Isoperimetric $2^m n$-gons applied to finding $\frac{1}{\pi}$ concisely by a new construction.

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FIGURE 27.

1. Let AB be the half-side of any n-gon, OB its in-radius (r), and OA its circum-radius (R). Draw OA_1 to bisect $\angle$AOB and $AA_1C \perp$ to it meeting OB in C. Then $A_1B_1 \|$ to AB is the half-side of a $2n$-gon having the same perimeter as the n-gon, OB_1 its in-radius (r_1), and OA_1 its circum-radius (R_1).

Since B_1 bisects BC and $\triangle OB_1A_1$ is similar to OA_1C

$$\left.\begin{aligned} 2OB_1 &= OB + OC = OB + OA, \quad \therefore\ 2r_1 = r + R \\ \text{and} \qquad OA_1^2 &= OB_1 \,.\, OC = OB_1 \,.\, OA, \quad \therefore\ R_1^2 = r_1 R \end{aligned}\right\} \quad \text{and } \therefore$$

$$2r_2 = r_1 + R_1, \quad 2r_3 = r_2 + R_2, \quad 2r_4 = \text{etc.}, \quad 2r_m = r_{m-1} + R_{m-1} \qquad \ldots \qquad (a)$$

$$R_2^2 = r_2R_1, \quad R_3^2 = r_3R_2, \quad R_4^2 = \text{etc.}, \quad R_m^2 = r_mR_{m-1}, \qquad \ldots \qquad (b)$$

where $R_2 = OA_2 = OC_2$, $R_3 = OA_3 = OC_3$, etc., $r_2 = OB_2$, $r_3 = OB_3$, etc., the new points being got by drawing $A_1A_2C_1$, $A_2A_3C_2$, etc., respectively $\perp$ to the successive bisectors OA_2, OA_3, etc., and A_2B_2, A_3B_3, etc., $\|$AB or A_1B_1.

Thus, given r and R, we find $r_m (= OB_m)$ and $R_m (= OC_m)$ the radii of a polygon of $2^m n$ sides which has the same perimeter as the original n-gon. The diagram shows (1) that as m increases the two points B_m and C_m approach nearer and nearer to an intermediate point K; and (2) that the line OK$(= k)$ is the radius of a circle having also the perimeter in question.

Choosing the simplest case, $n = 2$, then if the common peri-

https://doi.org/10.1017/S0013091500031576 Published online by Cambridge University Press

meter = 2 units, OB($=r$) vanishes, $R = OA = AB = \frac{1}{2}$ and Fig. 27 is modified to Fig. 28. Also, circumference of the circle $= 2 = 2\pi k$

$\therefore$ $k = \dfrac{1}{\pi}$. Then, applying (*a*) and (*b*), we have

$$\left.\begin{array}{ll} r_3 = \cdot 314,\ 208,\ 718,\ 257,\ 8(7) & R_3 = \cdot 320,\ 364,\ 430,\ 968 \\ r_4 = \cdot 317,\ 286,\ 574,\ 613,\ \ldots & R_4 = \cdot 318,\ 321,\ 788,\ 7\ldots \\ r_5 = \cdot 318,\ 054,\ 181,\ 6(5) & R_5 = \cdot 318,\ 437,\ 75\ \ \ldots \end{array}\right\},\ldots(c)$$

Thus for the $2^5 . 2$-gon the radii agree to only 3 places. When m is large the following results will greatly reduce the labour of finding k.

2. OA_2 bisects $\angle A_1OC_1$, $\therefore$ A_1C_1 bisects $\angle CA_1B_1$, and $CC_1 > 2B_1B_2$. Thus $BB_1 > 4B_1B_2$, $B_1B_2 > 4B_2B_3$, etc.,

$$\text{and } BB_1 + B_1B_2 + \text{ etc.}, > 4(B_1B_2 + B_2B_3 + \text{etc.}),$$

i.e. $$BK > 4B_1K \text{ or } k - r > 4(k - r_1)$$

$\therefore$ finally $k - r_{m-1} > 4(k - r_m)$ and for a close value

$$r_m < k < \tfrac{1}{3}(4r_m - r_{m-1}), \ \ldots \quad \ldots \quad \ldots \quad (d)$$

Thus by (*c*) $k = \frac{1}{3}(4r_6 - r_5) = \frac{1}{3}(2R_5 + r_5) = \cdot 318,\ 309,\ 89$.

FIGURE 28.

To find a similar relation between the circum-radii I draw $A_2D \parallel A_1C$ and A_2E bisecting $\angle C_1A_2D$. The four adjacent acute angles at A_2 are equal $\therefore$ $C_2C_1 < C_1E < ED$, CD or $C_1D > 2C_1C_2$. Thus $CC_1 > 4C_1C_2$, $C_1C_2 > 4C_2C_3$, etc., and, as before, CK or $R - k > 4C_1K$ or $4(R_1 - k)$. Finally $R_{m-1} - k > 4(R_m - k)$ and for a close value

$$R_m > k > \tfrac{1}{3}(4R_m - R_{m-1}), \ \ldots \quad \ldots \quad \ldots \quad (e)$$

Thus by (*c*) without using r_6

$$\tfrac{1}{3}(4R_5 - R_4) = \cdot 318,\ 309,\ 74,\ \ \tfrac{1}{3}(4r_5 - r_4) = \cdot 318,\ 310,\ 05$$

$\therefore$ $$k = \text{arith. mean} = \cdot 318,\ 309,\ 89 \text{ as above.}$$

When r_m and R_m agree to p places $p-1$ more can be found correctly by treating the new circum-radii as if they were in-radii.

https://doi.org/10.1017/S0013091500031576 Published online by Cambridge University Press

Hence a third method of contraction when k has to be found to a large number of decimals. Calling the radii (after R_m) $a_1, a_2, a_3 \ldots a_x$ we shall have

$$\left.\begin{array}{c} 2a_3 = a_1 + a_2 \\ 2a_4 = a_2 + a_3 \\ \text{etc.} \\ 2a_x = a_{x-2} + a_{x-1} \end{array}\right\} \therefore \text{ adding these } x-2 \text{ equations we get}$$

$$a_1 + 2a_2 = 2a_x + a_{x-1}$$

$$= 3a_x \qquad \text{very nearly}$$

$$\therefore\ k = \tfrac{1}{3}(r_m + 2R_m), \quad \ldots \quad \ldots \quad \ldots \quad *(f)$$

without finding the $x-3$ intervening terms.

3. A fourth contraction may be derived from (d), thus:

$$4(k - r_m) < k - r_{m-1} \text{ gives } u = \tfrac{1}{3}(4r_m - r_{m-1})$$

$$4(k - r_{m-1}) < k - r_{m-2} \quad ,, \quad v = \tfrac{1}{3}(4r_{m-1} - r_{m-2})$$

and $\qquad 16(k-u) < k - v$

$\therefore$ for a close value $k = \frac{1}{15}(16u - v)$, ... (g)

Thus, using (c)

$$u = \tfrac{1}{3}(4r_5 - r_4) = \cdot 318,\ 310,\ 050,\ 7$$

$$v = \tfrac{1}{3}(4r_4 - r_3) = \cdot 318,\ 312,\ 526,\ 7$$

$\therefore$ by (g)

$$k = \cdot 318,\ 309,\ 886, \text{ taking 9 places;}$$

whence $\qquad \dfrac{1}{k} = \pi = 3 \cdot 141,\ 592,\ 65\ (54)$ by division.

It may be noted that if in Fig. 28 a quadrant BKMN be drawn its arc is exactly measured by the line AB, and its area is exactly measured by the rectangle BN . BB_1.

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* This result is given by Gergonne (*Annales*, Vol. VI.) with a more complicated proof, and recently by MM. Rouché and Comberousse with a different proof still more intricate (*Traité de Géom.*, 6th ed., 1891). For knowledge of the latter I am indebted to John S. Mackay, LL.D.