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# Free Pre-Lie Algebras are Free as Lie Algebras

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*Abstract.* We prove that the $\mathfrak{S}$-module PreLie is a free Lie algebra in the category of $\mathfrak{S}$-modules and can therefore be written as the composition of the $\mathfrak{S}$-module Lie with a new $\mathfrak{S}$-module $X$. This implies that free pre-Lie algebras in the category of vector spaces, when considered as Lie algebras, are free on generators that can be described using $X$. Furthermore, we define a natural filtration on the $\mathfrak{S}$-module $X$. We also obtain a relationship between $X$ and the $\mathfrak{S}$-module coming from the anticyclic structure of the PreLie operad.

## 1 Introduction

A pre-Lie algebra is a vector space $V$ endowed with a bilinear map $\curvearrowleft\colon V \otimes V \to V$ such that

$$(x \curvearrowleft y) \curvearrowleft z - x \curvearrowleft (y \curvearrowleft z) = (x \curvearrowleft z) \curvearrowleft y - x \curvearrowleft (z \curvearrowleft y), \tag{1.1}$$

for all $x, y, z \in V$. This kind of algebra has been used for a long time in various areas, see [1] for a survey. In geometry, there is a pre-Lie product on the space of vector fields of any variety endowed with an affine structure. In algebra, this kind of product appears for instance in relation with deformation theory, operads, and vertex algebras.

In a previous article [7], the free pre-Lie algebras have been completely described in terms of rooted trees. The language of operads was a convenient setting for this.

In any pre-Lie algebra, the bracket $[x, y] = x \curvearrowleft y - y \curvearrowleft x$ satisfies the Jacobi identity and therefore defines a Lie algebra on the same vector space. This defines a morphism from the Lie operad to the PreLie operad and a structure of Lie algebra on PreLie in the category of $\mathfrak{S}$-modules.

The article is written using the language of $\mathfrak{S}$-modules. We will start by briefly recalling the main features of this theory. This is the natural category for operads and provides a clean way to deal with free algebras.

The main result is the fact that the $\mathfrak{S}$-module PreLie for pre-Lie algebras is isomorphic (as a Lie algebra in the category of $\mathfrak{S}$-modules) to the composition of the $\mathfrak{S}$-module Lie for Lie algebras with a $\mathfrak{S}$-module $X$.

This implies in turn that free pre-Lie algebras are free Lie algebras for the Lie bracket $[\,\cdot\,,\cdot\,]$. This result has been obtained before with different methods by Foissy [8]. Our result shows that the generators can be described using the $\mathfrak{S}$-module $X$.

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We prove our main result by using a spectral sequence; furthermore, this method gives a natural filtration on the $\mathfrak{S}$-module $X$.

We also give a description of the character of the $\mathfrak{S}$-module $X$. This description implies (at the level of characters) a direct relation between $X$ and the anticyclic structure on the PreLie operad. It is still not clear what the meaning of this link is and how to build an isomorphism that would explain it.

## 2 Notations and $\mathfrak{S}$-Modules

Let $\mathbb{K}$ be a fixed ground field of characteristic zero.

We will work in the Abelian category of $\mathfrak{S}$-modules over $\mathbb{K}$ or sometimes with complexes of $\mathfrak{S}$-modules over $\mathbb{K}$. Recall that a $\mathfrak{S}$-module $P$ can be seen as either a functor $P$ from the category of finite sets and bijections to the category of vector spaces over $\mathbb{K}$ or a sequence $(P(n))_{n\geq 0}$ of $\mathbb{K}$-modules over the symmetric groups $\mathfrak{S}_n$. We will use one or the other of these equivalent definitions freely.

The category of $\mathfrak{S}$-modules is symmetric monoidal for the following tensor product:

$$(F \otimes G)(I) = \bigoplus_{I=J\sqcup K} F(J) \otimes G(K),$$

where $I$ is a finite set and $J \sqcup K$ is the disjoint union of the sets $J$ and $K$.

There is another monoidal structure on $\mathfrak{S}$-modules that is nonsymmetric and defined by

$$(F \circ G)(I) = \bigoplus_{\simeq} F(I/\simeq) \otimes \bigotimes_{J\in I/\simeq} G(J),$$

where $I$ is a finite set and $\simeq$ runs over the set of equivalence relations on $I$.

An operad $Q$ is essentially a monoid in the monoidal category of $\mathfrak{S}$-modules with $\circ$ as tensor product, *i.e.*, the data of a map from $Q \circ Q$ to $Q$ which is associative.

We will consider $\mathfrak{S}$-modules endowed with various kinds of algebraic structures. One can define associative algebras, commutative algebras, exterior algebras, Lie algebras, Hopf algebras, and pre-Lie algebras in the category of $\mathfrak{S}$-modules by using the symmetric monoidal product $\otimes$ and the usual diagrammatic definitions. This works in particular for algebras over any operad $Q$. This kind of object is sometimes called a twisted algebra or a left-module over the corresponding operad. We will simply call them algebras.

If $Q$ is an operad and $P$ is a $\mathfrak{S}$-module, a structure of $Q$-algebra on $P$ is given by a map from $Q \circ P$ to $P$.

If $Q$ is an operad and $P$ is a $\mathfrak{S}$-module, the $\mathfrak{S}$-module $Q \circ P$ is the free $Q$-algebra on $P$. The map from $Q \circ Q \circ P$ to $Q \circ P$ is deduced from the map from $Q \circ Q$ to $Q$ defining the operad $Q$.

Let us consider the following $\mathfrak{S}$-modules.

Let $\lambda$ be a partition of an integer $n$. Then $\lambda$ is associated with an irreducible representation of the symmetric group $\mathfrak{S}_n$, denoted $S^\lambda$. This can be seen as a $\mathfrak{S}$-module concentrated in degree $n$. As a special case, $S^1$ is the trivial module over $\mathfrak{S}_1$.

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Let $S$ be the direct sum of all $\mathfrak{S}$-modules $S^n$ corresponding to trivial modules over the symmetric groups.

Let $T$ be the $\mathfrak{S}$-module defined as the direct sum of all the regular representations $T^n$ over the symmetric groups.

Let $\Lambda$ be the direct sum of all $\mathfrak{S}$-modules $\Lambda^n$ corresponding to alternating modules over the symmetric groups.

There are morphisms of $\mathfrak{S}$-modules $S \otimes S \to S$, $T \otimes T \to T$, and $\Lambda \otimes \Lambda \to \Lambda$ that define associative algebras in the category of $\mathfrak{S}$-modules. These algebras are essentially versions of the polynomial algebra, the tensor algebra, and the exterior algebra.

For example, if $I$ is a finite set, $T(I)$ is the vector space spanned by total orders on $I$. One can concatenate total orders on two sets $I$ and $J$ to get a total order on $I \sqcup J$. This defines the map from $T \otimes T$ to $T$.

Let $P$ be an $\mathfrak{S}$-module. The generating series $f_P$ associated with $P$ is defined by

$$f_P = \sum_{n \geq 0} \dim P(\{1, \ldots, n\}) \frac{x^n}{n!}.$$

There is a more refined object, which is a symmetric function $Z_P$, recording the action of the symmetric groups:

$$Z_P = \sum_{n \geq 0} \chi(P(\{1, \ldots, n\})),$$

where $\chi$ is the image of the character of the $\mathfrak{S}_n$-module $P(\{1, \ldots, n\})$ in the ring of symmetric functions. This symmetric function is a complete invariant in the sense that two $\mathfrak{S}$-modules are isomorphic if and only if they share the same symmetric function.

For more information on $\mathfrak{S}$-modules and operads, see [10, 11].

## 2.1 Relation with Usual Algebra

Each $\mathfrak{S}$-module $P$ defines a functor from vector spaces to vector spaces as follows:

$$P(V) = \bigoplus_{n \geq 0} P(n) \otimes_{\mathfrak{S}_n} V^{\otimes n}.$$

The tensor product $\otimes$ of $\mathfrak{S}$-modules has the property

$$(F \otimes G)(V) \simeq F(V) \otimes G(V),$$

and the tensor product $\circ$ of $\mathfrak{S}$-modules has the property

$$(F \circ G)(V) \simeq F(G(V)),$$

where $V$ is a vector space.

If the $\mathfrak{S}$-module $P$ is an operad and $V$ is a vector space, then $P(V)$ is the free $P$-algebra on $V$.

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Then if $P$ is a $\mathfrak{S}$-module with such an algebraic structure, *i.e.*, a $Q$-algebra for some operad $Q$, the vector space $P(V)$ is endowed with the corresponding structure of $Q$-algebra in the category of vector spaces.

The classical algebra structures on $S(V)$, $T(V)$ and $\Lambda(V)$ (respectively the polynomial algebra, the tensor algebra, and the exterior algebra) come from the morphisms of $\mathfrak{S}$-modules $S \otimes S \to S$, $T \otimes T \to T$, and $\Lambda \otimes \Lambda \to \Lambda$ that were introduced above.

## 3 The Pre-Lie $\mathfrak{S}$-Module $W$

A rooted tree on a set $I$ is a connected and simply-connected graph with vertex set $I$ together with a distinguished element of $I$ called the root. Note that a rooted tree can be canonically decomposed into its root and a set of rooted trees obtained when the root is removed. Let $W$ be the $\mathfrak{S}$-module which maps a finite set $I$ to the vector space spanned by the set of rooted trees on $I$.

It has been shown in [7] that $W$ is the operad describing pre-Lie algebras and in particular that $W(V)$ is the free pre-Lie algebra on a vector space $V$. There is a natural morphism $\curvearrowleft : W \otimes W \to W$ corresponding to the pre-Lie product. This means that $W$ is a pre-Lie algebra in the category of $\mathfrak{S}$-modules.

Let us recall briefly the definition of the map $\curvearrowleft$.

Given a rooted tree $S$ on a finite set $I$ and a rooted tree $T$ on a finite set $J$, one can define $S \curvearrowleft T$ (which is a sum of rooted trees on the set $I \sqcup J$) as follows: consider the disjoint union of $S$ and $T$, then add an edge between the root of $T$ and a vertex of $S$ and sum with respect to the chosen vertex of $S$. The root is taken to be that of $S$.

As there exists a morphism from the Lie operad to the PreLie operad induced by

$$[x, y] \mapsto x \curvearrowleft y - y \curvearrowleft x,$$

there is a morphism $[\cdot, \cdot] : W \otimes W \to W$ that makes $W$ into a Lie algebra in the category of $\mathfrak{S}$-modules.

The universal enveloping algebra of a Lie algebra is a well-defined associative algebra in the category of $\mathfrak{S}$-modules, which has most of the classical properties of the usual construction for vector spaces [13]. Let $\mathrm{U}(W)$ be the universal enveloping algebra of $W$.

In the sequel, a *$W$-module* is a module over the Lie algebra $W$ or equivalently a right module over $\mathrm{U}(W)$. This notion is defined in the obvious way in the symmetric monoidal category of $\mathfrak{S}$-modules.

There are two distinct $W$-module structures on the $\mathfrak{S}$-module $W$. The first one is given by the adjoint action of the Lie algebra $W$ on itself. It will be denoted by $W_{\mathrm{ad}}$. The other one is given by the pre-Lie product $\curvearrowleft$ and will be denoted by $W_{\mathrm{pL}}$. The fact that the map $\curvearrowleft$ is a right action results from the pre-Lie axiom (1.1).

Let us now recall results from [7, Thm. 3.3]. There is an isomorphism $\psi$ of $W$-modules between $W_{\mathrm{pL}}$ and the free $\mathrm{U}(W)$-module on the $\mathfrak{S}$-module $S^1$. By the description of $W$ in terms of rooted trees and the decomposition of a rooted tree into its root and its set of subtrees, the module $W_{\mathrm{pL}}$ is isomorphic as an $\mathfrak{S}$-module to $S^1 \otimes S \circ W$. The $\mathfrak{S}$-module $S \circ W$ is spanned by forests of rooted trees. The isomorphism $\psi$ can be written as $\mathrm{Id} \otimes \phi$ between $S^1 \otimes S \circ W$ and $S^1 \otimes \mathrm{U}(W)$. This defines

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an isomorphism $\phi$ of $W$-modules from $S \circ W$ to $\mathsf{U}(W)$, where the action $\triangleleft$ of $W$ on $S \circ W$ is deduced from the product $\curvearrowleft$ and is given by

$$(t_1 t_2 \dots t_k) \triangleleft t = t_1 t_2 \dots t_k t + \sum_{i=1}^{k} t_1 t_2 \dots t_{i-1} (t_i \curvearrowleft t) t_{i+1} \dots t_k, \tag{3.1}$$

where the $t$'s stand for some rooted trees.

From now on we will identify by the mean of $\phi$ the $W$-module $\mathsf{U}(W)$ with the $W$-module $S \circ W$ with this action $\triangleleft$. One can see from the explicit shape of the action $\triangleleft$ that the $W$-module $\mathsf{U}(W)$ has a decreasing filtration by the number of connected components of the forest.

The associated graded module is given by the action

$$(t_1 t_2 \dots t_k) \curvearrowleft t = \sum_{i=1}^{k} t_1 t_2 \dots t_{i-1} (t_i \curvearrowleft t) t_{i+1} \dots t_k, \tag{3.2}$$

where we have slightly abused notation by using the symbol $\curvearrowleft$ for the action. This is easily seen to be the natural $W$-module structure on $S \circ W_{\mathrm{pL}}$ obtained by extending the $W$-module $W_{\mathrm{pL}}$ by derivation. This is also the symmetric algebra on the $W$-module $W_{\mathrm{pL}}$.

One can see that there is only one other term in the filtered action (3.1), which is of degree 1 with respect to the graduation by the number of connected components and is just a product:

$$t_1 t_2 \dots t_k \otimes t \mapsto t_1 t_2 \dots t_k t. \tag{3.3}$$

## 4 Two Spectral Sequences

Let us consider the usual reduced complex computing the homology of the Lie algebra $W$ with coefficients in the $W$-module $\mathsf{U}(W)$. This is the tensor product of the exterior algebra on $W$ with the module $\mathsf{U}(W) \simeq S \circ W$.

As a $\mathfrak{S}$-module, this complex is $(S \circ W) \otimes (\Lambda \circ W)$. The differential $\partial$ is the usual Chevalley–Eilenberg map, which uses both the bracket or the action as a contraction:

$$\partial(x_1 x_2 \dots x_k \otimes y_1 \wedge y_2 \wedge \dots \wedge y_\ell) = \sum_{j=1}^{\ell} \pm (x_1 x_2 \dots x_k) \triangleleft y_j \otimes y_1 \wedge \dots \wedge \widehat{y_j} \wedge \dots \wedge y_\ell$$
$$+ \sum_{1 \le i < j \le \ell} \pm x_1 x_2 \dots x_k \otimes [y_i, y_j] \wedge y_1 \wedge \dots \wedge \widehat{y_i} \wedge \dots \wedge \widehat{y_j} \wedge \dots \wedge y_\ell,$$

where the signs are given by the Koszul sign rule.

But $\mathsf{U}(W)$ is a free $W$-module by definition, hence the homology is concentrated in homological degree 0 and is given by the $\mathfrak{S}$-module $S^0$.

Let us now use the filtration on $\mathsf{U}(W)$ to define two spectral sequences computing the same homology. In fact, we will first define a bicomplex and then consider its two associated spectral sequences.

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We have to introduce a triple grading on the complex $(S \circ W) \otimes (\Lambda \circ W)$. Let us denote by $n$ the internal degree of the $\mathfrak{S}$-module (seen as a collection of modules over $\mathfrak{S}_n$), by $p$ the degree with respect to the graduation of the symmetric algebra and by $q$ the homological degree with respect to the graduation in the exterior algebra. As $W$ has no component with $n = 0$, one has $p \geq 0$, $q \geq 0$, and $p + q \leq n$.

Let $r$ be $n - p - q$. We will use the triple grading by $(n, p, r)$. The differential $\partial$ is of degree 0 with respect to the first grading. Hence one can consider each part of fixed first degree $n$ separately.

The differential $\partial$ on $(S \circ W) \otimes (\Lambda \circ W)$ decomposes into two pieces according to the decomposition of the action $\triangleleft$ into the action $\curvearrowleft$ coming from (3.2) plus another term coming from (3.3).

The first part is defined as follows:

$$\partial_{\mathrm{pL}}(x_1 x_2 \dots x_k \otimes y_1 \wedge y_2 \wedge \dots \wedge y_\ell) = \sum_{j=1}^{\ell} \pm (x_1 x_2 \dots x_k) \curvearrowleft y_j \otimes y_1 \wedge \dots \wedge \widehat{y_j} \wedge \dots \wedge y_\ell$$
$$+ \sum_{1 \leq i < j \leq \ell} \pm x_1 x_2 \dots x_k \otimes [y_i, y_j] \wedge y_1 \wedge \dots \wedge \widehat{y_i} \wedge \dots \wedge \widehat{y_j} \wedge \dots \wedge y_\ell.$$

One can recognize the differential $\partial_{\mathrm{pL}}$ of degree $(0, 0, 1)$ which computes the homology of the Lie algebra $W$ with coefficients in the graded $W$-module $(S \circ W_{\mathrm{pL}}, \curvearrowleft)$.

The remaining terms of the differential are

$$\partial_{\mathrm{K}}(x_1 x_2 \dots x_k \otimes y_1 \wedge y_2 \wedge \dots \wedge y_\ell) = \sum_{j=1}^{\ell} \pm x_1 x_2 \dots x_k y_j \otimes y_1 \wedge \dots \wedge \widehat{y_j} \wedge \dots \wedge y_\ell.$$

The map $\partial_{\mathrm{K}}$ has degree $(0, 1, 0)$ and no longer uses the bracket of the Lie algebra $W$. This is nothing but the differential in the Koszul complex relating the exterior algebra $\Lambda \circ W$ on $W$ and the symmetric algebra $S \circ W$ on $W$ [12].

The two differentials $\partial_{\mathrm{pL}}$ and $\partial_{\mathrm{K}}$ are of degree $(0, 0, 1)$ and $(0, 1, 0)$ and their sum is also a differential. Hence they define a bicomplex, and one can consider the two spectral sequences associated with this bicomplex.

***Proposition 4.1*** *The spectral sequence beginning with $\partial_{\mathrm{K}}$ degenerates at the first step.*

**Proof** As it is known that the exterior and symmetric algebras are Koszul dual of each other and Koszul, the homology of $\partial_{\mathrm{K}}$ is concentrated in homological degree 0 and is given by $S^0$. ■

Before studying the other spectral sequence, one needs the two following results. Let us first recall a classical lemma.

***Lemma 4.2*** *Let $A$ be a Hopf algebra and let $N$ be a right $A$-module. Then $N \otimes A$ is isomorphic as a right $A$-module to the free right $A$-module generated by $N$.*

**Proof** The argument uses the antipode of $A$ to define an isomorphism, see for instance [9, §3.6 and §3.7]. ■

We will also need the following property of some $W$-modules.

***Proposition 4.3*** *Let $\lambda$ be a nonempty partition. Then $S^\lambda \circ W_{\mathrm{pL}}$ is a projective $\mathsf{U}(W)$-module.*

**Proof** Recall that $W_{\mathrm{pL}}$ is isomorphic to the free $\mathsf{U}(W)$-module $S^1 \otimes \mathsf{U}(W)$.

Hence, for each integer $k \geq 1$, the $W$-module $T^k \circ W_{\mathrm{pL}}$ is isomorphic to the module $T^k \otimes (T^k \circ \mathsf{U}(W))$, where $W$ acts on the right factor only. Here, we have used the property of the $\mathfrak{S}$-module $T$ that $T \circ (A \otimes B) \simeq (T \circ A) \otimes (T \circ B)$.

As $\mathsf{U}(W)$ is a Hopf algebra, one can apply Lemma 4.2, which implies that $T^k \circ \mathsf{U}(W)$ is a free $\mathsf{U}(W)$-module. It follows that $T^k \circ W_{\mathrm{pL}}$ is also a free $\mathsf{U}(W)$-module.

Now, for each partition $\lambda$, $S^\lambda$ is usually defined as a direct factor of $T^{|\lambda|}$, where $|\lambda|$ is the size of $\lambda$. So $S^\lambda \circ W$ is a direct factor of $T^{|\lambda|} \circ W$. As $T^{|\lambda|} \circ W$ is a free $\mathsf{U}(W)$-module, $S^\lambda \circ W$ is a projective $\mathsf{U}(W)$-module. ■

***Remark*** It may be that all the $\mathsf{U}(W)$-modules $S^\lambda \circ W$ are in fact free.

Let us now go back to the bicomplex. To illustrate the computation of the horizontal spectral sequence starting with $\partial_{\mathrm{pL}}$, we will draw the component of fixed first degree $n$ of the bicomplex in the first quadrant, with the $p$ grading increasing from bottom to top and the $r$ grading from left to right.

We have the following description of the homology with respect to $\partial_{\mathrm{pL}}$.

The bottom line ($p = 0$) of the bicomplex is the complex $S^0 \otimes (\Lambda \circ W)$ computing the homology of the Lie algebra $W$ with coefficients in the trivial module. This is what we would like to compute.

The other lines ($p > 0$) of the bicomplex are the complexes $(S^p \circ W_{\mathrm{pL}}) \otimes (\Lambda \circ W)$ computing the homology of the Lie algebra $W$ with coefficients in the modules $(S^p \circ W_{\mathrm{pL}})$. As we know that these modules are projective by Proposition 4.3, the homology is concentrated in degree $q = 0$ *i.e.*, $r = n - p$.

So the first step of the spectral sequence looks like

$$\begin{array}{cccccc} * & & & & & \\ 0 & * & & & & \\ 0 & 0 & * & & & \\ 0 & 0 & 0 & * & & \\ 0 & 0 & 0 & 0 & * & \\ * & * & * & * & * & *, \end{array}$$

where a $*$ represents a possibly nonzero homology group.

As the spectral sequence converges to $S^0$ because the homology of the total complex is $S^0$, one deduces from the shape above that the homology of the bottom row is concentrated in degree $q = 1$ *i.e.*, $r = n - 1$. So in fact, the first step looks like

$$\begin{array}{cccccc} * & & & & & \\ 0 & * & & & & \\ 0 & 0 & * & & & \\ 0 & 0 & 0 & * & & \\ 0 & 0 & 0 & 0 & * & \\ 0 & 0 & 0 & 0 & * & 0. \end{array} \tag{4.1}$$

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Let us denote these homology groups by $H(n, p, r)$.

***Proposition 4.4*** *Let $n \geq 1$. The dimension of $H(n, 0, n-1)$ is $(n-1)^{n-1}$. The dimension of $H(n, p, n-p)$ is $\binom{n}{p}(p-1)(n-1)^{n-p-1}$ for $1 \leq p \leq n$. There is a filtration on the unique nonvanishing homology group $H(n, 0, n-1)$ of the bottom line whose graded pieces are isomorphic to the homology groups $H(n, p, n-p)$ of the other lines.*

**Proof** Using once again the fact that the spectral sequence converges to $S^0$, one can see that the component of first degree $n$ of the horizontal spectral sequence cannot degenerate before the $n$-th step and that the successive pages of the spectral sequence provide the expected filtration on $H(n, 0, n-1)$. More precisely, all the higher differentials of the spectral sequence have to be surjective, and their successive kernels define decreasing subspaces of $H(n, 0, n-1)$, hence a filtration.

Let $f_W$ be the generating series associated with $W$:

$$f_W = \sum_{n \geq 1} n^{n-1} \frac{x^n}{n!}. \tag{4.2}$$

The function $-f_W(-x)$ is usually called the Lambert W function [4].

Let us consider now the graded $\mathfrak{S}$-module $S \circ W$. We introduce a variable $s$ to encode the $p$ grading. The associated generating series is $f_{S \circ W} = e^{s f_W}$.

For the graded exterior $\mathfrak{S}$-module $\Lambda \circ W$, we introduce a variable $-t$ to encode the $q$ grading. The minus sign in front of $t$ is convenient here ; specialization at $t = 1$ gives the Euler characteristic. The associated generating series is $f_{\Lambda \circ W} = e^{-t f_W}$.

The associated generating series for the bicomplex is then given by

$$f_{(S \circ W) \otimes (\Lambda \circ W)} = e^{s f_W} e^{-t f_W} = e^{(s-t) f_W} = 1 + (s-t) \sum_{n \geq 1} (n + s - t)^{n-1} \frac{x^n}{n!}.$$

Here the Taylor expansion is a classical result, see for example [4, Formula (2.36)].

Taking the horizontal Euler characteristic is given by substituting $t = 1$:

$$e^{s f_W} e^{-f_W} = e^{(s-1) f_W} = 1 + (s-1) \sum_{n \geq 1} (n + s - 1)^{n-1} \frac{x^n}{n!}. \tag{4.3}$$

As we know by (4.1) where the horizontal homology of the bicomplex is concentrated, computing the Euler characteristic is enough to get the dimension of the homology.

Let us first compute the constant term of (4.3) with respect to $s$. One gets

$$1 - \sum_{n \geq 1} (n-1)^{n-1} \frac{x^n}{n!}.$$

Hence the dimension of $H(n, 0, n-1)$ is $(n-1)^{n-1}$ as expected.

One can also easily compute the coefficient of $s^p$ for $p > 0$ and get the expected formula for the dimension of $H(n, p, n-p)$. ■

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***Remark*** The filtration on $H(n,0,n-1)$ with quotients $H(n,p,n-p)$ for $1 \leq p \leq n$ gives a nice interpretation of the classical identity

$$(n-1)^{n-1} = \sum_{p=1}^{n} \binom{n}{p} (p-1)(n-1)^{n-p-1},$$

which can be found for instance in [3, Prop. 2] and goes back to Cayley [2].

## 5 Description of the Symmetric Group Action

We will denote by $X$ the $\mathfrak{S}$-module corresponding to the collection of $\mathfrak{S}_n$-modules $H(n,0,n-1)$ for $n \geq 1$. Note that one does not include the degree 0 component in this definition.

One can get not just the dimensions of $H(n,0,n-1)$ but a description of the action of the symmetric groups on $X$.

In [6, Prop. 7.2], the symmetric function $Z_{\Lambda \circ W}$ was computed with a parameter $-t$ accounting for the cohomological grading. Putting $t = 1$ in the formula there and removing the constant term, one finds the symmetric function for $X$.

Let us denote by $p_\lambda$ the power-sum symmetric functions. If $\lambda$ is a partition, let $\lambda_k$ be the number of parts of size $k$ in $\lambda$ and $f_k(\lambda)$ be the number of fixed points of a permutation of type $\lambda$. Let $z_\lambda$ be the product over $k$ of $k^{\lambda_k}(\lambda_k)!$, a classical constant associated with a partition.

***Proposition 5.1*** *The symmetric function $Z_{\Lambda \circ W}$ has the following expression*

$$1 + (-t) \sum_{\lambda, |\lambda| \geq 1} (\lambda_1 - t)^{\lambda_1 - 1} \prod_{k \geq 2} \left( (f_k(\lambda) - t^k)^{\lambda_k} - k\lambda_k (f_k(\lambda) - t^k)^{\lambda_k - 1} \right) \frac{p_\lambda}{z_\lambda},$$

*and the symmetric function $Z_X$ has the following expression*

$$\sum_{\lambda, |\lambda| \geq 1} (\lambda_1 - 1)^{\lambda_1 - 1} \prod_{k \geq 2} \left( (f_k(\lambda) - 1)^{\lambda_k} - k\lambda_k (f_k(\lambda) - 1)^{\lambda_k - 1} \right) \frac{p_\lambda}{z_\lambda}, \tag{5.1}$$

*where the sums are over the set of non-empty partitions $\lambda$.*

Recall that it was proved in [5] that the PreLie operad is an anticyclic operad. This implies in particular that there is an action of the symmetric group $\mathfrak{S}_{n+1}$ on the space PreLie($n$). Let us denote by $\widehat{W}$ the corresponding $\mathfrak{S}$-module and let us compute the symmetric function $Z_{\widehat{W}}$ describing this action of the symmetric group $\mathfrak{S}_{n+1}$ on the space PreLie($n$).

From [5, Eq. (50)], this symmetric function is characterized by the relation

$$1 + Z_{\widehat{W}} = p_1(1 + Z_W + 1/Z_W). \tag{5.2}$$

***Proposition 5.2*** *The symmetric function $Z_{\widehat{W}}$ is given by*

$$\sum_{\lambda, |\lambda| \geq 1, \lambda_1 \neq 1} (\lambda_1 - 1)^{\lambda_1 - 2} \prod_{k \geq 2} \left( (f_k(\lambda) - 1)^{\lambda_k} - k\lambda_k (f_k(\lambda) - 1)^{\lambda_k - 1} \right) \frac{p_\lambda}{z_\lambda}. \tag{5.3}$$

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**Proof** One has (see [6])

$$\Big(p_1 \exp\Big(-\sum_{k\geq 1} p_k/k\Big)\Big) \circ Z_W = p_1.$$

Let us introduce new variables $y_\ell = p_\ell \circ Z_W$, for $\ell \geq 1$. Then the inverse map is given by $p_\ell = y_\ell \exp(-\sum_k y_{k\ell}/k)$.

Let $\lambda$ be a partition with longest part at most $r$. To compute the coefficient of $p_\lambda$ in the symmetric function $1 + Z_{\widehat{W}}$, it is enough to compute the residue

$$\iiint (1 + Z_{\widehat{W}}) \prod_{i=1}^{r} \frac{dp_i}{p_i^{\lambda_i+1}},$$

which is equal by formula (5.2) to

$$\iiint p_1(1 + Z_W + 1/Z_W) \prod_{i=1}^{r} \frac{dp_i}{p_i^{\lambda_i+1}}.$$

One can assume without restriction that all variables $y_j$ and $p_j$ vanish if $j > r$.

We will change the variables to get a residue in the variables $y$ instead. One has to use the formula

$$\prod_{i=1}^{r} dp_i = \exp\Big(-\sum_i \sum_k y_{ik}/k\Big) \prod_{i=1}^{r}(1 - y_i)dy_i.$$

We therefore have to compute the residue

$$\iiint y_1 \exp\Big(-\sum_k y_k/k\Big) \big(1 + y_1 + 1/y_1\big) \exp\Big(\sum_i \lambda_i \sum_k y_{ik}/k\Big) \prod_{i=1}^{r} \frac{(1 - y_i)}{y_i^{\lambda_i+1}} dy_i.$$

Gathering the exponentials and reversing the order of summation, one finds

$$\iiint \exp\Big(\sum_k (f_k(\lambda) - 1)y_k/k\Big) (1 + y_1 + y_1^2) \prod_{i=1}^{r} \frac{(1 - y_i)}{y_i^{\lambda_i+1}} dy_i.$$

This integral decomposes as a product of residues in each variable $y_i$.

Let us discuss first the integral with respect to $y_1$:

$$\oint \exp\big((\lambda_1 - 1)y_1\big) \frac{(1 - y_1^3)}{y_1^{\lambda_1+1}} dy_1.$$

This is the sum of two terms:

$$\oint \exp\big((\lambda_1 - 1)y_1\big) \frac{1}{y_1^{\lambda_1+1}} dy_1 = \frac{(\lambda_1 - 1)^{\lambda_1}}{\lambda_1!},$$

and

$$\oint \exp\big((\lambda_1 - 1)y_1\big)\,\frac{(-y_1^3)}{y_1^{\lambda_1+1}}\,dy_1. = -\frac{(\lambda_1-1)^{\lambda_1-3}}{(\lambda_1-3)!}.$$

Note that some care must be taken in the second term when $\lambda_1 \leq 2$. The resulting sum is zero if $\lambda_1 = 1$ and

$$\frac{(\lambda_1-1)^{\lambda_1-2}}{\lambda_1!} \text{ if } \lambda_1 \neq 1.$$

Let us then discuss the integral with respect to $y_k$ for $k \geq 2$.

$$\oint \exp\big((f_k(\lambda)-1)y_k/k\big)\,\frac{(1-y_k)}{y_k^{\lambda_k+1}}\,dy_k.$$

This is also the sum of two terms

$$(f_k(\lambda)-1)^{\lambda_k}/k^{\lambda_k}/\lambda_k! - (f_k(\lambda)-1)^{\lambda_k-1}/k^{\lambda_k-1}/(\lambda_k-1)!.$$

This can be rewritten as

$$\frac{(f_k(\lambda)-1)^{\lambda_k} - k\lambda_k(f_k(\lambda)-1)^{\lambda_k-1}}{k^{\lambda_k}\lambda_k!}.$$

Gathering all terms and removing the contribution of the empty partition gives the result. ■

From this, one deduces the following relation.

***Theorem 5.3*** *One has* $Z_X - p_1 = (p_1\partial_{p_1} - \mathrm{Id})Z_{\widehat{W}}$, i.e., *the action of* $\mathfrak{S}_n$ *on* $X(n)$ *is obtained from the action of* $\mathfrak{S}_n$ *on* $\widehat{W}(n)$ *by taking the inner tensor product with the reflection module of dimension* $n-1$ *of* $\mathfrak{S}_n$.

**Proof** This is a computation using Proposition 5.1 and 5.2. Indeed, the operator $(p_1\partial_{p_1} - \mathrm{Id})$ acts by multiplication of $p_\lambda$ by $\lambda_1 - 1$. There is one subtle point to check, though. In formula (5.3) for $Z_{\widehat{W}}$, the summation is over all partitions of size at least 2 with $\lambda_1 \neq 1$, whereas in Formula (5.1) for $Z_X - p_1$, the summation is over all partitions of size at least 2 without further condition. Let $\lambda$ be any partition of size at least 2 with $\lambda_1 = 1$. Let $k$ be the size of the next-to-smallest part of $\lambda$. Then $f_k(\lambda) = k\lambda_k + 1$, and hence the expression $(f_k(\lambda)-1)^{\lambda_k} - k\lambda_k(f_k(\lambda)-1)^{\lambda_k-1}$ vanishes. It follows that all such partitions do not contribute to $Z_X - p_1$ and that one can deduce the expected equation.

That the operator $(p_1\partial_{p_1} - \mathrm{Id})$ corresponds to the inner tensor product by the reflection module is a classical fact in the theory of symmetric functions. ■

## 6 Freeness from Homology Concentration

We will use the knowledge of the homology of the bottom line of the bicomplex to show that the free pre-Lie algebracs are free as Lie algebras.

## 6.1 General Setting

Let $P$ be an operad and assume that $P(1) = \mathbb{K}1$ and let $P^+$ be the $\mathfrak{S}$-module such that $P = \mathbb{K}1 \oplus P^+$. Let $A$ be a $P$-algebra in the category of $\mathfrak{S}$-modules.

The structure of $P$-algebra on $A$ is given by a morphism $\mu : P \circ A \to A$. Let us define for each $k \geq 0$ a subspace $A_{\geq k}$ of $A$. Let $A_{\geq 0}$ be $A$. By induction, let $A_{\geq k}$ be the image by $\mu$ of $P^+ \circ A_{\geq k-1}$. This is a decreasing filtration of $A$ by subspaces. By construction, this filtration is in fact a filtration of $P$-algebras. We will furthermore assume that this filtration is separating, which is true for instance if $A$ has some auxiliary grading concentrated in positive degrees.

Let us define $H_0(A)$ to be the degree 0 component $A_{\geq 0}/A_{\geq 1}$ of the associated graded $P$-algebra $grA$. Let us choose a section of $H_0(A)$ in $A$. Let $\mathrm{Free}_P(H_0(A))$ be the free $P$-algebra on $H_0(A)$. Then there exists a unique morphism $\theta$ of $P$-algebras from the free $P$-algebra $\mathrm{Free}_P(H_0(A))$ to $A$ extending the chosen section.

***Proposition 6.1*** *The morphism $\theta$ is surjective.*

**Proof** Because the filtration is assumed to be separating, it is enough to prove that the associated morphism of graded algebras is surjective.

This is done by induction on the degree associated with the filtration. It is true in degree 0 because the map $\theta$ comes from a section.

Now let $[x]$ be a class in $grA_k$ with $k \geq 1$. Pick a representative $x \in A_{\geq k}$ of the class $[x]$. Then by hypothesis, $x$ can be written as a sum $\sum_a \mu(y_a, z_a)$, where $y_a \in P^+$ and $z_a \in A_{\geq k-1}$.

Then it follows from the fact that we used a filtration of $P$-algebras that the class $[x]$ itself can be written as $\sum_a \mu(y_a, [z_a])$, where $y_a \in P^+$ and $z_a \in A_{\geq k-1}$. Each class $[z_a]$ in $grA_{k-1}$ belongs to the image of $\theta$ by induction. Therefore $[x]$ belongs to the image of $\theta$. ∎

To show that the morphism $\theta$ is an isomorphism, an argument of equality of dimension (in some appropriate sense) between $A$ and $\mathrm{Free}_P(H_0(A))$ is therefore sufficient.

## 6.2 Application

Let us apply the general setting above to the case that we are studying. The operad $P$ is the operad Lie. The $P$-algebra $A$ is the $\mathfrak{S}$-module $W$. As $W$ has no component in degree 0, one can apply the previous construction. The space $X$ is exactly the homology group $H_0(A)$.

***Theorem 6.2*** *The $\mathfrak{S}$-module $W$ is isomorphic as a Lie algebra in the category of $\mathfrak{S}$-modules to the free Lie algebra* $\mathrm{Lie} \circ X$ *on the $\mathfrak{S}$-module $X$.*

***Corollary 6.3*** *For any vector space $V$, the free pre-Lie algebra on $V$ is isomorphic as a Lie algebra to the free Lie algebra on $X(V)$.*

**Proof** As follows from Subsection 6.1, there is a map $\theta$ from $\mathrm{Lie} \circ X$ to PreLie which is surjective. To prove the theorem, it suffices to compare the dimensions.

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Let $f_X$ be the generating series

$$\sum_{n\geq 1}(n-1)^{n-1}\frac{x^n}{n!}, \tag{6.1}$$

which is associated with the $\mathfrak{S}$-module $X$ of indecomposable elements of PreLie as a Lie algebra. We have to check that the generating series of the free Lie algebra $\mathrm{Lie}\circ X$ on $X$ is equal to $f_W$. As the series $f_{\mathrm{Lie}}$ is $-\log(1-x)$, this amounts to the equality $-\log(1-f_X) \stackrel{?}{=} f_W$, which can be rewritten as $e^{-f_W} \stackrel{?}{=} 1 - f_X$. The constant term in $x$ is 1 on both sides. Therefore it is enough to compare the derivatives with respect to $x$:

$$-f_W' e^{-f_W} \stackrel{?}{=} -f_X' = -(1 + x f_W'),$$

where the right equality is by comparison of the Taylor expansions (4.2) and (6.1). But we have by definition $f_W = xe^{f_W}$, hence $f_W' = e^{f_W} + x f_W' e^{f_W}$. This proves the expected equality and hence the Theorem. ■

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