TOURNAMENTS WITH A GIVEN AUTOMORPHISM GROUP

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1. Introduction and summary. The set of all adjacency-preserving automorphisms of the vertex set of a graph form a group which is called the (automorphism) group of the graph. In 1938 Frucht **(2)** showed that every finite group is isomorphic to the group of some graph. Since then Frucht, Izbicki, and Sabidussi have considered various other properties that a graph having a given group may possess. (For pertinent references and definitions not given here see Ore **(4)**.) The object in this paper is to treat by similar methods a corresponding problem for a class of oriented graphs. It will be shown that a finite group is isomorphic to the group of some complete oriented graph if and only if it has an odd number of elements.

2. Definitions. A (round-robin) *tournament*, or a complete oriented graph, consists of a finite set of vertices $p, q, \ldots$ such that each pair of distinct vertices is joined by an arc oriented towards one of the vertices. If the arc joining p and q is oriented towards q, we say that "p defeats q" or, symbolically, $p \rightarrow q$. Let $d(p)$ denote the *degree* of p, i.e., the number of vertices q such that $p \rightarrow q$. Suppose that α is an orientation-preserving permutation of the vertices of a tournament so that $\alpha(p) \rightarrow \alpha(q)$ if and only if $p \rightarrow q$. It is readily verified that the set of all such permutations forms a group, the (automorphism) group of the tournament.

In the next section we answer the following question. Which finite groups are isomorphic to the group of some tournament?

3. Constructing a tournament with a given group. We first observe that no tournament T has a group G of even order. For if G is of even order, then it contains at least one self-inverse element α not equal to the identity element. Hence, there exist two distinct vertices p and q in T such that $\alpha(p) = q$ and $\alpha(q) = p$ and if, as we may suppose, $p \rightarrow q$, then $\alpha(q) \rightarrow \alpha(p)$. This is contrary to the definition of G. Thus, a necessary condition for a finite group to be isomorphic to the group of some tournament is that it be of odd order. We now show that this condition is also sufficient.

Let G be a group of odd order whose elements are $g_1, g_2, \ldots, g_n$. Suppose that $g_1, g_2, \ldots, g_h$ form a minimal set of generators, i.e., every element of G can be expressed as a finite product of powers of these h elements and no

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smaller set has this property. We shall temporarily assume that $h \geqslant 2$ and treat the cyclic groups later.

In constructing a tournament whose group is isomorphic to G, we begin, as could be expected, by forming what is essentially the Cayley colour graph T^* of G; see Coxeter and Moser (**1**, pp. 21–23). The vertices of T^* correspond to the elements of G. For convenience we use the same symbol for a vertex and its corresponding group element. With each generator g_j we associate a certain set of arcs in T^* which are said to have colour j. There is an arc of colour j, $j = 1, 2, \ldots, h$, going from p to q in T^* if and only if $pg_j = q$. At each vertex there is now one incoming and one outgoing arc for each generator. No vertex is joined by an arc to itself since the identity element is not one of the generators. No two vertices are joined by two arcs, one oriented in each direction, since the colours of these arcs would correspond to group elements which are the inverses of each other and no such elements are in the set of generators.

If two distinct vertices p and q are not joined by an arc in the above procedure we introduce one of the 0th colour which is oriented towards q or p according as the element $p^{-1}q$ or $q^{-1}p$ has the larger subscript in the original listing of the elements of G. It is clear that these products are not equal. If an arc of the 0th colour goes from p to q, then it is easily seen that an arc of the 0th colour goes from $q' = pq^{-1}p$ to p. Each pair of distinct vertices of T^* is now joined by a coloured arc and the orientations are such that $d(g) = \frac{1}{2}(n - 1)$ for each g.

The group of orientation- and colour-preserving automorphisms of T^* is isomorphic to G. Our problem is to maintain this property while transforming T^* into a tournament T whose arcs all have the same colour, or rather none at all. We accomplish this by introducing j new vertices for each arc of colour j, for $j = 0, 1, \ldots, h$.

The additional vertices are labelled $x_{j,k}^{(i)}$ (where $i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, h$; and $k = 1, 2, \ldots, j$). Consider any vertex g_i. If there is an arc oriented from q to g_i of colour j in T^*, i.e., if $q = g_i g_j^{-1}$, then in T $q \rightarrow x_{j,k}^{(i)} \rightarrow g_i$ if $2 \leqslant k \leqslant j$ but $x_{j,1}^{(i)}$ defeats both q and g_i. All the coloured arcs in T^* are replaced by their corresponding uncoloured arcs in T. The orientation of the arcs between the remaining pairs of vertices is determined as follows. All the vertices $x_{j,k}^{(i)}$ for a given i are said to "belong" to g_i. Also, g_i belongs to itself. If the vertices x and y belong to p and q, respectively, where $p \neq q$, then $x \rightarrow y$ if and only if $p \rightarrow q$. For distinct new vertices belonging to the same vertex we let $x_{j,k}^{(i)} \rightarrow x_{l,m}^{(i)}$ if and only if $j > l$ or $j = l$ and $k > m$.

This completes the definition of the tournament T. Appealing to the method of construction and recalling the exceptional character of the vertices $x_{j,1}^{(i)}$, it is not difficult to verify the following equalities:

$$\begin{aligned} d(g) &= (1 + 2 + \ldots + h)h + \tfrac{1}{2}(n - 2h - 1)[(1 + 2 + \ldots + h) + 1] \\ &= \tfrac{1}{2}(n - 1)\binom{h+1}{2} + \tfrac{1}{2}(n - 2h - 1) \end{aligned}$$

for each of the vertices associated with an element of G;

$$d(x_{j,k}^{(i)}) = d(g) + \binom{j}{2} + h + k, \qquad \text{if } 2 \leqslant k \leqslant j,$$

and

$$d(x_{j,1}^{(i)}) = d(x_{j,2}^{(i)}), \qquad \text{for all } i \text{ and } j.$$

The important fact here is that $d(x) > d(g)$ for every new vertex x and that

$$d(x_{j,k}^{(i)}) = d(x_{l,m}^{(t)})$$

if and only if $j = l$, and $k = m$ or $km = 2$. Since $d(p) = d(\alpha(p))$ for any admissible automorphism α of T, it follows, in particular, that the identity of the sets of original vertices g and new vertices x is preserved by α.

We now prove that the automorphism group of T is isomorphic to G. Let α be any automorphism which leaves some vertex g_i fixed, i.e. $\alpha(g_i) = g_i$. First we show that this implies that α leaves every vertex of T fixed.

Consider any vertex $x = x_{j,k}{}^{(i)}$ belonging to g_i where $k \geqslant 2$. At least one such vertex exists since $h \geqslant 2$. We recall that $x \to g_i$; hence $\alpha(x)$ could belong to a vertex g_l in $\alpha(T)$ such that $g_i \to g_l$ only if

$$\alpha(x) = x_{j,1}^{(l)}.$$

If this happens, there exists a vertex y—take

$$y = x_{2,2}^{(l)}$$

if $j = 1$ or

$$y = x_{j,2}^{(l)}$$

if $j \neq 1$—such that $g_i \to y \to \alpha(x)$. But there exists no such corresponding path from g_i to x in T, so this possibility is excluded. If $\alpha(x)$ belongs to a vertex g_l in $\alpha(T)$ such that $g_l \to g_i$, then $\alpha(x) \to g_l \to g_i$. But, again, there is no corresponding path of length two from x to g_i in T whose intermediate vertex is one of the vertices g. The only alternative is that $\alpha(x)$ belongs to g_i itself, and considering the degrees of such vertices and the types of paths of length two from x to g_i when $k = 2$ leads to the conclusion that $\alpha(x) = x$.

Now consider any remaining vertex

$$x = x_{j,1}^{(i)}$$

belonging to g_i. By the same argument as that given above, it follows that $\alpha(x)$ cannot belong to a vertex g_l in $\alpha(T)$ such that $g_i \to g_l$. But if $\alpha(x)$ belongs to g_l where $g_l \to g_i$, then

$$\alpha(x) \to \alpha(x_{h,2}^{(i)}) = x_{h,2}^{(i)},$$

contradicting the fact that α preserves the orientation of the arc going from $x_{h,2}{}^{(i)}$ to x. Hence, $\alpha(x)$ belongs to g_i, and considering the degrees involved and the results in the preceding paragraph leads again to the conclusion that $\alpha(x) = x$.

For any $j = 1, 2, \ldots, h$ consider the vertex $g = g_i g_j^{-1}$. Since

$$x_{j,1}^{(i)} \rightarrow g \rightarrow g_i$$

(this is where the exceptional property of $x_{j,1}^{(i)}$ is needed) it must be that

$$x_{j,1}^{(i)} \rightarrow \alpha(g) \rightarrow g_i$$

since $x_{j,1}^{(i)}$ and g_i are fixed under α. But g itself is the only vertex associated with an element of G which has this property. Hence, $\alpha(g) = g$ for each such g.

The tournament T is strongly connected, i.e., it is possible to reach any vertex from any other vertex by passing along a sequence of similarly oriented arcs. This follows from the method of constructing T from T^*, which itself is strongly connected (**4**, p. 243). Hence, by repeating the above argument as often as is necessary, we eventually conclude that if $\alpha(g_i) = g_i$, then $\alpha(p) = p$ for every vertex p in T.

The rest of the argument is standard. We have seen that if α is not the identity element, then $\alpha(g) \neq g$ for every vertex associated with an element of G. From this it follows that for any two such vertices g_u and g_v there can be at most one automorphism α of T such that $\alpha(g_u) = g_v$. But, the group element $\alpha = g_v g_u^{-1}$ induces such an automorphism defined as follows: $\alpha(g) = \alpha g$ for all vertices of T associated with an element of G, and

$$\alpha(x_{j,k}^{(i)}) = x_{j,k}^{(l)},$$

where $g_l = \alpha(g_i)$, for all i, j, and k. From these results it follows that the automorphism group of T is isomorphic to G.

It remains to treat the case where $G = C_n$, the cyclic group of (odd) order n. It is not difficult to see that C_n is isomorphic to the group of the tournament whose vertices are $p_1, p_2, \ldots, p_n$ and in which $p_i \rightarrow p_j$ if and only if $0 < j - i \leqslant \frac{1}{2}(n - 1)$, where the subtraction is (mod n). Alternatively, if the tournament arising from letting $h = 1$ in the general construction given before is modified so that

$$x_{1,1}^{(i)} \rightarrow g$$

if and only if $g = g_i$ or $g_i \rightarrow g$, for each i, then a simple argument shows that C_n is also isomorphic to the group of this tournament.

In constructing T from T^*, j new vertices were introduced for each arc of colour j. This procedure can easily be extended to form other tournaments having the same non-trivial group by introducing mj new vertices for each arc of colour j for any integer $m \geqslant 2$. There exist tournaments with an arbitrary number of vertices whose group consists only of the identity element, e.g., the tournament whose vertices are $p_1, p_2, \ldots, p_n$ and in which $p_1 \rightarrow p_n$ but $p_j \rightarrow p_i$ if and only if $j > i$ otherwise. It is clear that all of these tournaments are strongly connected. Combining these results completes the proof of the following theorem.

THEOREM. *If G is a group of odd order, then there exist infinitely many strongly connected tournaments whose group is isomorphic to G.*

4. Concluding remarks. If G is a group of odd order n generated by h of its elements, then the construction in §3 yields a tournament with

$$n\binom{h+1}{2} + n$$

vertices whose group is isomorphic to G. The smallest tournament with this property undoubtedly has far fewer vertices in general. For example, let G be the smallest odd non-abelian group. In this case $n = 21$ and $h = 2$ (**1**, p. 134). So the above construction gives a tournament with 84 vertices whose group is isomorphic to G. But there exists a tournament with only 7 vertices whose group is isomorphic to G, namely the tournament in which $p_i \to p_j$ if and only if $j - i$ is a quadratic residue (mod 7), for $i, j = 1, 2, \ldots, 7$ and $i \neq j$.

For abelian groups the original problem has an almost trivial solution, which in a sense is much more efficient than the one given. We have seen that every cyclic group of odd order n is isomorphic to the group of some tournament with not more than n vertices. This statement can be extended to any odd abelian group upon recalling that any abelian group is isomorphic to the direct product of cyclic groups and observing that if A and B are two disjoint tournaments with groups G and H, respectively, then the group of the tournament formed by joining every vertex in A to every vertex in B by an arc oriented towards the vertex in B is isomorphic to the direct product of G and H. Unfortunately, the tournaments obtained in this manner are not strongly connected except when the original group was itself cyclic. Hence, they are not very representative of tournaments in general as it is known (**3**) that the proportion of tournaments which are strongly connected tends rapidly to one as the number of vertices increases.

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