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THE 2-DIMENSIONAL CALABI FLOW

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Abstract. In this paper, based on a Harnack-type estimate and a local Sobolev constant bounded for the Calabi flow on closed surfaces, we extend author's previous results and show the long-time existence and convergence of solutions of 2-dimensional Calabi flow on closed surfaces. Then we establish the uniformization theorem for closed surfaces.

§1. Introduction

Let (Σ, g_0) be a closed Riemann surface with a given conformal class $[g_0]$. In author's previous paper [Ch2], we consider the so-called Calabi flow on $(\Sigma, [g_0])$:

$$\frac{\partial g_{ij}}{\partial t} = (\Delta R) g_{ij}, \quad g_{ij} \in [g_0]. \tag{1.1}$$

If $g = e^{2\lambda} g_0$, for a smooth function

$$\lambda : \Sigma \times [0, \infty) \longrightarrow \mathbf{R},$$

then the equations (1.1) reduce to the following initial value problem of fourth order parabolic equation on $(\Sigma, [g_0])$

$$\begin{cases} \dfrac{\partial \lambda}{\partial t} = \dfrac{1}{2} \triangle R \\ \lambda(p, 0) = \lambda_0(p) \\ g = e^{2\lambda} g_0 \\ \displaystyle\int_\Sigma e^{2\lambda_0} d\mu_0 = \int_\Sigma d\mu_0, \end{cases} \tag{1.2}$$

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where $\Delta = \Delta_g$, $\Delta_0 = \Delta_{g_0}$, R is the scalar curvature with respect to the metric g, R_0 is the scalar curvature with respect to the metric g_0, $d\mu_0$ is the volume element of g_0 and $d\mu$ is the volume element of g.

For the background metric g_0 with constant Gaussian curvature, P. T. Chruściel proved that the following result ([Chru]).

PROPOSITION 1.1. *Let (Σ, g_0) be a Riemann surface with constant Gaussian curvature. For any given smooth initial value λ_0, there exists a smooth solution $\lambda(t)$ of (1.2) on $\Sigma \times [0, \infty)$. Furthermore, the metric converges to a constant curvature metric.*

Remark 1.1. Since there always exists a constant Gaussian curvature metric, due to the uniformization theorem in a Riemann surface, Chruściel's proof appears to be satisfactory for most purposes. However motivated by many reasons such as the study of higher-dimensional Calabi flow, it is desirable to remove this assumption. We refer to the author's review paper [Ch4] for more details. Moreover, X. X. Chen ([Chen]) has provided a new proof of Chruściel's result from such a motivation (from a viewpoint which is quite different from ours). But he still needed to assume the uniformization theorem.

Later, we proved the long-time existence and asymptotic convergence of solutions of (1.2) on $\Sigma \times [0, \infty)$ for (Σ, g_0) with $h = \text{genus}(\Sigma) \geq 2$. Namely, we obtained the next result.

PROPOSITION 1.2. ([Ch2]) *Let (Σ, g_0) be a closed surface of genus $h \geq 2$ with any arbitrary background metric g_0. For any given smooth initial value λ_0, there exists a smooth solution $\lambda(t)$ of (1.2) on $\Sigma \times [0, \infty)$. Furthermore, there exists a subsequence of solution, say $\lambda(t_j)$, such that $g = e^{2\lambda(t_j)} g_0$ converges to the constant negative curvature metric g_∞ as $t_j \to \infty$.*

In this paper, we will extend Proposition 1.2 to more general cases. Our main results are the following two theorems.

THEOREM 1.3. *Let (Σ, g_0) be a closed Riemann surface of genus $h = 1$. For any given smooth initial value λ_0, there exists a smooth solution $\lambda(t)$ of (1.2) on $\Sigma \times [0, \infty)$. Furthermore, there exists a subsequence of solution $\lambda(t_j)$ such that $g = e^{2\lambda(t_j)} g_0$ converges to a zero scalar curvature g_∞ as $t_j \to \infty$.*

For a closed Riemann surface of genus $h = 0$, there exists a metric g_1 with $R_1 > 0$. Moreover, from [KW, Lemma 6.1], it follows that there is a metric g_0 which is conformal equivalent to g_1 (i.e. g_0 is pointwise conformal to $\varphi^* g_1$ for some diffeomorphism φ) with $R_0 \geq 0$, $R_0 \not\equiv 0$ and $R_0 = 0$ in a ball $B(p_0, \rho_0)$. Now we consider the Calabi flow on such a surface $(\Sigma, [g_0])$.

THEOREM 1.4. *Let (Σ, g_0) be a closed Riemann surface of genus $h = 0$ with $R_0 \geq 0$, $R_0 \not\equiv 0$ and $R_0 = 0$ in a ball $B(p_0, \rho_0)$. For any given smooth initial value λ_0, there exists a smooth solution $\lambda(t)$ of (1.2) on $\Sigma \times [0, \infty)$. Furthermore, there exists a subsequence of solutions which converges to a positive constant scalar curvature metric as $t_j \to \infty$.*

In view of Proposition 2.2 below, we reduce the proof of our main Theorems to finding a uniformly lower bound of $\lambda(t)$ as in Section 2.

Remark 1.2. Recently, we showed the global existence and convergence of solutions of the Calabi flow on Einstein 4-manifolds which is an extention of Proposition 1.1 to 4-dimensional manifolds ([Ch3]).

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§2. A uniformly lower bound

For $g = e^{2\lambda} g_0$, $R_0 = R_{g_0}$, we have the following formulae for the quantities appearing in (1.2) and related ones:

$$R = R_g = e^{-2\lambda}(R_0 - 2\Delta_0 \lambda), \tag{2.1}$$

$$\Delta R = e^{-2\lambda} \Delta_0 R, \quad \text{where } \Delta_0 = \Delta_{g_0},\ \Delta = \Delta_g, \tag{2.2}$$

$$d\mu = e^{2\lambda} d\mu_0, \quad \text{where } d\mu_0 = d\mu_{g_0},\ d\mu = d\mu_g, \tag{2.3}$$

$$\frac{\partial}{\partial t} d\mu = \triangle R d\mu, \tag{2.4}$$

$$\int_\Sigma d\mu = \int_\Sigma e^{2\lambda} d\mu_0 = \int_\Sigma e^{2\lambda_0} d\mu_0 = \int_\Sigma d\mu_0. \tag{2.5}$$

Remark 2.1. (2.5) implies the volume is fixed by the flow (1.2).

Then we have

LEMMA 2.1. *Under the flow* (1.2), *we have*

$$\int_\Sigma R^2 d\mu \leq C(R_0, \lambda_0),$$

for $0 \leq T \leq \infty$.

In Chruściel's proof for Proposition 1.1, the crucial step is the so-called Bondi mass loss formula, i.e.

$$\frac{d}{dt}\int_\Sigma e^{3\lambda} d\mu_0 \leq 0$$

if the background metric g_0 has constant Gaussian curvature. Then, by using elliptic estimate and Moser inequality, he got a C^0-estimate. In general, this method does not work for any arbitrary background metric g_0.

Here we generalize Bondi mass loss formula to the case of surfaces (Σ, g_0) with any arbitrary background metric g_0. In our situation, Chruściel's method can not be applied directly. The main difficulty for any arbitrary background metric g_0, the Bondi mass may not decay. Instead we follow the following alternative approach. First we get a kind of Harnack estimate ([Ch2]) on the Bondi mass $\int_\Sigma e^{3\lambda} d\mu_0$

$$\begin{aligned}(2.6)\qquad \frac{d}{dt}\int_\Sigma e^{3\lambda} d\mu_0 &\leq C_1 + C_2 \int_\Sigma e^{-\lambda} d\mu_0 \\ &\quad - C_3 \int e^{\lambda}\left[2|\overset{0}{\nabla}{}^2 e^{-\lambda}|^2 - (\Delta_0 e^{-\lambda})^2\right] d\mu_0.\end{aligned}$$

Then, from (2.6), one can show the next theorem.

PROPOSITION 2.2. ([Ch2]) *Let* (Σ, g_0) *be a closed Riemann surface with any arbitrary background metric* g_0. *For any given smooth initial value* λ_0, *if*

$$(2.7)\qquad \lambda(t) \geq -H$$

for the positive constant H *which is independent of* t, *then there exists a smooth solution* $\lambda(t)$ *of* (1.2) *on* $\Sigma \times [0, \infty)$. *Furthermore, there exists a subsequence of solution, say* $\lambda(t_j)$, *such that* $g = e^{2\lambda(t_j)} g_0$ *converges to one of the constant curvature metric* g_∞ *as* $t_j \to \infty$.

Remark 2.2. The similar results hold for the 3-dimensional Calabi flow. We refer to [CW] for details.

Hence, in view of Proposition 2.2, in order to show the main Theorems, all we need is to find a uniformly lower bound on λ. In the sequel, we will follow the notion as in [Ch2].

DEFINITION 2.1. We say that $\lambda(t)$ satisfies the property $(*)$ if there is a point $x \in \Sigma$ and positive constants ρ, ε, C such that, for $g = e^{2\lambda} g_0$ the inequality

$$\int_{B(x,\rho)} e^{-\varepsilon\lambda(t)} d\mu_0 \le C \tag{$*$}$$

holds.

LEMMA 2.3. ([Ch2]) *For a fixed conformal class $(\Sigma, e^{2\lambda} g_0)$ where volume and $\int_\Sigma R^2 d\mu$ are bounded and such that λ satisfies the property $(*)$. Then there are positive constants C_0 and δ_0 such that*

$$\int_\Sigma e^{-\delta_0\lambda} d\mu_0 \le C_0'. \tag{2.8}$$

As a consequence, there is a constant C_0 such that

$$\lambda \ge -C_0. \tag{2.9}$$

In fact, for $h \ge 2$, one can show easily that

LEMMA 2.4. ([Ch2]) *For a fixed conformal class $(\Sigma, e^{2\lambda} g_0)$ where volume and $\int_\Sigma R^2 d\mu$ are bounded and such that genus $h \ge 2$. Then λ satisfies* (2.8) *and then has a uniformly lower bound.*

Remark 2.3. In the case where $h = 1$ and $h = 0$, first we will show that λ satisfies $(*)$ and then we can show (2.9) using the Lemma 2.3.

For $h = 1$, we have the next lemma,

LEMMA 2.5. *For a fixed conformal class $(\Sigma, e^{2\lambda} g_0)$ where volume and $\int_\Sigma R^2 d\mu$ are bounded and such that genus $h = 1$. Then λ satisfies $(*)$.*

Proof. Given $x \in \Sigma$, define the mass of x by

$$m(x) = \text{mass of } x = \lim_{\rho \to 0} \limsup_{t \to T} \int_{B(x,\rho)} e^{2\lambda} d\mu_0$$

and put

$$E(x) = \lim_{\rho \to 0} \limsup_{t \to T} \int_{B(x,\rho)} R^2 d\mu.$$

Remark 2.4. A point $x \in \Sigma$ will have large mass $m(x)$ if e^{λ} concentrates at x. On the other hand, if $m(x)$ is small enough, e^{λ} will be bounded in a small neighborhood of x.

Following [G, Proposition 2.1] or [Chen], for $g \in [g_0]$ we put $g = e^{2\lambda} g_0$. If $\int d\mu \leq V$ and $\int_{\Sigma} R^2 d\mu \leq \beta^2$, for some positive constants V, β, then for a given $x \in \Sigma$, either one of the following holds:

$$m(x) = 0$$

or

$$m(x) \geq \frac{4\pi}{E(x)} \geq \frac{4\pi}{\beta^2}.$$

Hence one has either ([CW])

(i)

$$\max_{\Sigma} \lambda \leq C\Big(\int d\mu, \int R^2 d\mu \Big), \tag{2.10}$$

or that

(ii) there is a nonempty finite set $S = \{x_1, \ldots, x_k\}$ and a subsequence $\{t_j\}$ such that, given a compact set $K \subset\subset \widetilde{\Sigma} = \Sigma - S$,

$$\max_{K} \lambda \leq C\Big(K, \int d\mu, \int R^2 d\mu \Big). \tag{2.11}$$

Moreover, $w = \lim_{t_j \to T} \lambda$ is defined on $\widetilde{\Sigma}$ and the inequality

$$w \leq C\Big(\int d\mu, \int R^2 d\mu \Big) \tag{2.12}$$

holds.

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Now with respect to g_0, from [C] and [Ch1, Lemma 3.2], we have the local Sobolev constant $A_0 = A_0(n)$, i.e., for $\varphi = e^{\lambda/2} f$, $\varphi \in C_0^\infty(B_\rho)$, $\rho < i_0/2$,

$$\left(\int_{B_\rho} |\varphi|^{2l} d\mu_0\right)^{1/l} \le A_0 \left[\int_{B_\rho} |\overset{0}{\nabla}\varphi|^2 d\mu_0\right], \tag{2.13}$$

where i_0 is the injectivity radius with respect to g_0 and

$$l = \begin{cases} n/(n-2), & n > 2, \\ < \infty, & n = 2. \end{cases}$$

Next we are able to estimate the local Sobolev constant with respect to g in the fixed conformal class. For simplicity, we study only the case where $l = 2$. Now for $B_\rho \subset K$ where $\max_K \lambda \le C\big(K, \int d\mu, \int R^2 d\mu\big)$ and $\varphi = e^{\lambda/2} f$, $\varphi \in C_0^\infty(B_\rho)$, $\rho < i_0/2$, from (2.13) we have

$$\begin{aligned}
\left(\int_{B_\rho} |f|^4 d\mu\right)^{1/2} &= \left(\int_{B_\rho} |\varphi|^4 e^{-2\lambda} d\mu\right)^{1/2} \\
&= \left(\int_{B_\rho} |\varphi|^4 d\mu_0\right)^{1/2} \\
&\le A_0 \left(\int_{B_\rho} |\overset{0}{\nabla}\varphi|^2 d\mu_0\right) \\
&\le C \int_{B_\rho} f^2 |\overset{0}{\nabla}\lambda|^2 e^{\lambda} d\mu_0 + C \int_{B_\rho} |\overset{0}{\nabla} f|^2 e^{\lambda} d\mu_0 \\
&\le C_7 \int_{B_\rho} f^2 |\overset{0}{\nabla}\lambda|^2 e^{\lambda} d\mu_0 + C_4 \int_{B_\rho} |\overset{0}{\nabla} f|^2 d\mu_0.
\end{aligned} \tag{2.14}$$

Now in order to have the local Sobolev constant bound as in (2.18) and (2.19), we need to get a suitable estimate as in (2.17) for $\int_{B_\rho} f^2 |\overset{0}{\nabla}\lambda|^2 e^{\lambda} d\mu_0$ in (2.14).

In fact since $2\Delta_0 \lambda = R_0 - e^{2\lambda} R$ and $B_\rho \subset K$, we have

$$\int_{B_\rho} (\Delta_0 \lambda)^2 d\mu_0 \le C + C \int_{B_\rho} R^2 d\mu \le C_5.$$

Then

$$\lambda - \overline{\lambda} \in W^{2,2}(B_{\rho/2}) \subset W^{1,p}(B_{\rho/2}),$$

for any $p < \infty$ and $\overline{\lambda} = \int_{B_\rho} \lambda d\mu_0 / \int_{B_\rho} d\mu_0$. Therefore we get

$$\int_{B_{\rho/2}} |\overset{0}{\nabla}\lambda|^p d\mu_0 \leq C_6(p), \quad \text{for any } p < \infty. \tag{2.15}$$

Let $E_b = \{x \in B_{\rho/2} : |\overset{0}{\nabla}\lambda|(x) \geq b\}$, $b \gg 1$. The we get the estimate

$$\begin{aligned} \int_{B_{\rho/2}} f^2|\overset{0}{\nabla}\lambda|^2 e^{\lambda} d\mu_0 &= \int_{E_b^c} f^2|\overset{0}{\nabla}\lambda|^2 e^{-\lambda} d\mu + \int_{E_b} f^2|\overset{0}{\nabla}\lambda|^2 e^{-\lambda} d\mu \\ &\leq b^2\Big(\int_{B_{\rho/2}} f^4 d\mu\Big)^{1/2}\Big(\int_{B_{\rho/2}} d\mu_0\Big)^{1/2} \\ &\quad + \Big(\int_{B_{\rho/2}} f^4 d\mu\Big)^{1/2}\Big(\int_{B_{\rho/2}} |\overset{0}{\nabla}\lambda|^4 d\mu_0\Big)^{1/2}. \end{aligned} \tag{2.16}$$

From (2.15), we get for $p > 4$, that

$$C_6 \geq \int_{E_b} |\overset{0}{\nabla}\lambda|^p d\mu_0 \geq b^{p-4}\int_{E_b} |\overset{0}{\nabla}\lambda|^4 d\mu_0$$

and then

$$\int_{E_b} |\overset{0}{\nabla}\lambda|^4 d\mu \leq C_6 b^{4-p}.$$

From (2.16),

$$\int_{B_{\rho/2}} f^2|\overset{0}{\nabla}\lambda|^2 e^{2\lambda} d\mu_0 \leq \left[b^2\Big(\int_{B_{\rho/2}} d\mu_0\Big)^{1/2} + (C_6 b^{4-p})^{1/2}\right]\Big(\int_{E_b} f^4 d\mu\Big)^{1/2}. \tag{2.17}$$

This and (2.14) imply

$$\begin{aligned} \Big(\int_{B_{\rho/2}} |f|^4 d\mu\Big)^{1/2} &\leq C_7\left[b^2\Big(\int_{B_{\rho/2}} d\mu_0\Big)^{1/2} + (C_6 b^{4-p})^{1/2}\right]\Big(\int_{E_b} f^4 d\mu\Big)^{1/2} \\ &\quad + C_4 \int_{B_{\rho/2}} |\overset{0}{\nabla} f|^2 d\mu_0. \end{aligned} \tag{2.18}$$

Choose b large enough (depending on p, A_0) and $\int_{B_{\rho/2}} d\mu_0$ small enough (depending on b). We can absorb the first term on the right-hand side of the above inequality into the left-hand side. On the other hand, for $n = 2$

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and $f \in C_0^\infty(B_{\rho/2})$, we have $\int_{B_{\rho/2}} |\overset{0}{\nabla} f|^2 d\mu_0 = \int_{B_{\rho/2}} |\nabla f|^2 d\mu$, and hence we have

$$\Big(\int_{B_{\rho/2}} |f|^4 d\mu\Big)^{1/2} \leq A_0'\Big(\int_{B_{\rho/2}} |\nabla f|^2 d\mu\Big), \tag{2.19}$$

for $B_{\rho/2} \subset K$.

In general, for any $1 < l < \infty$, $\varphi = e^{\lambda/l} f$, $\varphi \in C_0^\infty(B_\rho)$, doing the same trick as above, one can show that

$$\Big(\int_{B_{\rho/2}} |f|^{2l} d\mu\Big)^{1/l} \leq A_0'\Big(\int_{B_{\rho/2}} |\nabla f|^2 d\mu\Big), \tag{2.20}$$

for $B_{\rho/2} \subset K$ with $\int_{B_{\rho/2}} d\mu_0$ small enough.

Next we study the inequality

$$-\Delta e^{-\lambda} = e^{-\lambda}(\Delta\lambda - |\nabla\lambda|^2) \leq \frac{1}{2} e^{-\lambda}(e^{-2\lambda} R_0 - R).$$

In the case where $R_0 = 0$ on Σ, we have

$$-\Delta f \leq bf \tag{2.21}$$

for $f = e^{-\lambda}$ and $b = \frac{1}{2}|R|$.

But

$$\int f^2 d\mu \leq C \quad \text{and} \quad \int b^2 d\mu \leq C. \tag{2.22}$$

From (2.20), (2.21) and (2.22), we can apply Moser iteration as in [CW, Section 3]) for $n = 2$. It follows that, for $\int_{B_{\rho/2}} d\mu_0$ small enough,

$$\sup_{B_{\rho/4}} e^{-\lambda} \leq C_8.$$

Hence λ satisfies $(*)$.

In the case where $R_0 \neq 0$, we may assume R_0 is negative on some $B_\rho \subset K$ with small enough $\int_{B_{\rho/2}} d\mu_0$. Then the differential inequality (2.21) still holds. Again by the same method as above, we have

$$\sup_{B_{\rho/4}} e^{-\lambda} \leq C_9.$$

Now we have proved that $\lambda(t)$ satisfies $(*)$. □

For a fixed conformal class where volume and $\int_\Sigma R^2 d\mu$ are bounded and such that $h = 0$, one has

Lemma 2.6. *For (S^2, g_0) where volume and $\int_\Sigma R^2 d\mu$ are bounded. If $R_0 \geq 0$, $R_0 \neq 0$ and $R_0 = 0$ in a ball $B(p_0, \rho_0)$, then $\lambda(t)$ does satisfy $(*)$.*

Proof. As before we have

$$-\Delta f \leq bf$$

on $B(p_0, \rho_0)$ and $f = e^{-\lambda}$, $b = \frac{1}{2}|R|$. Then, based on the same arguments as in the previous lemma, we have

$$\sup_{B_\rho} e^{-\lambda} \leq C_{10}$$

for some ball $B_\rho \subset B(p_0, \rho_0)$.

It follows that $\lambda(t)$ satisfies $(*)$. ∎

Then Theorem 1.3 and Theorem 1.4 follow easily from Lemma 2.3, Lemma 2.5, Lemma 2.6 and Proposition 2.2.

References

[C] C. B. Croke, *Some isoperimetric inequalities and eigenvalue estimates*, Ann. Sci. Ec. Norm. Super., **13** (1980), 419–435.

[Ca1] E. Calabi, *Extremal Kähler metrics*, Seminars on Differential Geometry (S. T. Yau, ed.), Princeton Univ. Press and Univ. of Tokyo Press, Princeton, New York (1982), pp. 259–290.

[Ca2] E. Calabi, *Extremal Kähler metrics II*, Differential Geometry and Complex Analysis (I. Chavel and H. M. Farkas, eds.), Springer-Verlag, Berlin-Heidelberg-New York-Tokyo (1985), pp. 95–114.

[Ch1] S.-C. Chang, *Critical Riemannian 4-manifolds*, Math. Z., **214** (1993), 601–625.

[Ch2] S.-C. Chang, *Global existence and convergence of solutions of Calabi flow on surfaces of genus $h \geq 2$*, J. of Mathematics of Kyoto University, **40** (2000), no. 2, 363–377.

[Ch3] S.-C. Chang, *Global existence and convergence of solutions of the Calabi flow on Einstein 4-manifolds*, Nagoya Math. J., **163** (2001), 193–214.

[Ch4] S.-C. Chang, *Recent developments on the Calabi flow*, Contemporary Mathematics, **367** (2005), 17–42.

[Chen] X. X. Chen, *Calabi flow in Riemann surfaces revisited: a new point of view*, IMRN (2001), no. 6, 275–297.

[Chru] P. T. Chruściel, *Semi-global existence and convergence of solutions of the Robinson-Trautman (2-dimensional Calabi) equation*, Commun. Math. Phys., **137** (1991), 289–313.

[CW] S.-C. Chang and J.-T. Wu, *On the existence of extremal metrics for L^2-norm of scalar caurvature on closed 3-manifolds*, J. of Mathematics of Kyoto University, **39** (1999), no. 3, 435–454.

[G] M. J. Gursky, *Compactness of conformal metrics with integral bounds on curvature*, Duke Math. J., **72** (1993), no. 2, 339–367.

[GT] D. Gilbarg and N. S. Trudinger, Elliptic partial differential equation of second order, Springer-Verlag, New York, 1983.

[KW] J. L. Kazdan and F. W. Warner, *Existence and conformal deformation of metrics with prescribed Gaussian and scalar curvatures*, Ann. of Math., **101 (2)** (1975), 317–331.

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