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NON-EXISTENCE OF ODD PERIODIC MAPS ON CERTAIN SPACES WITHOUT FIXED POINTS

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In this paper, we show that the fixed point set of Z_p-actions, p an odd prime, on a finitistic space X of type (a,b) is non-empty, whenever $b \equiv 0 \pmod{p}$. We also prove a similar result for circle group actions on finitistic spaces of $(a,0)$ type.

1. Statement of main results

Let X be a finitistic space, that is, X is paracompact Hausdorff and each open cover of it has a finite dimensional open refinement. We say that a space X has type (a,b) if

$$H^{in}(X;Z) \simeq Z \ , \quad i = 0,1,2,3$$

are the only non-trivial cohomology groups and there are generators $u_i \in H^{in}(X;Z)$, $i = 0,1,2,3$ such that

$$u_1^2 = au_2 \ , \ u_1u_2 = bu_3 \ , \ a,b \in Z \ .$$

For arbitrary integers a and b , there are spaces of type (a,b) [6]. Here, by $H^*(Y;\Lambda)$ we mean the sheaf cohomology of the space Y with

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closed supports on Y and coefficients in the constant sheaf associated with a given ring Λ, in the sense of [1]. It is easy to see that the Universal Coefficient formula for Z_p-coefficients holds in general. Therefore, we have

$$H^{in}(X;Z_p) \approx Z_p\ , \quad i = 0,1,2,3\ .$$

Thus X is a Poincaré duality space over Z_p, if $b \not\equiv 0 \pmod p$, having cohomology ring isomorphic to that of $S^n \times S^{2n}$ or a cohomology projective space of height 3, according as $a \equiv 0 \pmod p$ or $a \not\equiv 0 \pmod p$. The fixed point sets of Z_p-actions and S^1-actions on such spaces have been studied in detail (for example see [2, Chapter VII]). We consider the remaining cases here. In fact we prove the following

THEOREM 1. *Let* $G = Z_p$, p *an odd prime, act continuously on a finitistic space* X *of* (a,b) *type; with fixed point set* F. *If* $b \equiv 0 \pmod p$ *then* F *is non-empty.*

For circle group actions, we prove the following.

THEOREM 2. *Let* $G = S^1$ *act continuously with finitely many orbit types on a finitistic space* X *of* $(a,0)$ *type. Then the fixed point set* $F = X^G$ *is non-empty.*

We generalise some results in §2 and prove the theorems in §3.

2. A criterion for the existence of fixed points

Let a topological group G act continuously on a space X and let $E_G \to B_G$ be a universal principal G-bundle. The quotient space of $X \times E_G$ under the diagonal action of G is denoted by X_G. We have the associated bundle

$$X \longrightarrow X_G \xrightarrow{\ \pi\ } B_G\ ,$$

over B_G with fiber X and structural group G. For a compact Lie group G, B_G is a CW-complex with finite N-skeleton B_G^N for all N. If E_G^N is the inverse image of B_G^N, then E_G^N is compact and N-universal,

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that is, $H^j(E_G^N;\Lambda) = 0$ for $j < N$. Let

$$X_G^N = X \times_G E_G^N ,$$

which is the associated bundle over B_G^N with fiber X. The equi-variant cohomology of the G-space X is defined by

$$H_G^*(X) = H^*(X_G) .$$

For Hausdorff spaces X, it is easily seen that

$$H_G^j(X) \simeq H^j(X_G^N) \quad \text{for} \quad j < N ,$$

(for example see [5]). Thus we may assume that E_G and $B_G = E_G/G$ are locally contractible and X_G is paracompact whenever X is.

The projection $\pi : X_G \to B_G$ induces the homomorphism

$$\pi^* : H^*(B_G) \longrightarrow H^*(X_G)$$

and thus $H_G^*(X)$ can be regarded as a module over the ring $H^*(B_G)$ via the cup product.

Let $S \subset H^*(B_G)$ be a multiplicative system. Then the sets X^S are defined by

$$X^S = \{x \in X \mid \text{no element of } S \text{ is mapped to zero in } H^*(B_G) \longrightarrow H^*(B_{G_x})\} .$$

The inclusion $X^S \subset X$ induces an $H^*(B_G)$-homomorphism

$$H_G^*(X) \longrightarrow H_G^*(X^S) .$$

By localizing at S, we have the homomorphism

$$S^{-1} H_G^*(X) \longrightarrow S^{-1} H_G^*(X^S) .$$

In [3] we proved the following result.

THEOREM 2.1. *Let a compact Lie group G act on a finitistic space X with finitely many orbit types. If S is a multiplicative system in $H^*(B_G;\Lambda)$ and Λ is a prime field, then the localized restriction homomorphism*

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$$S^{-1}\,H_G^*(X;\Lambda) \longrightarrow S^{-1}\,H_G^*(X^S;\Lambda)$$

is an isomorphism.

We use this theorem to prove the following

PROPOSITION 2.2. *Let* $G = Z_p^k$ *act on a finitistic space* X *and* F *be the fixed point set. If* S *is the multiplicative system* $\Lambda[t_1,\ldots,t_k] - \{0\}$ *where* $\Lambda[t_1,\ldots,t_k]$ *is the polynomial part of* $H^*(B_G;\Lambda)$, $\Lambda = Z_p$, *then the localized restriction homomorphism*

$$S^{-1}\,H_G^*(X;\Lambda) \longrightarrow S^{-1}\,H_G^*(F;\Lambda) \approx H^*(F;\Lambda) \otimes S^{-1}\,H^*(B_G;\Lambda)$$

is an ismorphism.

This also holds for $G = T^k$ and $\Lambda = Q$, if the number of orbit types is finite.

Proof. We need to show that $X^S = F$, and our Proposition, then, follows immediately from Theorem 2.1. It is obvious that $F \subset X^S$. To prove the inclusion $X^S \subset F$, assume that $x \notin F$. Then $G_x = Z_p^\ell$ $\ell < k$ when $G = Z_p^k$; and $G_x = H \times T^\ell$, $\ell < k$ and H a finite group, when $G = T^k$. Thus the polynomial part of $H^*(B_{G_x})$ is generated by ℓ variables while that of $H^*(B_G)$ is generated by k variables. Therefore some generators t_j of $\Lambda[t_1,\ldots,t_k]$ map to zero under the homomorphism

$$H^*(B_G) \longrightarrow H^*(B_{G_x}) .$$

So $x \notin X^S$. Together with the fact $F \subset X^S$, this implies that $X^S = F$. The isomorphism

$$S^{-1}\,H_G^*(F) \approx H^*(F) \otimes S^{-1}\,H^*(B_G)$$

follows from the Künneth rule. □

The following corollary gives us a criterion for the existence of fixed points of actions of p-tori or tori on finitistic spaces.

COROLLARY 2.3. *Let* $G = Z_p^k$ *act on a finitistic space* X . *Then the fixed point set* $F = X^G$ *is non-empty if and only if*

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$$H_G^*(pt;\Lambda) \longrightarrow H_G^*(X;\Lambda)$$

is a monomorphism, where $\Lambda = Z_p$.

This also holds for $G = T^k$ and $\Lambda = Q$, if there are only finitely many orbit types.

Proof. Let F be non-empty and $x \in F$. Then the composite

$$B_G \longrightarrow \{x\}_G \longrightarrow X_G \longrightarrow B_G$$

is a homeomorphism of B_G onto itself. Therefore the composite homomorphism

$$H^*(B_G) \longrightarrow H^*(X_G) \longrightarrow H^*(B_G)$$

is an isomorphism, and hence

$$H^*(B_G) \longrightarrow H_G^*(X)$$

is a monomorphism.

Conversely, if the above homomorphism is a monomorphism then $1 \in H_G^*(X)$ is torsion-free and hence $S^{-1} H_G^*(X) \neq 0$. By Proposition 2.2, $H^*(F) \neq 0$, which holds only if F is non-empty. □

3. Proofs of Theorems 1 and 2

Let a compact Lie group G act on a paracompact Hausdorff space X . We consider the Leray spectral sequence of the map $\pi : X_G \to B_G$ with coefficients in the constant sheaf associated with a given ring Λ and closed supports on both X_G and B_G . Its E_2-term is given by

$$E_2^{k,j} = H^k(B_G;H^j(X;\Lambda)) .$$

The coefficients $H^j(X;\Lambda)$ are locally constant, but are twisted via the canonical action of $\pi_0(G)$ on $H^j(X,\Lambda)$. The spectral sequence converges to $H_G^*(X;\Lambda)$ in the sense that there exists a decreasing filtration F^k of $H_G^*(X)$ such that

$$E_\infty^{k,j} = F^k(H_G^{k+j}(X))/F^{k+1}(H_G^{k+j}(X)) .$$

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In particular

$$E_\infty^{k,0} \simeq F^k(H_G^k(X))$$

for each k, since $F^{k+1}(H_G^k(X)) = 0$.

We first prove the following:

PROPOSITION 3.1. *Let $G = Z_p$, p an odd prime, act on a finitistic space X of type (a,b). If $b \equiv 0 \pmod p$, then the Leray spectral sequence of the map $\pi : X_G \to B_G$, with coefficients in $\Lambda = Z_p$, degenerates on the base, that is, $E_2^{k,0} = E_\infty^{k,0}$ for all k.*

Proof. By the Universal Coefficient theorem, we have

$$H^{in}(X;Z_p) \simeq Z_p , \quad i = 0,1,2,3$$

Also, we can choose generators $v_i \in H^{in}(X;Z_p)$, $i = 1,2,3$ such that

$$v_1^2 = \bar{a}v_2 \text{ and } v_1v_2 = \bar{b}v_3$$

where $\bar{a}$ and $\bar{b}$ denote modulo p reductions of integers a and b, respectively. Since Z_p has no automorphism of period p, it follows that Z_p acts trivially on $H^*(X)$. Therefore

$$E_2^{k,j} = H^k(B_G) \otimes H^j(X) ,$$

where $H^*(B_{Z_p};Z_p) = Z_p[s,t]/(s^2)$, deg $s = 1$ and deg $t = 2$.

Assume that $b \equiv 0 \pmod p$. Then $v_1v_2 = 0$. Now there are two cases depending on whether $a \not\equiv 0$ or $a \equiv 0$ modulo p.

First we consider the case $a \not\equiv 0 \pmod p$. Thus the mod p cohomology ring of X satisfies

$$v_1^2 \neq 0 \text{ and } v_1v_2 = 0 .$$

Since p is odd, n must be even. If possible, suppose

$$d_{n+1}(1 \otimes v_1) \neq 0 .$$

Without any loss in generality, we may assume that

$$d_{n+1}(1 \otimes v_1) = s \otimes 1 ,$$

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where

$$d_{n+1} : E_{n+1}^{0,n} \longrightarrow E_{n+1}^{n+1,0} .$$

If $d_{n+1}(1 \otimes v_i) = 0$ for some $i = 2,3$, then

$$0 = d_{n+1}(1 \otimes v_1 v_i) = s \otimes v_i \neq 0 ,$$

a contradiction. But the assumption $d_{n+1}(1 \otimes v_3) = s \otimes v_2$ implies that

$$0 = d_{n+1}(1 \otimes v_1 v_3) = s \otimes v_3 + s \otimes v_1 v_2 = s \otimes v_3 \neq 0$$

again a contradiction. Therefore we must have $d_{n+1}(1 \otimes v_1) = 0$.

Now suppose that

$$d_{in+1}(1 \otimes v_i) \neq 0 , \quad \text{for} \quad i = 2, \text{ or } 3 .$$

Let $d_{in+1}(1 \otimes v_i) = As \otimes 1$, $0 \neq A \in Z_p$. Obviously $d_{in+1}(1 \otimes v_1) = 0$, so that we have

$$0 = d_{in+1}(1 \otimes v_1 v_i) = As \otimes v_1 \neq 0 ,$$

a contradiction. Therefore $d_{in+1}(1 \otimes v_i) = 0$ for $i = 1,2,3$, in this case.

Now we consider the case $a \equiv 0 \pmod p$. Thus the generators v_1, v_2, v_3 of mod p cohomology ring of X satisfy the relations

$$v_1^2 = 0 , \text{ and } v_1 v_2 = 0 .$$

If possible, suppose that

$$d_{n+1}(1 \otimes v_1) \neq 0 .$$

We then notice that n must be odd, for otherwise, we may assume that $d_{n+1}(1 \otimes v_1) = As \otimes 1$ for some $0 \neq A \in Z_p$ which implies that

$$0 = d_{n+1}(1 \otimes v_1^2) = 2A(s \otimes v_1) \neq 0 .$$

Hence, we can write

$$d_{n+1}(1 \otimes v_1) = t^q \otimes 1 .$$

If $d_{n+1}(1 \otimes v_i) = 0$ for $i = 2$ or 3, then we have

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$$0 = d_{n+1}(1 \otimes v_1 v_i) = t^q \otimes v_i \neq 0$$

a contradiction. And, if $d_{n+1}(1 \otimes v_i) \neq 0$ for some $i = 2,3$, then we may assume that $d_{n+1}(1 \otimes v_i) = t^{q'} \otimes v_{i-1}$. This implies that

$$0 = d_{n+1}(1 \otimes v_1 v_i) = t^q \otimes v_i \neq 0 ,$$

again a contradiction. Therefore we must have $d_{n+1}(1 \otimes v_1) = 0$. As in the first case, we see that

$$d_{2n+1}(1 \otimes v_2) = 0 \quad \text{and} \quad d_{3n+1}(1 \otimes v_3) = 0 \quad \text{for} \quad n \quad \text{even} .$$

For odd n, the assumption

$$d_{3n+1}(1 \otimes v_3) = A\, t^q \otimes 1 , \; 0 \neq A \in Z_p$$

implies that

$$0 = d_{3n+1}(1 \otimes v_1 v_3) = -A\, t^q \otimes v_1 \neq 0 .$$

So $d_{3n+1}(1 \otimes v_3) = 0$ in this case also.

It is now clear that the differentials

$$d_r : E_r^{0,r-1} \longrightarrow E_r^{r,0} , \; r \geq 2$$

are zero. Hence, it follows that the differentials

$$d_r : E_r^{k,r-1} \longrightarrow E_r^{k,0} , \; r \geq 2 \quad \text{and} \quad k \geq 0$$

are also zero and this completes the proof of the proposition. □

Proof of Theorem 1. It is obvious from Proposition 3.1 that

$$H^*(B_G) = F^*(H_G^*(X)) \subset H_G^*(X)$$

and thus we have a monomorphism $H^*(B_G) \longrightarrow H_G^*(X)$. It is easily seen that this homomorphism is induced by the projection $X_G \xrightarrow{\pi} B_G$. Hence it follows from Corollary 2.3, that the fixed point set F is non-empty. □

Proof of Theorem 2. Since there are only finitely many orbit types, we can choose a prime p so large that $Z_p \subset S^1$ is contained in no proper isotropy subgroup of S^1. Then $X^{Z_p} = X^{S^1} = F$. Now it follows from Theorem 1, that F is non-empty. □

REMARK. It remains to determine the possible cohomology structures of components of the fixed point set as has been done in case of product of spheres and cohomology projective spaces.

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