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ON THE "LARGENESS" OF ONE-RELATOR GROUPS

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1. Introduction

If G is a one-relator group on at least 3 generators, or is a one-relator group with torsion on at least 2 generators, then it follows from results in [**1**] and [**6**] that G has a subgroup of finite index which can be mapped homomorphically onto F_2, the free group of rank 2. In the language of [**2**], G is equally as large as F_2, written $G \simeq F_2$. This leaves open the following question, raised as a problem in [**2**]:

Let G be a two-generator, torsion-free, one-relator group. Under what conditions is G equally as large as F_2?

Examples of two-generator one-relator groups which are not equally as large as F_2 are given in [**2**] (Examples 3.2, 3.3). In this paper we shall prove the following positive results.

Theorem 1. *Let* $G = \langle a, b; a^{\alpha_1} b^{\beta_1} a^{\alpha_2} b^{\beta_2} \dots a^{\alpha_m} b^{\beta_m} \rangle$, *where* $m \geqq 2$, α_i, β_i *are non-zero integers* $(1 \leqq i \leqq m)$, *and* $\sum_{k=1}^{m} \beta_k = 0$. *Suppose that there is a pair* i, j $(1 \leqq i < j \leqq m)$ *and an integer* $p > 1$ *such that:*

(1) $p \mid \alpha_k$ *for* $k \neq i, j$ *and* $hcf(p, \alpha_i) = hcf(p, \alpha_j) = 1$;

(2) $\left| \sum_{i \leqq k < j} \beta_k \right| > 1$; *and*

(3) $\left(\left| \sum_{i \leqq k < j} \beta_k \right|, p \right) \neq (2, 2)$.

Then $G \simeq F_2$.

Theorem 2. *Let* $G = \langle a, b; a^{\alpha_1} b^{\beta_1} \dots a^{\alpha_m} b^{\beta_m} \rangle$ *where the* α_i, β_i $(1 \leqq i \leqq m)$ *are non-zero integers. Assume that there is a factorisation* $m = rl$ *of* m $(r, l > 1)$, *and integers* $p, q > 1$ *such that the following hold:*

(4) *For* $1 \leqq i \leqq j < r$, *the number* $\beta_i + \dots + \beta_j$ *is not divisible by* q;

(5) $s = hcf\{\alpha_i + \alpha_{r+i} + \dots + \alpha_{(l-1)r+i} : 1 \leqq i \leqq r\} > 1$;

(6) $pq \mid \beta_1 + \beta_2 + \dots + \beta_r$;

(7) $\beta_k \equiv \beta_{k+r} \equiv \beta_{k+2r} \equiv \dots \equiv \beta_{k+(l-1)r}$ *(mod pq) for* $1 \leqq k \leqq r$; *and*

(8) $(s, p) \neq (2, 2)$.

Then $G \simeq F_2$.

It is clear that if the relator $a^{\alpha_1}b^{\beta_1}\dots a^{\alpha_m}b^{\beta_m}$ satisfies the conditions (4), (6) and (7) then so does the relator

$$a^{\alpha_1}b^{\beta_1+n_1pq}a^{\alpha_2}b^{\beta_2+n_2pq}\dots a^{\alpha_m}b^{\beta_m+n_mpq}$$

for any choice of integers n_i, $1\leqq i\leqq m$. Thus any particular example of a group satisfying the conditions of Theorem 2 will generate an infinite family of further such examples.

We also note that condition (5) of Theorem 2 cannot in general be removed. For example let $G_0=\langle a,t;t^{-1}a^nt=a^{n+1}\rangle$, $n>1$, and let G_1 be the *HNN* extension

$$\langle a,b,t;b^{-1}ab=t,t^{-1}a^nt=a^{n+1}\rangle=\langle a,b;a^{-1}ba^nb^{-1}aba^{-(n+1)}b^{-1}\rangle.$$

This latter presentation does not satisfy condition (5). Moreover, G_1 is not equally as large as F_2 ([**2**], Example 3.3).

2. Notation and definitions

Most of the definitions and results in this section are fairly standard. Further details can be found in [**3**, p. 115–120].

A **1-complex** or **graph** consists of two disjoint sets V,E together with three functions, $\iota:E\to V$, $\tau:E\to V$, $^{-1}:E\to E$ satisfying $\iota(e^{-1})=\tau(e)$, $(e^{-1})^{-1}=e,e^{-1}\neq e$ for all e in E. When representing 1-complexes diagrammatically we follow the convention adopted in [**5**, p. 13].

If γ is a closed path in a 1-complex then the set of cyclic permutations of γ, denoted $\{\gamma\}^*$, is called a **cycle**. If c is the cycle $\{\gamma\}^*$ then c^{-1} is defined to be $\{\gamma^{-1}\}^*$.

A **2-complex** $\mathbf{C}$ consists of three disjoint sets V,E,C where V,E together constitute a 1-complex (called the **1-skeleton** $\mathbf{C}^1$ of $\mathbf{C}$), and where there are two maps ∂ from C to the set of cycles in C^1, and $^{-1}:C\to C$ satisfying $\partial\Delta^{-1}=(\partial\Delta)^{-1},(\Delta^{-1})^{-1}=\Delta,\Delta\neq\Delta^{-1}$ for each Δ in C. The members of C are called **2-cells** and $\partial\Delta$ is the boundary of Δ.

The fundamental group of $\mathbf{C}$ at a vertex $v\in V$ is denoted by $\pi_1(\mathbf{C},v)$. All 2-complexes shall be connected, so $\pi_1(\mathbf{C},v)$ is independent of v up to isomorphism. Thus we can speak of *the* fundamental group of $\mathbf{C}$, which we denote by $\pi_1(\mathbf{C})$.

A **mapping** from the 2-complex $\mathbf{C}$ to the 2-complex $\mathbf{L}$ consists of three functions, one from the vertex set of $\mathbf{C}$ to the vertex set of $\mathbf{L}$, one from the edge set of $\mathbf{C}$ to the set of paths of $\mathbf{L}$, and one from the set of 2-cells of $\mathbf{C}$ to the set of 2-cells of $\mathbf{L}$. Denoting all three functions by the same latter, say ϕ, we require that given an edge e in $\mathbf{C}$, we have $\phi(\iota(e))=\iota(\phi(e))$, $\phi(\tau(e))=\tau(\phi(e))$, $\phi(e^{-1})=\phi(e)^{-1}$, and, for a given 2-cell Δ of $\mathbf{C}$, $\phi(\partial\Delta)=\partial\phi(\Delta)$, and $\phi(\Delta^{-1})=\phi(\Delta)^{-1}$. (Here, if $\partial\Delta=\{\gamma\}^*$, then $\phi(\partial\Delta)$ is defined to be $\{\phi(\gamma)\}^*$.) If v is a vertex in $\mathbf{C}$ and if $\phi(v)=u$, we have a well-defined induced group homomorphism ϕ_* from $\pi_1(\mathbf{C},v)$ to $\pi_1(\mathbf{L},u)$.

Let $\langle\mathbf{X};\mathbf{R}\rangle$ be a presentation of the group G. Then we can associate a 2-complex $\mathbf{K}$ with this presentation as follows. The 2-complex $\mathbf{K}$ has a single vertex v, and an edge e_x for each x in $\mathbf{X}$ together with e_x^{-1}. Furthermore, for each defining relator $x_1^{\varepsilon_1}\dots x_n^{\varepsilon_n}$ in $\mathbf{R}$ ($x_i\in\mathbf{X}$, $\varepsilon_i=\pm1$, for $1\leqq i\leqq n$), we introduce a 2-cell Δ with boundary $\delta\Delta=\{e_{x_1}^{\varepsilon_1}\dots e_{x_n}^{\varepsilon_n}\}^*$, together with Δ^{-1}. The fundamental group of the associated 2-complex $\mathbf{K}$ is isomorphic to G.

We can represent a given subgroup H of $G=\langle\mathbf{X};\mathbf{R}\rangle$ by a 2-complex $\mathbf{K_H}$. Firstly identify G with $\pi_1(\mathbf{K})$. Then $\mathbf{K_H}$ is constructed as follows. The verticies of $\mathbf{K_H}$ are the right cosets $H\gamma$ of H in G. (Strictly speaking the elements of $\pi_1(\mathbf{K})$ are equivalence classes of paths, but we will abuse notation and represent a class by a path in it.) The edges of $\mathbf{K_H}$ are $(H\gamma, e)$ where e is an edge in $\mathbf{K}$. We define $\iota(H\gamma, e)$ to be $H\gamma$ and $\tau(H\gamma, e)$ to be $H\gamma e$. Also, $(H\gamma, e)^{-1} = (H\gamma e, e^{-1})$. Let $\alpha = e_1 e_2 \dots e_n$ be a path in $\mathbf{K}^1$. Then the **lift** of α to $\mathbf{K}^1_\mathbf{H}$ at the vertex $H\gamma$ is the path

$$(H\gamma, e_1)(H\gamma e_1, e_2)\dots(H\gamma e_1 \dots e_{n-1}, e_n).$$

For the 2-cells of $\mathbf{K_H}$ let $\tilde{R}_{i,H\gamma}$ be the lift of R_i ($R_i \in \mathbf{R}$) at $H\gamma$. Then $\tilde{R}_{i,H\gamma}$ is a closed path in $\mathbf{K}^1_\mathbf{H}$. Attach a 2-cell $\tilde{\Delta}_{i,H\gamma}$ to $\mathbf{K}^1_\mathbf{H}$ with boundary $\{\tilde{R}_{i,H\gamma}\}^*$, and attach an inverse 2-cell $\tilde{\Delta}^{-1}_{i,H\gamma}$ with boundary $\{\tilde{R}^{-1}_{i,H\gamma}\}^*$. There is a mapping $\phi: \mathbf{K_H} \to \mathbf{K}$ which sends the edge $(H\gamma, e)$ to e, and the 2-cell $\Delta^\varepsilon_{i,H\gamma}$ to Δ^ε_i, $\varepsilon = \pm 1$, where $\partial\Delta_i = \{R_i\}^*$ for $R_i \in \mathbf{R}$. The induced homomorphism $\phi_*: \pi_1(\mathbf{K_H}) \to \pi_1(\mathbf{K})$ is injective, and $\phi_*(\pi_1(\mathbf{K_H})) = H$, whence the isomorphism $\pi_1(\mathbf{K_H}) \cong H$.

We remark that the edges of $\mathbf{K}^1_H$ with second co-ordinate $e_i^{\pm 1}$ will sometimes be called e_i-edges.

If we collapse each edge of a maximal subtree in $\mathbf{K}^1$ to a point and make the corresponding alterations to the 2-cells of $\mathbf{K}$, then the fundamental group of the 2-complex which remains will be isomorphic to $\pi_1(\mathbf{K})$. These collapses are examples of mappings of 2-complexes and, by consideration of the induced homomorphisms, it is clear that further collapsing of edges will give 2-complexes whose fundamental groups are homomorphic images of $\pi_1(\mathbf{K})$.

3. Proof of Theorem 1

By cyclically permuting the relator of G, if necessary, we may assume that $i = 1$. Also, replacing b by b^{-1}, if necessary, we may assume that $\sum_{1 \leq k < j} \beta_k > 0$. For simplicity, denote $\sum_{1 \leq k < j} \beta_k$ by h.

Let $\mathbf{K}$ be the 2-complex associated with the given presentation of G and let H be the normal closure in G of $\{a, b^h\}$. Then $\mathbf{K_H}$, the covering of $\mathbf{K}$ corresponding to H, has 1-skeleton:

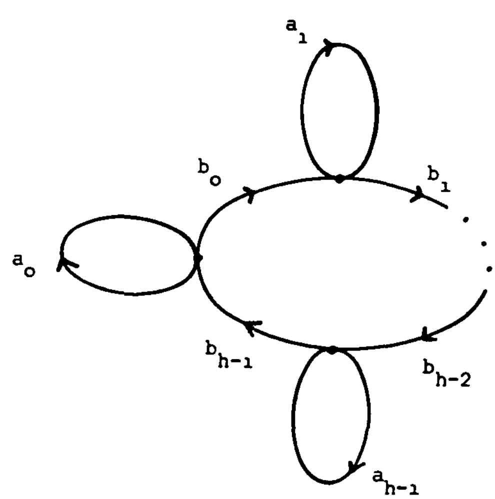

FIGURE 1

The life of the relator of G at the vertex μ has the form

$$a_\mu^{\alpha_1}\omega_\mu a_\mu^{\alpha_j}v_\mu,$$

where $\sigma_{b_{h-1}}(\omega_\mu)=1$, $\sigma_{b_{h-1}}(v_\mu)=-1$. Collapse all the b-edges apart from b_{h-1}, obtaining a 2-complex $\bar{\mathbf{K}}_{\mathbf{H}}$. Then, since we have collapsed a maximal tree, $\pi_1(\bar{\mathbf{K}}_{\mathbf{H}})\cong H$.

Let L be the homomorphic image of $\pi_1(\bar{\mathbf{K}}_{\mathbf{H}})$ obtained by adding the relations $a_\mu^p=1$ $(0\leqq\mu<h)$. Then, using (1), and writing z for b_{h-1}, we have

$$L=\langle a_0,\dots,a_{h-1},z;a_\mu^p,a_\mu^{\alpha_1}za_\mu^{\alpha_j}z^{-1}(0\leqq\mu<h)\rangle.$$

Again making use of (1), this may be rewritten as

$$L=\langle a_0,\dots,a_{h-1},z;a_\mu^p=1,za_\mu z^{-1}=a_\mu^q(0\leqq\mu<h)\rangle$$

for some integer q with $hcf(p,q)=1$. Adding the relation $z^{\phi(p)}=1$, where ϕ is the Euler function, then gives a group which is a split extension of the free product $\bar{L}$ of h cyclic groups of order p, by a cyclic group of order $\phi(p)$. Since $\bar{L}\simeq F_2$ (using conditions (2), (3) and **[4]**, Theorem 3.7), the result follows.

4. Proof of Theorem 2

Our proof of Theorem 2 takes the form of a discussion and we shall introduce each of the conditions as we require them. In order to do this it is more convenient to replace conditions (6) and (7) by the condition:

(9) *For each i $(1\leqq i\leqq m)$, $pq\,|\,\beta_i+\cdots\beta_{i+(r-1)}$, where subscripts are reduced mod m to lie between* 1 *and* m.

It is straightforward to check that conditions (4), (6) and (7) are together equivalent to conditions (4) and (9).

Let $G=\langle a,b;a^{\alpha_1}b^{\beta_1}\dots a^{\alpha_m}b^{\beta_m}\rangle$ where α_i,β_i are non-zero integers $(1\leqq i\leqq m)$. Assume that m is composite, say $m=rl$, $(r,l>1)$. Now suppose that there is an integer $q>1$ such that:

(10) *For each i $(1\leqq i\leqq m)$, the numbers $\beta_i+\cdots+B_{i+(r-1)}$, where subscripts are reduced mod m to lie between* 1 *and m, are divisible by q.*

Assume also that:

(11) *There is an integer $\lambda>1$ such that λ divides $\sum_{i=1}^{m}\alpha_i$.*

Let $\mathbf{K}$ be the 2-complex associated with the given presentation for G and let H be the normal closure in G of $\{a^\lambda,b^q,[a,b]\}$. Then $\mathbf{K}_{\mathbf{H}}$, the covering of $\mathbf{K}$ corresponding to H, has 1-skeleton:

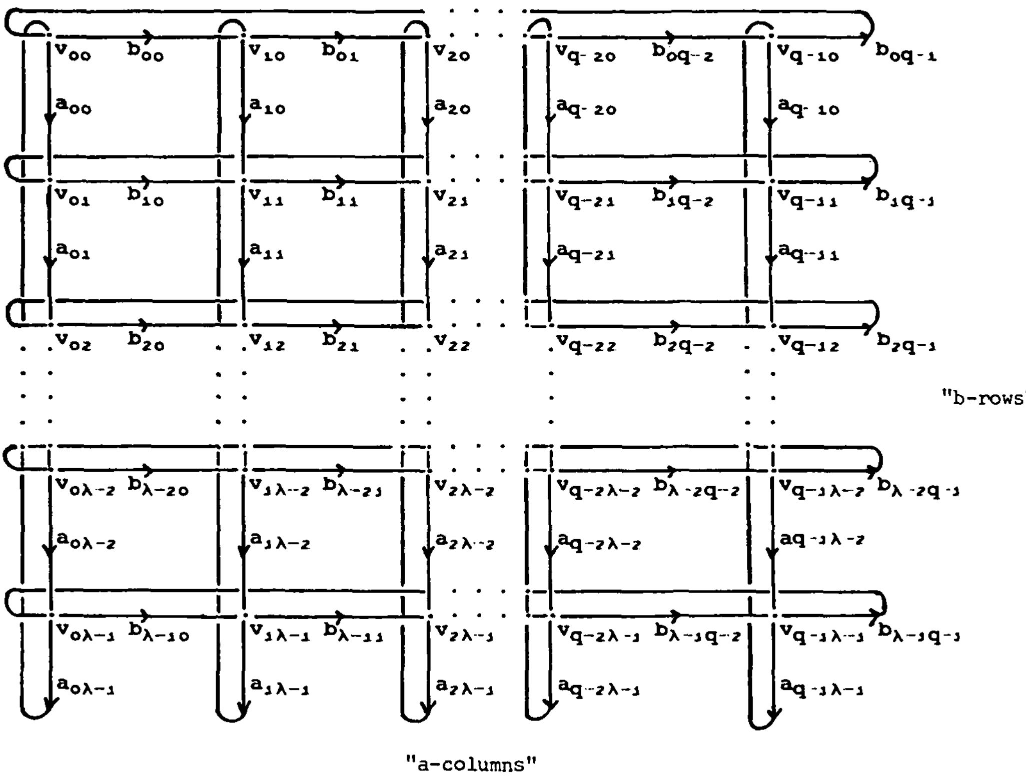

FIGURE 2

Consider the lift $R_{(\nu,\mu)}$ of the relator of G at the vertex (ν,μ). By (10), this will involve edges from at most r a-columns, and will involve edges from the a_0-column $(=\{a_{0i}^{\pm 1}\colon 0 \leqq i < \lambda\})$ provided

$$\nu + \beta_1 + \cdots + \beta_k \equiv 0 \bmod q$$

for some $k \in \{1, \ldots, r\}$. We want to guarantee that the value k, if it exists, is unique; that is, the lift $R_{(\nu,\mu)}$ involves edges from precisely r a-columns. This is achieved by requiring:

(4) *For $1 \leqq i \leqq j < r$, the number $\beta_i + \cdots + \beta_j$, is not divisible by q.*

It follows from (11) and (4) that the number of edges of the a_0-column involved in $R_{(\nu,\mu)}$ will either be zero or will be

$$|\alpha_k| + |\alpha_{k+r}| + \cdots + |\alpha_{k+(l-1)r}|$$

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where the number $k \in \{1,\ldots,r\}$. Moreover,

$$\sum_{i=0}^{\lambda-1} \sigma_{a_{0i}}(R_{(\nu,\mu)}) = \alpha_k + \alpha_{k+r} + \cdots + \alpha_{k+(l-1)r}.$$

We want the numbers $\sum_{i=0}^{\lambda-1} \sigma_{a_{0i}}(R_{(\nu,\mu)})$, $(0 \leqq \nu \leqq q-1, 0 \leqq \mu \leqq \lambda-1)$, to have a common factor greater than 1. By what we have just observed, this will be achieved by requiring:

(5) $s = hcf\{\alpha_i + \alpha_{r+i} + \cdots + \alpha_{(l-1)r+i} : 1 \leqq i \leqq r\} > 1$

(Note that (5) implies (11) with $\lambda = s$.)

We shall also want information about $\sum_{i=0}^{\lambda-1} \sigma_{b_{i0}}(R_{(\nu,\mu)})$. Assume therefore that:

(9) *There is an integer $p > 1$ such that for each i $(1 \leqq i \leqq m)$, pq divides the number $\beta_i + \cdots + \beta_{i+(r-1)}$, where subscripts are reduced* mod m *to lie between* 1 *and* m.

(*Note that* (9) *implies* (10).)

Then (9) ensures that p divides the numbers $\sum_{i=0}^{\lambda-1} \sigma_{b_{i0}}(R_{(\nu,\mu)})$, $0 \leqq \nu \leqq q-1$, $0 \leqq \mu \leqq \lambda-1$. Moreover, suppose that $R_{(\nu,\mu)}$ involves edges from the a_0-column, then (9) and (4) together imply that:

($\ast$) *if $a_{0i}^{\pm 1} \omega a_{0j}^{\pm 1}$ $(i,j \in \{0,\ldots,\lambda-1\})$ is a subword of some cyclic permutation of $R_{(\nu,\mu)}$, where ω does not involve any edges from the a_0-column, then p divides $\sum_{i=0}^{\lambda-1} \sigma_{b_{i0}}(\omega)$.*

Let $\bar{\mathbf{K}}_{\mathbf{H}}$ be the 2-complex obtained by collapsing all the edges of $\mathbf{K}_{\mathbf{H}}$ except $\{a_{0i}^{\pm 1}, b_{i0}^{\pm 1} : 0 \leqq i \leqq \lambda-1\}$, and let the corresponding images of $R_{(\nu,\mu)}$ be denoted by $\bar{R}_{(\nu,\mu)}$. Since we have collapsed a maximal tree in passing from $\mathbf{K}_{\mathbf{H}}$ to $\bar{\mathbf{K}}_{\mathbf{H}}$, $\pi_1(\bar{\mathbf{K}}_{\mathbf{H}})$ is a homomorphic image of H, and has the presentation

$$\pi_1(\bar{\mathbf{K}}_{\mathbf{H}}) = \langle a_{0i}, b_{i0}\, (0 \leqq i \leqq \lambda-1); \bar{R}_{(\nu,\mu)}\, (0 \leqq \nu \leqq q-1, 0 \leqq \mu \leqq \lambda-1) \rangle.$$

Moreover, the choice of edge-set collapsed in obtaining $\bar{R}_{(\nu,\mu)}$ from $R_{(\nu,\mu)}$ implies that, for each (ν,μ),

$$\sum_{i=0}^{\lambda-1} \sigma_{a_{0i}}(\bar{R}_{(\nu,\mu)}) = \sum_{i=0}^{\lambda-1} \sigma_{a_{0i}}(R_{(\nu,\mu)}),$$

$$\sum_{i=0}^{\lambda-1} \sigma_{b_{i0}}(\bar{R}_{(\nu,\mu)}) = \sum_{i=0}^{\lambda-1} \sigma_{b_{i0}}(R_{(\nu,\mu)}),$$

and

$$\sum_{i=0}^{\lambda-1} \sigma_{b_{i0}}(\bar{\omega}) = \sum_{i=0}^{\lambda-1} \sigma_{b_{i0}}(\omega),$$

where ω is the subword of $R_{(\nu,\mu)}$ as described in ($\ast$) above.

Consequently, if L is the homomorphic image of $\pi_1(\bar{K}_H)$ obtained by adding the relations $a_{00}=a_{01}=\cdots=a_{0\lambda-1}$, $b_{00}=b_{10}=\cdots=b_{\lambda-10}$, $a_{00}^s=1$, and $b_{00}^p=1$, then $L=\langle a_{00}, b_{00}; a_{00}^s=b_{00}^p=1\rangle$. Thus if:

(8) $(s,p)\neq(2,2)$,

then $G\simeq F_2$ ([**4**], Theorem 3.7).

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REFERENCES

1. B. Baumslag and S. J. Pride, Groups with two more generators than relators, *J. London Math. Soc.* (2) **17** (1978), 425–426.

2. M. Edjvet and S. J. Pride, The concept of "largeness' in group theory II, *Proc. Conf. on Group Theory*, Korea 1983 (Lecture Notes in Mathematics **1098**, Springer, Berlin-Heidelberg-New York, 1984).

3. R. C. Lyndon and P. E. Schupp, *Combinatorial Group Theory* (Springer, Berlin-Heidelberg-New York, 1977).

4. S. J. Pride, The concept of "largeness" in group theory in *Proc. Conf. on Word and Decision Problems in Group Theory and Algebra*, Oxford, 1976.

5. J.-P. Serre, *Trees* (Springer, Berlin-Heidelberg-New York, 1980).

6. R. Stohr, Groups with one more generator than relators, *Math. Z.* **182** (1983), 45–47.

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