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First Steps of Local Contact Algebra

Dedicated to H. S. M. Coxeter

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Abstract. We consider germs of mappings of a line to contact space and classify the first simple singularities up to the action of contactomorphisms in the target space and diffeomorphisms of the line. Even in these first cases there arises a new interesting interaction of local commutative algebra with contact structure.

1 Introduction

The belief that all simple (having no continuous moduli) objects in nature are controlled by Coxeter groups is a kind of religion. The corresponding theorem in singularity theory is due to A. B. Givental [9]. It classifies simple singularities of caustics and wave fronts, defined by projections of Lagrange and Legendre subvarieties of symplectic and contact manifolds, in terms of Coxeter euclidean reflections groups, extending to the case of singular varieties my previous A-D-E-classification [1] (corresponding to smooth submanifolds).

The present work is an attempt to start the classification of singular simple curves in contact manifolds.

The idea that every reasonable mathematical theory should have symplectic and contact versions is also based on the success of Coxeter's extension of linear algebra (considered as the theory of the root systems A) to other mirror configurations. The application of this idea to the calculus has led to the foundation of symplectic and contact topologies (see [2]).

In the present article the same idea is applied to a modest local problem. It is astonishing that this problem—the classification of simple curves in a contact space—is rather difficult and leads to interesting new interactions of the local commutative algebra with symplectic and contact structures.

A contact structure on an odd-dimensional manifold is a field of hyperplanes in the tangent spaces, which is completely nonintegrable. The classical Darboux-Givental theorem claims that a germ of a smooth submanifold of a contact manifold is well defined (up to a contactomorphism preserving the contact structure) by the induced structure on the submanifold (see [8]).

The present paper shows that at a singular point of a curve there exist more invariants—some ghost of the contact structure persists. It would be interesting to describe this ghost algebraically, in terms of the local algebra of the singularity and of the Poisson brackets. Such a formula is missing in the paper. I just calculate the normal forms, showing the existence of the ghost.

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A singularity of a curve is a germ of a smooth (holomorphic, formal, ...) mapping of the line into a smooth (holomorphic, formal, ...) manifold at a singular point (where the derivative of the mapping vanishes). The singularities are considered up to the diffeomorphisms of the images in the target manifold (that is, up to the right-left equivalence in terms of singularity theory).

A singularity is called *simple* if all the singularities of the neighbouring mappings belong to a finite set of equivalence classes. The classification of simple singularities of curves is described in [3].

The codimension of the set of nonsimple singularities of mappings of a line to $\mathbf{C}^N$ equals to $6N-6$ if N is large (nonsimple singularities first occur at some points of some curve of generic families, if the number of parameters reaches $6N-7$).

The list of simple singularities of curves starts with the series

$$A_2 \leftarrow A_4 \leftarrow \cdots, \quad A_{2k} = (s^2, s^{2k+1}, 0, \dots, 0).$$

These are all the singularities whose Taylor series contain nonzero quadratic terms (neglecting those which are infinitely degenerate). The list of the simple singularities contains also the (finitely degenerate) singularities, whose Taylor expansions start with nonzero cubic terms (they form two 3-indices series, related to E_6 and to E_8). There are also seven 1-index series of finitely degenerate singularities, whose 6-jet has the form $(s^4, s^6, 0, \dots, 0)$.

The remaining 32 *sporadic* simple singularities have Taylor series starting with $(s^4, s^5, 0, \dots, 0)$ (7 curves), with $(s^4, s^7, 0, \dots, 0)$ (15 curves), with $(s^4, s^9, 0, \dots, 0)$ (1 curve), with $(s^5, 0, \dots, 0)$ (6 curves) and with $(s^6, 0, \dots, 0)$ (3 curves). They live in spaces of dimension at least 2 (5 curves), at least 3 (13 curves), at least 4 (9 curves), at least 5 (4 curves), and at least 6 (one curve).

I mention here this information, hoping that it might help someone to find the missing relation of this list of simple curves to other simple objects. The special role of the number 6 suggests that the affine version of E_6 might be responsible for the first nonsimple singularities generation (like in the A-D-E theory) and the cardinality 32 suggest the possibility of a relation to the complex reflection groups.

The main results of the present paper are the classifications of the simple singularities of integral curves of series A_{2k} in contact manifolds and of the nonintegral curves with a semicubical singularity A_2. The solution of the last problem is based on the study of the simplest objects of the symplectic and contact local algebra—of the Diraciens of the ideals in the algebras of functions on symplectic manifolds and of their contact versions.

2 Integral Curves of Series A

The A_{2k}-singularity is the singularity diffeomorphic to that of the plane curve $x^{2k+1} = y^2$ at the origin.

Consider the contact space $\mathbf{C}^{2n+1}$ equipped with the Darboux coordinate system and with the contact structure $dz - \sum p_i dq_i$. Consider the integral curves $A_{2k,r}$ whose projections to the symplectic $2n$-space are

$$A_{2k,0} : (q_1 = s^2, p_1 = s^a, p_i = q_i = 0 \text{ for } i > 1);$$

$$A_{2k,r} : (q_1 = s^2, q_2 = s^a, p_1 = s^b, p_2 = 0, p_i = q_i = 0 \text{ for } i > 2),$$

where $a = 2k+1$, $b = a+2r$, $r > 0$; of course $z = (2/b+2)s^{b+2}$ along such integral curves.

Theorem 1 *Almost every integral curve in a contact space whose quadratic term is nonzero is simple and contact equivalent to one of the curves $A_{2k,r}$ $(0 \le r \le 2k)$ above.*

The exceptional curves are nonsimple and form a subset of codimension infinity consisting of infinitely degenerate curves.

One might replace s^b by 0 in the normal form $A_{2k,2k}$—the corresponding integral curves are diffeomorphic.

The stratum $A_{2k,r}$ has codimension $(2n-1)(k-1)+r$ in the space of integral curves in $\mathbf{C}^{2n+1}$ having a singular point at the origin. In 3-space only the series $A_{2,0} \leftarrow A_{4,0} \leftarrow \cdots$ occurs, $\operatorname{codim}(A_{2k,0}) = k-1$.

3 Semicubical Singular Curves in Contact Spaces

A semicubical singularity is the singularity diffeomorphic to that of the curve $x^3 = y^2$ at the origin.

This plane curve is the image of the mapping $x = s^2$, $y = s^3$. A singularity of a mapping into a contact space is *simple*, if the singularities of the images of the neighbouring mappings are contactomorphic to those of a finite set of curves.

Theorem 2 *There are exactly 5 simple semicubical singularity types in the contact space $\mathbf{C}^{2n+1}$. They are contactomorphic to the singularities*

$$\begin{array}{ccccccccc}
a^0 & \longleftarrow & b^1 & \longleftarrow & c^2 & \longleftarrow & e^3 & \longleftarrow & f^4 \\
 & & & & \uparrow & & \uparrow & & \uparrow \\
 & & & & (d^3) & \longleftarrow & (h^4) & \longleftarrow & (j^5),
\end{array}$$

where

$$\begin{aligned}
a^0:&\quad z = s^2, \quad q_1 = s^3, \quad p = 0, \quad q_{>1} = 0;\\
b^1:&\quad z = s^3, \quad q_1 = s^2, \quad p = 0, \quad q_{>1} = 0;\\
c^2:&\quad z = s^4, \quad q_1 = s^3, \quad p_1 = s^3, \quad p_{>1} = q_{>1} = 0;\\
e^3:&\quad z = s^4, \quad q_1 = s^2, \quad q_2 = s^3, \quad p_1 = s^5, \quad p_{>1} = q_{>2} = 0;\\
f^4:&\quad z = s^4, \quad q_1 = s^2, \quad q_2 = s^3, \quad p = 0, \quad q_{>2} = 0.
\end{aligned}$$

Here the superscript denotes the codimension in the space of singular curves in $\mathbf{C}^{2n+1}$. The classes in brackets are not simple. For $n = 1$ (that is the contact 3-space) only 3 simple singularities $a^0 \leftarrow b^1 \leftarrow c^2 \leftarrow (d^3)$ exist.

The nonsimple singularities of semicubical curves in a contact space form a set of codimension 3 in the space of singular curves. Such singularities occur in m-parametric families of generic curves in the contact $2n+1$-space if $m \ge 2n+3$. In the contact 3-space the nonsimple semicubical singularities first occur when the number of parameters equals 5.

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4 Classification of Integral Curves A_{2k}

Consider the projection $(z, p, q) \mapsto (p, q)$ of the contact space $\mathbf{C}^{2n+1}$ onto the symplectic space $\mathbf{C}^{2n}$. We may suppose, without loss of generality, that the z-axis at the singular point of our curve having the A_{2k}-singularity, does not belong to the tangent 2-plane of the curve at the singular point.

The projected curve has then also the A_{2k}-singularity. The symplectic classification of such singularities is given in [4]. It is just the list of $A_{2k,r}$ in Theorem 1 above ($0 \le r \le 2k$).

The symplectomorphism $h(p, q) = (P, Q)$, reducing the projection to the symplectic normal form, can be lifted to a contactomorphism g of the (z, p, q)-space by the formula

$$g(z; p, q) = \big(z + S(p, q); P(p, q), Q(p, q)\big).$$

The generating function S is defined here by the relation $dS = PdQ - pdq$.

The contactomorphism g sends the integral curves onto the integral curves. It reduces the initial integral curve γ to the normal form of Theorem 1.

The integral curves $A_{2k,r}$ are pairwise contact nonequivalent. Indeed, k is a diffeomorphism invariant. The contact invariance of the index $r > 0$ follows from its description as of *the order of the closeness of γ to the closest smooth integral 2-surface*.

To prove this description, consider the isotropic 2-surface providing the closest approximation of the projected curve in the symplectic space. The distance to this surface is vanishing along the projected curve as s^b if $r < 2k$ and vanishes for $r = 2k$ (according to [4]). Integrating pdq we lift this isotropic 2-surface to an integral surface approximating the initial integral curve at the same accuracy. Indeed, the difference between the values of z along two intersecting integral curves whose projections are b-th order tangent in the symplectic space is $b + 1$-th order tangent to 0.

5 Diracians and Contactians

The classification of simple nonintegral curves in contact spaces is based on some algebraic preliminaries which I shall give in a slightly more general situation than is needed for the proof of Theorem 2.

Let I be an ideal in the algebra of smooth (holomorphic, formal, ...) functions on a symplectic (or Poisson) manifold.

Definition The *Diracian DI* of the ideal I consists of those elements of I, whose Poisson brackets with all the functions of I belong to I.

Remark It seems, that Dirac has not considered these ideals. However his classification of constraints is a predecessor of the local symplectic algebra.

Proposition 1 *The Diracian of an ideal is an ideal.*

Proof Let $H \in DI$, $B \in I$. Then for any A one has $\{AH, B\} = A\{H, B\} + H\{A, B\} \in I$, since $\{H, B\} \in I$ and $H \in I$. Therefore, $AH \in DI$.

Proposition 2 *The Diracian of an ideal contains the square of this ideal.*

Proof Let $A, B, C \in I$. Then $\{AB, C\} = A\{B, C\} + B\{A, C\} \in I$. Therefore $AB \in DI$ and $I^2 \subset DI$.

Proposition 3 *The Diracian of an ideal is a Lie algebra (with respect to the Poisson bracket).*

Proof Let $A, B \in DI, C \in I$. According to the Jacoby identity, $\{\{A, B\}, C\} = \{A, \{B, C\}\} + \{B, \{C, A\}\}$. Since $B \in DI$, we get $\{B, C\} \in I$. Since $A \in DI$, we conclude that $\{A, \{B, C\}\} \in I$. Similarly, $\{B, \{C, A\}\} \in I$. Therefore $\{\{A, B\}, C\} \in I$, whence $\{A, B\} \in DI$.

Proposition 4 *Let the ideal I be generated by the generators F_i. Then $H \in DI$ if and only if $\{H, F_i\} \in I$.*

Proof $\{H, AF_i\} = \{H, A\}F_i + \{H, F_i\}A$. If $\{H, F_i\} \in I$, we get $\{H, AF_i\} \in I$, since $F_i \in I$.

Proposition 5 *Let the ideal I be generated by the generators F_i. Then the function $F = \sum A_i F_i$ belongs to the Diracian of I if and only if $\sum A_i\{F_i, F_j\} \in I$ for every j.*

Proof According to Proposition 4, we should verify the conditions $\{F, F_j\} \in I$. But we have

$$\{F, F_j\} = \sum A_i\{F_i, F_j\} + \sum F_i\{A_i, F_j\}$$

and the last term belongs to I.

In the three examples that follow we consider the ideals in the $\mathbf{C}$-algebra of the germs of holomorphic functions of 4 coordinates $x = q_1$, $y = q_2$, $u = p_1$, $v = p_2$ at the origin of the space $\mathbf{C}^4$ equipped with the symplectic srtucture $dp_1 \wedge dq_1 + dp_2 \wedge dq_2$. The word "generated" means "generated as a module over this algebra".

Example 1 Consider the ideal I of the functions, vanishing on the semicubical curve $A_{2,0}$, defined by the equations $u^2 = x^3$, $y = v = 0$. It is generated by the three functions $(H = u^2 - x^3, y, v)$.

Proposition 6 *The Diracian DI is generated by (H, v^2, uv, y^2).*

Proof Let $F = AH + Bv + Cy \in DI$. Since $\{H, v\} = \{H, y\} = 0$, $\{v, y\} = 1$, the cirterion of Proposition 5 takes the form $B \in I$, $C \in I$, whence $F = AH + Kv^2 + Lvy + My^2$, as required.

Example 2 Let I be the ideal of functions vanishing on the semicubical parabola $x^3 = y^2$, $u = v = 0$. Then I is generated by three functions $(H = y^2 - x^3, u, v)$.

Proposition 7 *The Diracian DI is generated by I^2 and two more functions*

$$H_e = 2xu + 3yv, \quad H_h = 2yu + 3x^2v$$

(and hence by $(H^2, H_e, H_h, u^2, uv, v^2)$).

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Proof Since I^2 belongs to DI, we may replace all the functions by linear (inhomogeneous) functions in u and v. A linear homogeneous function $B(x, y)u + C(x, y)v$ may be interpreted as a vector field $\xi = B\frac{\partial}{\partial x} + C\frac{\partial}{\partial y}$ on the plane (x, y) (considering u and v as the coordinates of the cotangent vector $udx + vdy$).

Applying the Proposition 5 to $F = AH + Bu + Cv$, we first find the Poisson brackets

$$\{H, u\} = 3x^2, \quad \{H, v\} = -2y, \quad \{u, v\} = 0.$$

Then we rewrite the criteria of Proposition 5

$$-3x^2B + 2yC \in I, \quad 3x^2A \in I, \quad -2yA \in I,$$

(where the coefficients A, B, C depend only on x and y and I denotes the ideal generated by H in the algebra of functions of x and y).

The last conditions imply $A \in I$, $AH \in I^2$. The first condition has the form $\xi H \in I$, which means that the vector field ξ is tangent to the curve $H = 0$.

Two tangent fields are the Euler quasihomogeneous field $\xi_e = 2x\frac{\partial}{\partial x} + 3y\frac{\partial}{\partial y}$ and the Hamilton field $\xi_h = 2y\frac{\partial}{\partial x} + 3x^2\frac{\partial}{\partial y}$. All other solutions are their combinations $\xi = \alpha(x, y)\xi_e + \beta(x, y)\xi_h$.

Indeed, $\xi_e H = 6H$, hence, choosing α, we may obtain any value of the coefficient G in $\alpha\xi_e H = GH \in I$. To find all the solutions ξ of the equation $\xi H = GH$, it remains to solve in B_0 and C_0 the linear homogeneous equation $\xi_0 H = 0$, which has the form $-3x^2B_0 + 2yC_0 = 0$. We obtain $B_0 = 2y\beta$ and $C_0 = 3x^2\beta$. Therefore $\xi_0 = \beta\xi_h$, $\xi = \alpha\xi_e + \beta\xi_h$.

Example 3 Let I be the ideal of functions vanishing on the semicubically singular curve $x^3 = y^2$, $u = xy$, $v = 0$. I is generated by three functions ($H = y^2 - x^3$, $L = u - xy$, v).

Proposition 8 *The Diracian DI is generated by I^2 and by two more functions*

$$H_e = 2x^2u + 3xyv + a, \quad H_h = 2yu + 3x^2v + b,$$

where $a = y^3 - 3x^3y$, $b = -x^4 - x^2y$.

I shall only use that $H_e, H_h \in DI$. This fact follows from the vanishing of the Poisson brackets of these functions with H, L and v (according to Proposition 4).

Remark The contact Hamilton functions are sections of the line bundle defined by the contact structure rather than functions. Therefore the contact Diracians are submodules rather than ideals.

In the standard contact space one may identify contact Hamilton functions with ordinary functions (see [5]). *The contactian CI* of an ideal of such functions consists of those functions belonging to I whose contact flows preserve I. This ideal, intermediate betwen I and I^2, is a Lie subalgebra with respect to the contact Poisson brackets $\{.,\}_C$ (see [5]).

The condition $K \in CI$ for a function K of an ideal generated by A_i may be written in the form $\{K, A_i\}_C \in I$.

6 Semicubical Nonintegrable Curves

The tangent 2-plane of a semicubical singularity of a curve in a contact space is generically transversal to the contact hyperplane at the singular point. Otherwise it belongs to the contact hyperplane.

Case 1 Let it be transversal. Then the smooth 2-surface, containing the curve, can be transformed into the coordinate (z, q_1)-plane by a contactomorphism (according to the Darboux-Givental theorem, see [8]).

The structure, induced by the contact structure on this plane is the fibration $dz = 0$. The fibered diffeomorphisms of the plane are extendable to contactomorphisms.

The normal forms of the semicubical singularities of curves in a fibered plane have been obtained already in [6], [7] (using the "invariants convolution" operation for the Coxeter group A_2).

These normal forms are $q^2 = z^3$ (the generic case—the fiber direction $dz = 0$ is not tangent to the curve at the origin) and $z^3 = q^2$ (the degenerate case—the fiber is tangent to the curve).

We thus get the cases a^0 and b^1 of Theorem 2.

Remark These two normal forms are also related to the Coxeter groups D_5 and E_7 corresponding to the functions $z(q^2 - z^3)$ and $z(q^3 - z^2)$.

Case 2 The tangent 2-plane of the curve at the singular point belongs to the contact hyperplane.

Project the curve to the symplectic space along the z-direction. Reduce the resulting A_2-curve to one of the three normal forms (c, e, f) of Theorem 2 by a symplectomorphism (as in Section 4). Lifting it to a contactomorphism, we get $z = h(s)$, where $h(s) = cs^4 + \cdots$ (since the tangent 2-space belongs to the contact hyperplane).

Suppose that $c \neq 0$ (as it is generically). Rescaling the coordinates, we can construct a contactomorphism and a change of the coordinate s such that c will be equal to 1.

Theorem 3 **(c,e,f)** *Every curve, defined by the above projection c, e or f and by* $h = s^4 + \cdots$*, is contactomorphic to the curve with the same projection to the symplectic space, for which* $h = s^4$.

These three theorems are proved in Sections 7, 8 and 9.

Suppose now that $h(s) = o(s^4)$ (cases d, h, j of Theorem 2).

Theorem 4 *No curve, defined in contact 3-space with the contact structure* $dz + p\,dq = 0$ *by the equations* $(q = s^2, p = s^3, z = cs^5)$ *is simple: c is a modulus.*

Proof The contact vector field defined by the contact Hamilton function K, has the form (see [5]):

$$\dot{p} = -K_q + pK_z, \quad \dot{q} = K_p, \quad \dot{z} = K - pK_p.$$

The existence of the required equivalence would imply the solvability of the homological

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equations

$$\begin{cases} 2su(s) + K_p = 0, \\ 3s^2u(s) - K_q + pK_z = 0, \\ 5cs^4u(s) + K - pK_p = s^5 \end{cases}$$

in the unknowns u and K (representing the deformations of the independent and dependent variables).

The function K should here be evaluated at the point $(q = s^2, p = s^3, z = cz^5)$. One also should have $u(0) = K(0) = K_p(0) = K_q(0) = 0$ (the singular point remaining at the origin).

These linear equations are contradictory. Indeed, they are quasihomogeneous (for the weights of variables $\deg s = 1$, $\deg q = 2$, $\deg p = 3$, $\deg z = 5$). Equating terms of degrees 2, 3 and 5 in the first, second and third equation, we get for the coefficients of the Taylor series $u(s) = as + \cdots$, $K = Apq + Bz + \cdots$ the relations $2a + A = 0$, $3a + (B - A) = 0$, $5ca + Bc = 1$, which are contradictory.

Corollary *There are no simple singularities among the semicubical curves with $h(s) = o(s^4)$.*

Proof Consider the class (d) of curves, whose projection to the symplectic plane is of type c but for which $h(s) = cs^5 +$ (higher order terms).

There are no simple curves is this class, since the contradiction, described above, remains when the higher order terms are taken into account.

But all the semicubical singularities for which $h(s) = o(s^4)$ are adjacent to the curves of class (d) (since classes e and f are adjacent to c). Whence the corollary.

Remark The result remains true in any dimension, since the nonsimple curves of class (d) remain nonsimple in any dimension (which follows, for instance, from the Darboux-Givental theorem but can also be deduced from the calculations above).

7 Proof of Theorem 3c

We need the following three statements on the representations of holomorphic functions by combinations of other holomorphic functions. They remain true if the given functions depend holomorphically on the parameters (of which the representation coefficients will then depend holomorphically).

1° *If $A(s^2, s^3) = 0$, one has $A(x, y) = B(x, y)(y^2 - x^3)$.*

Indeed, one always can write $A = a(x) + yb(x) + B(x, y)(y^2 - x^3)$, and if $A(s^2, s^3) = 0$ one should have $a \equiv b \equiv 0$.

2° *Any function $s^2 f(s)$ is the value of a holomorphic function $F(s^2, s^3)$.*

Indeed, if $f(s) = u(s^2) + sv(s^2)$, it suffices to take $F = xu(s) + yv(x)$.

3° Lemma *If $R(s^2, s^3) = s^a\big(c + sr(s)\big)$, $c \neq 0$, and $M(s^2, s^3) = s^{a+2}m(s)$, then there exists a representation*

$$M(x, y) = B(x, y)R(x, y) + C(x, y)(y^2 - x^3).$$

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Indeed, $s^2m(s)\big(c - sr(s)\big) = F(s^2, s^3)$, $c^2 - s^2r^2(s) = G(s^2, s^3)$, $G(0) \neq 0$, according to 2^o. For $B = F/G$ one gets $(M - BR)(s^2, s^3) = 0$ and the existence of C follows from 1^o. Lemma 3^o is proved.

Now consider the curve Γ in $\mathbf{C}^3$ with the contact structure $dz + p\,dq = 0$ (note the sign), defined by the equations $H(p, q) = 0$, $z = F(p, q)$ (we are mostly interested in the case $H = p^2 - q^3$, $F = q^2$, of the standard curve ($q = s^2$, $p = s^3$, $z = s^4$)).

The contact Hamiltonian K defines the contact vector field (see [5, appendix 4], for the choice of the sign of z):

$$\text{(1)} \qquad \frac{dp}{dt} = -K_q + pK_z, \quad \frac{dq}{dt} = K_p, \quad \frac{dz}{dt} = K - pK_p.$$

We choose $K = A(p, q) + \big(z - F(p, q)\big)B(p, q)$. Denote by $P(t)$, $Q(t)$, $Z(t)$ the motion, starting at $t = 0$, from a point (p, q, z) of Γ, defined by the contact Hamiltonian K.

Define the initial velocity $\dot{H} = (\frac{d}{dt})_{t=0}\Big(H\big(P(t), Q(t)\big)\Big)$. We find from (1)

$$\text{(2)} \qquad \dot{H} = \{A, H\} + B(\{H, F\} + pH_p),$$

where $\{I, J\} = I_pJ_q - I_qJ_p$ is the Poisson bracket of I and J.

Indeed, on Γ we have

$$\text{(3)} \qquad K = A, \quad K_p = A_p - F_pB, \quad K_q = A_q - F_qB, \quad K_z = B,$$

therefore $H_p\dot{p} + H_q\dot{q} = H_p(-A_q + F_qB + pB) + H_q(A_p - F_qB)$.

Now suppose, that $\dot{H} = 0$ on Γ. Write the vertical component of the moving curve as $Z(t) = F\big(P(t), Q(t), t\big)$ and calculate the initial velocity of the change of the function F:

$$\frac{\partial F}{\partial t} = \left(\frac{\partial}{\partial t}\right)_{t=0} F(P, Q, t).$$

We get from (1) and (3) the formula

$$\text{(4)} \qquad \frac{\partial F}{\partial t} = \{F, A\} + A - pA_p.$$

Indeed, $\dot{z} = F_p\dot{p} + F_q\dot{q} + \frac{\partial F}{\partial t}$, hence

$$\frac{\partial F}{\partial t} = \dot{z} - F_p\dot{p} - F_q\dot{q} = A - pA_p + pF_pB - F_p(-A_q + F_qB + pB) - F_q(A_p - F_pB),$$

as required.

To kill the perturbation m in the equation of the curve

$$q = s^2, \quad p = q^3, \quad z = s^4 + m(s), \quad m = s^5n(s),$$

using the homotopy method, we shall construct a (time-dependent) contact Hamilton function $K = A + (z - F)B$, where, at moment t, $F(s^2, s^3) = s^4 + tm(s)$ and where the coefficients A and B verify:

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(a) the condition $\dot{H} = 0$ on Γ_t of the preservation of the semicubical projection of the curve to the symplectic plane,
(b) the homological equation $\frac{\partial F}{\partial t} = m$ on Γ_t,
(c) the origin fixing conditions $A(0) = A_p(0) = A_q(0) = B(0) = 0$.

The conditions (b) and (a) can be written as equations

$$(b) \qquad \{F, A\} + A - pA_p = m(s),$$

$$(a) \qquad \{A, H\} + B(\{H, F\} + pH_p) = 0$$

along semicubical parabola $q = s^2$, $p = s^3$.

To solve the linear system (b), (a) in the unknowns A and B, consider the quasihomogeneous grading $\deg q = 2$, $\deg p = 3$, $\deg H = 6$. First we find the solutions, verifying (a) exactly but (b) only approximately:

$$(b') \qquad \{F, A\} + A - pA_p = s^n \bmod s^{n+1}, \quad n \geq 5.$$

We shall find a solution of this equation of quasihomogeneous degree $n + 1$. Since the equation (b) is linear, we thus reduce the equation (b) to a similar equation with a high order zero of the right hand side m, which we shall solve explicitly.

To find the solution of the approximating equation (b'), we replace F by its lower order part q^2 (which is also the unperturbed F). The lowest order term of the left hand side is $-2qA_p$. To make it equal to s^n we choose $A = p^2q^{a-1}/4$ if n is odd and $A = pq^{b-1}/2$ if n is even ($\deg A = n + 1 \geq 6$). We thus get the required solution A of equation (b').

Choose B verifying the condition (a), which we can write in the form

$$M = BR + CH, \quad M = \{H, A\}, \quad R = \{H, F\} + pH_p.$$

For $F = q^2$ we get $R = 4s^5 + O(s^6)$. The higher order terms in F do not change the principal part $4s^5$ of R. We also have, for the quasihomogeneous A, that

$$\deg M \leq \deg H + \deg A - \deg p - \deg q = \deg A + 1 \geq 7.$$

Equation (a) is thus solvable in B according to Lemma 3^o above.

Suppose now that the right hand side m of equation (b) is divisible by s^7. In this case we put $A = fH$ and get for f the equation $-fR = m$ along the semicubical parabola. This equation is solvable according to Lemma 3^o. Now $\{A, H\} = 0$ along the parabola, therefore equation (a) is satisfied for $B = 0$.

We have thus solved the homological equation for any m divisible by s^5. The flow of the contact vector field that we have constructed sends the nonperturbed curve $(z = s^4)$ onto the perturbed one $\big(z = s^4 + m(s)\big)$.

8 Proof of Theorem 3e

Consider the curve Γ ($x = s^2$, $y = s^3$, $u = v = 0$) in the space $\mathbf{C}^4$ equipped with the Darboux coordinates $q_1 = x$, $q_2 = y$, $p_1 = u$, $p_2 = v$. We shall reduce to the contact normal form the curve in the 5-space, equipped with the contact structure $dz + pdq = 0$, whose projection to $\mathbf{C}^4$ is Γ and for which $z = s^4 + m(s)$, $m = s^5u(s)$.

We should construct the contact homotopy over Γ of the curves $z = F(x, y, t) = x^2 + tM(x, y)$, $M(s^2, s^3) = m(s)$. Our contact time-dependent Hamilton function K will not depend on z. To preserve the projection Γ we shall choose K in the Diracian of the ideal I of the curve Γ.

The elements of I^2 are not moving the curve. Hence we try

$$K = fH_e + gH_h, \quad H_e = 2xu + 3yv, \quad H_h = 2yu + 3x^2v$$

(according to Example 2 of Section 5), where $f = f(x, y)$, $g = g(x, y)$. This choice implies $\dot{H} = 0$ on Γ. The velocity of the deformation of function F, defining the moving curve, is calculated as in equation (4) of Section 7. Taking into account that we have now $K = A$, $B = 0$, $A = pA_p$, $H_e = H_h = 0$ on Γ, we get along Γ

$$\frac{\partial F}{\partial t} = f\{F, H_e\} + g\{F, H_h\}.$$

For the unperturbed curve $F_0 = x^2$, we find $\{F_0, H_e\} = -4x^2$ and $\{F_0, H_h\} = -4xy$. Therefore, for the perturbed curve, we get along Γ at any moment

$$\{F, H_e\} = -4s^4 + \cdots, \quad \{F, H_h\} = -4s^5 + \cdots,$$

where the dots mean higher order terms. Now it is easy to solve the homological equation in f, g

$$f\{F, H_e\} + g\{F, H_h\} = m(s), \quad m = s^5n(s).$$

We choose $g = -n(0)/4$ and get for f the equation $f(-4s^4+\cdots) = s^6k(s)$ along Γ, solvable in $f(x, y)$ according to Lemma 3^o of Section 7.

The contact flow defined by contact Hamiltonian that we have constructed sends the nonperturbed curve onto the perturbed one.

9 Proof of Theorem 3f

Now Γ is defined in $\mathbf{C}^4$ by the equations $x = s^2$, $y = s^3$, $u = s^5$, $v = 0$. Choose $H = y^2 - x^3$, $L = u - xy$. As in Section 8, use the generators of the $D(I)/I^2$ provided by Example 3 of Section 5:

$$K = f(x, y)H_e + g(x, y)H_h,$$
$$H_e = 2x^2u + 3xyv + a, \quad H_h = 2yu + 3x^2v + b,$$
$$a = y^3 - 3x^3y, \quad b = -x^4 - xy^2.$$

Then along Γ we have

$$H = L = H_e = H_h = \{H, H_e\} = \{H, H_h\} = \{L, H_e\} = \{L, H_h\} = 0.$$

We also find $K - pK_p = af + bg$, and, along Γ, $a = -2s^9$, $b = -2s^8$.

Denote the coordinate that we wish to normalise by the homotopy method as $z = F(x, y, t) = x^2 + tM(x, y)$, $M(s^2, s^3) = s^5 n(s)$.

The homological equation $\{F, K\} + K - pK_p = M$ along Γ takes the form

$$f(\{F, H_e\} + a) + g(\{F, H_h\} + b) = m(s), \quad m = s^5 n(s). \tag{$*$}$$

For the unperturbed curve $F_0 = x^2$ we get $\{F_0, H_e\} = -4x^3$, $\{F_0, H_h\} = -4xy$. Therefore along Γ we have

$$\{F, H_e\} + a = -4s^6 \bmod s^7, \quad \{F, H_h\} + b = -4s^5 \bmod s^6.$$

To solve the homological equation $(*)$ we first solve it $\bmod s^6$, choosing $f = 0$, $g = -n(0)/4$. Then we solve $\bmod s^7$ the equation with the right hand side $s^6 k(s)$, choosing $g = 0$, $f = -k(0)/4$.

Finally, to solve the remaining equation with right hand side $s^7 l(s)$ we choose $f = 0$ and find $g(x, y)$ using Lemma 3^o of Section 7.

We thus obtain the required contact flow, sending the nonperturbed curve onto the perturbed one.

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