ker \rho_1)\eta.$$

By Lemma 3.2 (for $H = T \cap \text{IA}(M_3)$ and $N = \ker \rho_1$), $\Delta \in E_2(\mathcal{L}_P(\mathbb{F}_p A_3))$. But, by Lemma 3.6, $\Delta \in \text{SL}_2(\mathcal{L}_P(\mathbb{F}_p A_3)) \setminus E_2(\mathcal{L}_P(\mathbb{F}_p A_3))$ and so, ϕ is a non-tame automorphism of M_3 .

EXAMPLES 4.3. We shall give a family of examples of non-tame automorphisms of M_3 for $p=3$. It is enough to construct irreducible elements π and σ in $\mathbb{F}_3 A_2$ subject to all conditions of Theorem 4.2 are satisfied. The polynomial $\pi = s_1^3 - s_1 - 1$ is irreducible in $\mathbb{F}_3[s_1]$. It is easily verified that $\pi \notin P_{s_3}\Psi$. By Remark 3.4, $\pi \nmid x$ for all $x \in P_{s_3}\Psi$. Let I be the ideal in $\mathbb{F}_3 A_1$ generated by π , and let $K = \mathbb{F}_3 A_1/I$. Then K is a field of 27 elements. Let $q=13$. It is easily verified that $s_1^2 - 1 \notin I$. Let $b = s_1 + I$. Since the polynomial $s_2^{13} - b$ has no root in K , we obtain $s_2^{13^n} - b$ is irreducible in $K[s_2]$ for all $n \geq 1$ (see [12, Theorem 3.75 and page 145]). The natural mapping ϑ from $\mathbb{F}_3 A_1$ onto K induces a ring epimorphism ϑ_1 from $\mathbb{F}_3 A_2$ onto $K[s_2^{\pm 1}]$ in a natural way. Since $P_{s_3}\Psi\vartheta_1$ is a multiplicative closed subset of $K[s_2^{\pm 1}]$, and $0 \notin P_{s_3}\Psi\vartheta_1$, the epimorphism ϑ_1 induces a ring epimorphism $\tilde{\vartheta}_1$ from $\mathcal{L}_{P_{s_3}\Psi}(\mathbb{F}_3 A_2)$ onto $\mathcal{L}_{P_{s_3}\Psi\vartheta_1}(K[s_2^{\pm 1}])$ by defining $\frac{u}{v}\tilde{\vartheta}_1 = \frac{u\vartheta_1}{v\vartheta_1}$ for all $u \in \mathbb{F}_3 A_2$ and $v \in P_{s_3}\Psi$. But $s_2^{13^n} - b \notin P_{s_3}\Psi\vartheta_1$ and $s_2^{13^n} - b$ is not invertible in $\mathcal{L}_{P_{s_3}\Psi\vartheta_1}(K[s_2^{\pm 1}])$ for all n . Write $\sigma_n = s_2^{13^n} - s_1$. It is easy to verify that σ_n is irreducible in $\mathbb{F}_3 A_2$. In addition, $\sigma_n \nmid y$ for all $y \in P_{s_3}\Psi$, and π and σ_n are relatively prime in $\mathbb{F}_3 A_2$. Thus, for all $n \geq 1$, π and σ_n satisfy all the conditions of Theorem 4.2.

In the next few lines, we shall prove that the IA-automorphism group of M_3 is not finitely generated. Although the aforementioned result was stated in [16], we shall apply the aforementioned method to fill a gap to complete the proof. To get a contradiction, we assume that $\text{IA}(M_3)$ is finitely generated. We have the following short exact sequence

$$1 \rightarrow \ker \rho_2 \rightarrow \text{IA}(M_3) \xrightarrow{\rho_2} A_3 \rightarrow 1,$$

where $\phi\rho_2 = \det J_\phi = s_1^{\mu_1} s_2^{\mu_2} s_3^{\mu_3}$, $\mu_1, \mu_2, \mu_3 \in \mathbb{Z}$. Applying Lemma 3.2 for $H = \text{IA}(M_3)$ and $N = \ker \rho_2$, there exists a multiplicative monoid P of $\mathbb{F}_p A_3$ such that $(d_{ij}) \in E_2(\mathcal{L}_P(\mathbb{F}_p A_3))$ for all $(d_{ij}) \in (\ker \rho_2)\eta$. By the proof of Lemma 3.2 (and Remark 3.4), we may choose a (monic) irreducible polynomial π of degree m in $\mathbb{F}_p[s_1]$ subject to $\pi \nmid x$ for all $x \in P_{s_3}\Psi$, and there exists q an odd prime divisor of $p^m - 1$. Let I be the ideal of $\mathbb{F}_p A_1$ generated by π . By Lemma 2.5, $K = \mathbb{F}_p A_1/I$ is a field of p^m elements. By Lemma 3.5, there exists $b \in K$ such that $s_2^q - b$ is irreducible in $K[s_2^{\pm 1}]$ for all $n \geq 1$, and, for infinitely many n , $s_2^{q^n} - b$ is not invertible in $\mathcal{L}_{P_{s_3}\Psi\vartheta_1}(K[s_2^{\pm 1}])$. The natural mapping ϑ from $\mathbb{F}_p A_1$ onto K induces a ring epimorphism ϑ_1 from $\mathbb{F}_p A_2$ onto $K[s_2^{\pm 1}]$ in a natural way. Since $P_{s_3}\Psi\vartheta_1$ is a multiplicative closed subset of $K[s_2^{\pm 1}]$, and $0 \notin P_{s_3}\Psi\vartheta_1$, the epimorphism ϑ_1 induces a ring epimorphism $\tilde{\vartheta}_1$ from $\mathcal{L}_{P_{s_3}\Psi}(\mathbb{F}_p A_2)$ onto $\mathcal{L}_{P_{s_3}\Psi\vartheta_1}(K[s_2^{\pm 1}])$ by defining $\frac{u}{v}\tilde{\vartheta}_1 = \frac{u\vartheta_1}{v\vartheta_1}$ for all $u \in \mathbb{F}_p A_2$ and $v \in P_{s_3}\Psi$. Choose $s_2^{q^n} - b$ an irreducible element in $K[s_2]$ subject to $s_2^{q^n} - b$ is not invertible in $\mathcal{L}_{P_{s_3}\Psi\vartheta_1}(K[s_2^{\pm 1}])$. Let

c be an element of $\mathbb{F}_p A_1$ such that $c\vartheta = b$. Then $\sigma = s_2^{q^n} - c$ is an irreducible element in $\mathbb{F}_p A_1[s_2]$. Hence σ is irreducible in $\mathbb{F}_p A_2$. Furthermore $\sigma \nmid y$ for all $y \in P_{s_3}\Psi$. It is easily verified that π and σ are relatively prime elements in $\mathbb{F}_p A_2$. Let

$$(a_{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ \sigma^2 b_3 s_3^{-1} - \sigma \pi b_1 b_2 s_3^{-1} & 1 + \sigma \pi b_1^2 s_3^{-1} & -\sigma^2 b_1 s_3^{-1} \\ -\pi^2 b_1^2 b_2 s_3^{-1} + \sigma \pi b_1 b_3 s_3^{-1} & \pi^2 b_1^3 s_3^{-1} & 1 - \sigma \pi b_1^2 s_3^{-1} \end{pmatrix}.$$

Since $\det(a_{ij}) = 1$ and the rows of (a_{ij}) satisfy the conditions (15), we obtain $(a_{ij}) \in (\ker \rho_2)\zeta$. Since ζ is a group monomorphism, there exists $\phi \in \ker \rho_2$ such that $(a_{ij}) = J_\phi$. To get its image in $\text{GL}_2(\mathcal{L}_S(\mathbb{F}_p A_3))$, we conjugate it by

$$(c_{ij}) = \begin{pmatrix} b_1 & 0 & 0 \\ b_2 & b_1^{-1} & 0 \\ b_3 & 0 & 1 \end{pmatrix}$$

which implies that

$$\Delta = \begin{pmatrix} 1 + \sigma \pi b_1^2 s_3^{-1} & -\sigma^2 b_1^2 s_3^{-1} \\ \pi^2 b_1^2 s_3^{-1} & 1 - \sigma \pi b_1^2 s_3^{-1} \end{pmatrix} \in (\ker \rho_2)\eta.$$

Thus, by Lemma 3.2, Δ is an element of $E_2(\mathcal{L}_P(\mathbb{F}_p A_3))$. By Lemma 3.6, $\Delta \in \text{SL}_2(\mathcal{L}_P(\mathbb{F}_p A_3)) \setminus E_2(\mathcal{L}_P(\mathbb{F}_p A_3))$ which is a contradiction. Therefore $\text{IA}(M_3)$ is not a finitely generated group.

Appendix

- $(J_{\tau_{13}}, s_3^{-1}(s_1 - 1), 0), (J_{\tau_{23}}, 0, s_3^{-1}(s_2 - 1)), (J_{\tau_{123}}, s_1 s_2^{-1} s_3^{-1}(s_2 - 1), 0),$
- $(J_{\tau_{213}}, 0, s_1^{-1} s_2 s_3^{-1}(s_1 - 1)), (J_{\tau_{23}^{-1} \tau_{13}}, s_3^{-1}(s_1 - 1), 1 - s_2), (J_{(\tau_{12}, \tau_{13})}, (1 - s_1)(s_2 - 1), 0),$
- $(J_{(\tau_{12}, \tau_{23})}, s_3^{-1}(s_1 - 1)(s_2 - 1), 0), (J_{(\tau_{12}, \tau_{123})}, -s_1 s_2^{-1} s_3^{-1}(s_2 - 1)^2, 0),$
- $(J_{(\tau_{12}, \tau_{213})}, s_1^{-1} s_2 s_3^{-1}(s_1 - 1)^2(1 - (s_1^{-1} - 1)(s_3^{-1} - 1)), s_1^{-2} s_2 s_3^{-2}(s_1 - 1)^2(s_3 - 1)),$
- $(J_{(\tau_{13}, \tau_{21})}, 0, s_3^{-1}(s_1 - 1)(1 - s_2)), (J_{(\tau_{13}, \tau_{31})}, -s_1^{-1} s_3^{-1}(s_1 - 1)^2, 0),$
- $(J_{(\tau_{13}, \tau_{32})}, -s_2^{-1}(s_1 - 1)(s_2 - 1), 0), (J_{(\tau_{13}, \tau_{123})}, s_1 s_2^{-1} s_3^{-1}(1 - s_2)(s_3 - 1), 0),$
- $(J_{(\tau_{13}, \tau_{213})}, 0, s_1^{-1} s_2 s_3^{-2}(s_3 - 1)(s_1 - 1)), (J_{(\tau_{13}, \tau_{312})}, -s_1^{-1} s_2^{-1}(s_1 - 1)^2(s_2 - 1), 0),$
- $(J_{(\tau_{21}, \tau_{23})}, 0, -(s_1 - 1)(s_2 - 1)),$
- $(J_{(\tau_{21}, \tau_{123})}, s_1 s_2^{-2} s_3^{-2}(s_2 - 1)^2(s_3 - 1), s_1 s_2^{-1} s_3^{-1}(s_2 - 1)^2(1 - (s_2^{-1} - 1)(s_3^{-1} - 1)),$
- $(J_{(\tau_{21}, \tau_{213})}, 0, -s_1^{-1} s_2 s_3^{-1}(s_1 - 1)^2), (J_{(\tau_{23}, \tau_{31})}, 0, s_1^{-1}(s_2 - 1)(1 - s_1)),$
- $(J_{(\tau_{23}, \tau_{32})}, 0, -s_2^{-1} s_3^{-1}(s_2 - 1)^2),$
- $(J_{(\tau_{23}, \tau_{123})}, s_1 s_2^{-1} s_3^{-2}(s_2 - 1)(s_3 - 1), 0), (J_{(\tau_{23}, \tau_{213})}, 0, s_1^{-1} s_2 s_3^{-1}(s_1 - 1)(1 - s_3)),$
- $(J_{(\tau_{23}, \tau_{312})}, 0, s_1^{-1} s_2^{-1}(s_1 - 1)(s_2 - 1)^2),$

$$\begin{aligned}
& (J_{(\tau_{31}, \tau_{123})}, s_1 s_2^{-1} s_3^{-1} (1 - s_2) (s_1^{-1} (1 - s_1) + s_2^{-1} s_3^{-1} (s_2 - 1) (s_3 - 1)), 0), \\
& (J_{(\tau_{31}, \tau_{213})}, 0, s_1^{-2} s_2 s_3^{-1} (s_1 - 1)^2), (J_{(\tau_{32}, \tau_{123})}, s_1 s_2^{-2} s_3^{-1} (s_2 - 1)^2, 0), \\
& (J_{(\tau_{32}, \tau_{213})}, 0, s_1^{-1} s_2 s_3^{-1} (1 - s_1) (s_2^{-1} (1 - s_2) + s_1^{-1} s_3^{-1} (s_1 - 1) (s_3 - 1))), \\
& (J_{(\tau_{123}, \tau_{213})}, -s_3^{-2} (s_1 - 1) (s_3 - 1) + s_1 s_2^{-1} s_3^{-3} (s_2 - 1) (s_3 - 1)^2 - s_3^{-4} (s_1 - 1) (s_3 - 1)^3, \\
& \quad -s_1^{-1} s_2 s_3^{-3} (s_3 - 1)^2 (s_1 - 1) + s_3^{-2} (s_2 - 1) (s_3 - 1)) \\
& (J_{(\tau_{123}, \tau_{312})}, -s_1 s_2^{-3} s_3^{-1} (s_2 - 1)^3, 0), (J_{(\tau_{213}, \tau_{312})}, 0, s_1^{-3} s_2 s_3^{-1} (s_1 - 1)^3),
\end{aligned}$$

Set

$$\delta_1 = 1 + s_3,$$

$$\delta_2 = s_2 (s_1 - 1) (s_3 - 1) - s_1 s_3 (s_2 - 1),$$

$$\delta_3 = s_2 s_3 (1 - s_1) + s_1 (s_2 - 1) (s_3 - 1),$$

$$\delta_4 = s_1 s_3 (1 - s_2) + s_2 (s_1 - 1) (s_3 - 1)$$

and

$$\begin{aligned}
\delta_5 = & 2s_1 s_2 s_3^2 (s_1 - 1) (s_2 - 1) - s_1^2 s_3 (s_2 - 1)^2 (s_3 - 1) \\
& + s_1 s_2 (s_1 - 1) (s_2 - 1) (s_3 - 1)^2 - s_2^2 s_3 (s_1 - 1)^2 (s_3 - 1).
\end{aligned}$$

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