10

Tensor Tomography

This chapter solves the tensor tomography problem for simple surfaces following Paternain et al. (2013). We shall in fact prove a stronger result in which the absence of conjugate points is replaced by the assumption that I_0^* is surjective. In order to do this we introduce the notion of holomorphic integrating factors and prove their existence, which will be important in later chapters.

10.1 Holomorphic Integrating Factors

Let (M,g) be a compact non-trapping surface having strictly convex boundary, and consider the geodesic X-ray transform I_m that acts on symmetric m-tensor fields. Recall that the solenoidal injectivity of I_m is equivalent with a uniqueness statement for the transport equation (see Proposition 6.4.4). We will focus on proving this uniqueness statement.

Suppose that $u \in C^\infty(SM)$ solves

$$Xu = -f \text{ in } SM, \qquad u|_{\partial SM} = 0, \tag{10.1}$$

where f has degree m. For simplicity, assume that $f \in \Omega_m$. By Lemma 6.1.3, in the special coordinates (x,θ) on SM we may write

$$f(x,\theta) = \tilde{f}(x)e^{im\theta}.$$

Recall that we already know how to deal with the case where $m = 0$ (this is the injectivity of I_0 proved in Theorem 4.4.1). Let us try to reduce to this case simply by multiplying the equation (10.1) by $e^{-im\theta}$. This gives a new transport equation for $e^{-im\theta}u$:

$$(X + a)(e^{-im\theta}u) = -\tilde{f}(x), \qquad e^{-im\theta}u|_{\partial SM} = 0, \tag{10.2}$$

https://doi.org/10.1017/9781009039901.013 Published online by Cambridge University Press

where $a := -e^{im\theta}X(e^{-im\theta})$. Note that $a \in \Omega_{-1} \oplus \Omega_1$, since $X = \eta_+ + \eta_-$ and

$$e^{im\theta}\eta_\pm(e^{-im\theta}) \in \Omega_{\pm 1}.$$

We have now reduced the equation (10.1), where the right-hand side has degree m, to a new transport equation (10.2) where the right-hand side has degree 0. However, the price to pay is that the new equation has a nontrivial attenuation factor a. One could ask if there is another reduction that would remove this factor. The next example gives such a reduction in elementary ODE theory.

Example 10.1.1 (Integrating factor) Consider the ODE

$$u'(t) + a(t)u(t) = f(t), \qquad u(0) = 0.$$

The standard method for solving this ODE is to introduce the *integrating factor* $w(t) = \int_0^t a(s)\,ds$, so that the equation is equivalent with

$$\left(e^w u\right)'(t) = \left(e^w f\right)(t), \qquad \left(e^w u\right)(0) = 0.$$

Using an integrating factor has removed the zero-order term from the equation, which can now be solved just by integration. The solution is

$$u(t) = e^{-w(t)} \int_0^t \left(e^w f\right)(s)\,ds.$$

In geodesic X-ray transform problems, we are often dealing with equations such as

$$Xu + au = -f \text{ in } SM, \qquad u|_{\partial SM} = 0,$$

where $a \in C^\infty(SM)$ is an attenuation factor and $f \in C^\infty(SM)$. We would like to use an integrating factor $w \in C^\infty(SM)$ satisfying $Xw = a$ in SM, which reduces the equation to

$$X\left(e^w u\right) = -e^w f \text{ in } SM, \qquad e^w u|_{\partial SM} = 0.$$

This can always be done, for instance by choosing $w = u^{-a}$ (which may not be smooth at $\partial_0 SM$). However, in many applications one has special structure, in particular, f often has finite degree (e.g. $f = \tilde{f}(x)$ as in (10.2)). The problem with applying an arbitrary integrating factor is that multiplication by e^w may destroy this special structure. For example, if $f = \tilde{f}(x)$, then $e^w f$ could have Fourier modes of all degrees.

In this section we prove an important technical result about the existence of a certain solution of the transport equation $Xw = a$ when $a \in \Omega_{-1} \oplus \Omega_1$ (i.e. a represents a 1-form on M), where w is fibrewise *holomorphic* in the

https://doi.org/10.1017/9781009039901.013 Published online by Cambridge University Press

sense of Definition 6.1.14. This provides some control on the Fourier support of $e^w f$; e.g. if $f = \tilde{f}(x)$, then $e^w f$ is at least holomorphic. This result, which goes back to Salo and Uhlmann (2011) in the case of simple surfaces with $a \in \Omega_0$, will unlock the solution to several geometric inverse problems in two dimensions.

Proposition 10.1.2 (Holomorphic integrating factors, part I) *Let (M,g) be a compact non-trapping surface with strictly convex boundary. Assume that I_0^* is surjective. Given $a_{-1}+a_1 \in \Omega_{-1} \oplus \Omega_1$, there exists $w \in C^\infty(SM)$ such that w is holomorphic and $Xw = a_{-1} + a_1$. Similarly there exists $\tilde{w} \in C^\infty(SM)$ such that $\tilde{w}$ is anti-holomorphic and $X\tilde{w} = a_{-1} + a_1$.*

Proof We do the proof for w holomorphic; the proof for $\tilde{w}$ anti-holomorphic is analogous (or can be obtained by conjugation).

First we note that one can find $f_0 \in C^\infty(M)$ satisfying $\eta_+ f_0 = -a_1$. Indeed, by Remark 3.4.17 M is diffeomorphic to the closed unit disk $\overline{\mathbb{D}}$ and there are global special coordinates (x,θ) in SM. By Lemma 6.1.8 one has in these coordinates

$$\eta_+ f_0 = e^{-\lambda}\partial_z(f_0)e^{i\theta}, \qquad a_1 = \tilde{a}_1(x_1,x_2)e^{i\theta}.$$

Thus it is enough to find $f_0 \in C^\infty(\overline{\mathbb{D}})$ solving the equation

$$\partial_z(f_0) = -e^{\lambda}\tilde{a}_1 \quad \text{in } \mathbb{D}.$$

This equation can be solved for instance by extending the function on the right-hand side smoothly as a function in $C_c^\infty(\mathbb{C})$, and then by applying a Cauchy transform (inverse of ∂_z).

Since I_0^* is surjective, there exists $q \in C^\infty(SM)$ such that $Xq = 0$ and $q_0 = f_0$ (see Theorem 8.2.2). Recalling that $X = \eta_+ + \eta_-$ and looking at Fourier coefficients of Xq, we see that $\eta_+ q_{k-1} + \eta_- q_{k+1} = 0$ for all k. Hence

$$X(q_2 + q_4 + \cdots) = \eta_- q_2 = -\eta_+ q_0 = a_1. \tag{10.3}$$

Next, we solve $\eta_- g_0 = a_{-1}$ and use surjectivity of I_0^* to find $p \in C^\infty(SM)$ such that $Xp = 0$ and $p_0 = g_0$. Hence

$$X(p_0 + p_2 + \cdots) = \eta_- p_0 = a_{-1}. \tag{10.4}$$

Combining (10.3) and (10.4) and setting $w = \sum_{k\geq 0} p_{2k} + \sum_{k\geq 1} q_{2k}$, we see that w is holomorphic and $Xw = a_{-1} + a_1$. □

10.2 Tensor Tomography

Our main result gives a positive answer to the tensor tomography problem in the case of surfaces with I_0^* surjective.

Theorem 10.2.1 (Tensor tomography) *Let (M,g) be a compact non-trapping surface with strictly convex boundary and I_0^* surjective. The transform I_m is s-injective for any $m \geq 0$.*

We note that for the case of (M,g) simple and $m = 2$, solenoidal injectivity of I_2 was proved in Sharafutdinov (2007) using the solution to the boundary rigidity problem. We begin with a simple observation that holds in any dimension.

Lemma 10.2.2 *Let (M,g) be a compact non-trapping manifold with strictly convex boundary. If $I_0^* \colon C_\alpha^\infty(\partial_+ SM) \to C^\infty(M)$ is surjective, then $I_0 \colon C^\infty(M) \to C^\infty(\partial_+ SM)$ is injective.*

Proof Suppose that $f \in C^\infty(M)$ satisfies $I_0 f = 0$. If I_0^* is surjective, there is $w \in C_\alpha^\infty(\partial_+ SM)$ such that $I_0^* w = f$. Hence we can write

$$\|f\|^2 = (f, I_0^* w)_{L^2(M)} = (I_0 f, w)_{L^2_\mu(\partial_+ SM)} = 0,$$

and thus $f = 0$. □

The next result is the master result from which tensor tomography is derived. It asserts, in terms of the transport equation, that $I|_{\Omega_m} : \Omega_m \to C^\infty(\partial_+ SM)$ is injective whenever I_0^* is surjective.

Theorem 10.2.3 (Injectivity of $I|_{\Omega_m}$) *Let (M,g) be a compact non-trapping surface with strictly convex boundary and I_0^* surjective. Assume that $m \in \mathbb{Z}$, and let $u \in C^\infty(SM)$ be such that*

$$Xu = -f \in \Omega_m, \qquad u|_{\partial SM} = 0.$$

Then $u = 0$ and $f = 0$.

The proof is based on another important injectivity result, where the fact that f has one-sided Fourier support is used to deduce that u has one-sided Fourier support. A more precise result in this direction will be given in Proposition 10.2.6.

Proposition 10.2.4 *Let (M,g) be a compact non-trapping surface with strictly convex boundary and I_0 injective. If $u \in C^\infty(SM)$ is odd and satisfies*

$$Xu = -f \text{ in } SM, \qquad u|_{\partial SM} = 0,$$

https://doi.org/10.1017/9781009039901.013 Published online by Cambridge University Press

where f is holomorphic (respectively anti-holomorphic), then u is holomorphic (respectively anti-holomorphic).

Proof We prove the case where f is holomorphic. Write $q := \sum_{k=-\infty}^{-1} u_k$. Since f is holomorphic, we have $(Xu)_k = 0$ for $k \leq -1$, and using the decomposition $X = \eta_+ + \eta_-$ this gives that $\eta_+ u_{k-1} + \eta_- u_{k+1} = 0$ for $k \leq -1$. Thus we obtain that

$$Xq = \eta_+ u_{-1}, \qquad q|_{\partial SM} = 0.$$

Now $\eta_+ u_{-1}$ only depends on x, and hence the injectivity of I_0 implies that $\eta_+ u_{-1} = 0$. This proves that $q = 0$ showing that u is holomorphic. □

Proof of Theorem 10.2.3 We follow the approach described at the beginning of Section 10.1. Let $r := e^{-im\theta}$ and observe that $r^{-1}Xr \in \Omega_{-1} \oplus \Omega_1$ since

$$e^{im\theta}\eta_\pm(e^{-im\theta}) \in \Omega_{\pm 1}.$$

By Proposition 10.1.2, there is a holomorphic $w \in C^\infty(SM)$ and anti-holomorphic $\tilde{w} \in C^\infty(SM)$ such that $Xw = X\tilde{w} = -r^{-1}Xr$. Since $r^{-1}Xr$ is odd, without loss of generality we may replace w and $\tilde{w}$ by their even parts so that w and $\tilde{w}$ are even. A simple calculation shows that

$$X\left(e^w ru\right) = e^w\left(X - r^{-1}Xr\right)(ru) = -e^w rf \tag{10.5}$$

with a similar equation for $\tilde{w}$. Since $rf \in \Omega_0$, $e^w rf$ is holomorphic and $e^{\tilde{w}} rf$ is anti-holomorphic.

Assume now that m is even, the proof for m odd being very similar. Then we may assume that u is odd and thus $e^w ru$ and $e^{\tilde{w}} ru$ are odd. By Proposition 10.2.4, since we have

$$X\left(e^w ru\right) = -e^w rf, \qquad e^w ru|_{\partial SM} = 0,$$

we see that $e^w ru$ is holomorphic and thus $ru = e^{-w}(e^w ru)$ is holomorphic. Arguing with $\tilde{w}$ we deduce that ru is also anti-holomorphic. Thus one must have $ru \in \Omega_0$. This implies that $u \in \Omega_m$, and using that $Xu \in \Omega_m$ we see that $Xu = 0$ and finally $u = f = 0$ as desired. □

One can explicitly compute $r^{-1}Xr$ in the proof above using isothermal coordinates in which the metric is $e^{2\lambda}(dx_1^2 + dx_2^2)$:

Exercise 10.2.5 Show that

$$r^{-1}Xr = m\eta_+(\lambda) - m\eta_-(\lambda).$$

By inspecting the proof of Proposition 10.1.2 show that the conclusion of Theorem 10.2.3 still holds if we assume that I_0 is injective and there is a

https://doi.org/10.1017/9781009039901.013 Published online by Cambridge University Press

smooth q such that $Xq = 0$ with $q_0 = \lambda$. Hence surjectivity of I_0^* is only needed for the function λ!

We will give two corollaries of Theorem 10.2.4.

Proposition 10.2.6 *Let (M, g) be a compact non-trapping surface with strictly convex boundary and I_0^* surjective. Let $u \in C^\infty(SM)$ be such that*

$$Xu = -f, \qquad u|_{\partial SM} = 0.$$

Suppose $f_k = 0$ for $k \geq m+1$ for some $m \in \mathbb{Z}$. Then $u_k = 0$ for $k \geq m$. Similarly, if $f_k = 0$ for $k \leq m-1$ for some $m \in \mathbb{Z}$, then $u_k = 0$ for $k \leq m$.

Proof Suppose $f_k = 0$ for $k \geq m+1$. Let $w := \sum_m^\infty u_k$. Using the equation $Xu = -f$ and the hypothesis on f, we see that

$$Xw = \eta_- u_m + \eta_- u_{m+1} \in \Omega_{m-1} \oplus \Omega_m.$$

Applying Theorem 10.2.3 to the even and odd parts of w, we deduce that $w = 0$ and thus $u_k = 0$ for $k \geq m$. Similarly, arguing with $\sum_{-\infty}^m u_k$ we deduce that $u_k = 0$ for $k \leq m$ if $f_k = 0$ for $k \leq m-1$. $\square$

The next corollary is an obvious consequence of the previous proposition.

Corollary 10.2.7 (Tensor tomography, transport version) *Let (M, g) be a non-trapping surface with strictly convex boundary and I_0^* surjective. Let $u \in C^\infty(SM)$ be such that*

$$Xu = f, \qquad u|_{\partial SM} = 0.$$

Suppose $f_k = 0$ for $|k| \geq m+1$ for some $m \geq 0$. Then $u_k = 0$ for $|k| \geq m$ (when $m = 0$, this means $u = f = 0$).

By Proposition 6.4.4, the previous result also proves Theorem 10.2.1.

10.3 Range for Tensors

In this section we explain how some of the ideas of the previous section can be employed to give a description of the range for the X-ray transform acting on symmetric tensors of any rank, pretty much in the spirit of Theorem 9.6.2.

Let (M, g) be a non-trapping surface with strictly convex boundary. Pick a function $h\colon SM \to S^1 \subset \mathbb{C}$ such that $h \in \Omega_1$. Such a function always exists: for instance, in global isothermal coordinates we may simply take $h = e^{i\theta}$. Our description of the range will be based on this choice of h. Define the 1-form

$$A := -h^{-1}Xh.$$

Observe that since $h \in \Omega_1$, then $h^{-1} = \bar{h} \in \Omega_{-1}$. Also $Xh = \eta_+ h + \eta_- h \in \Omega_2 \oplus \Omega_0$, which implies that $A \in \Omega_1 \oplus \Omega_{-1}$. It follows that A is the restriction to SM of a purely imaginary 1-form on M.

First we will describe the range of the geodesic ray transform I restricted to Ω_m:

$$\mathbf{I}_m := I|_{\Omega_m} \colon \Omega_m \to C^\infty(\partial_+ SM, \mathbb{C}).$$

Observe that if u solves the transport equation $Xu = -f$ where $f \in \Omega_m$ and $u|_{\partial_- SM} = 0$, then $h^{-m}u$ solves $(X - mA)(h^{-m}u) = -h^{-m}f$ and $h^{-m}u|_{\partial_- SM} = 0$. Also note that $h^{-m}f \in \Omega_0$. Thus

$$I_{-mA}(h^{-m}f) = \left(h^{-m}|_{\partial_+ SM}\right) \mathbf{I}_m(f), \tag{10.6}$$

where the left-hand side is an attenuated X-ray transform with attenuation $-mA$ as given in Definition 5.3.3. The relation in (10.6) is telling us that if we know how to describe the range of I_A acting on $C^\infty(M)$, where A is a purely imaginary 1-form, then we would know how to describe the range of $\mathbf{I}_m$. It turns out that this is possible to do even in much greater generality, namely when A is a *connection* (cf. Theorem 14.5.5). We will return to this topic in later chapters; for the time being we content ourselves with a description of the results.

Let $Q_m \colon C(\partial_+ SM, \mathbb{C}) \to C(\partial SM, \mathbb{C})$ be given by

$$Q_m w(x,v) := \begin{cases} w(x,v) & \text{if } (x,v) \in \partial_+ SM, \\ (e^{-m\int_0^{\tau(x,v)} A(\varphi_t(x,v))\,dt} w) \circ \alpha(x,v) & \text{if } (x,v) \in \partial_- SM, \end{cases}$$

and let $B_m \colon C(\partial SM, \mathbb{C}) \to C(\partial_+ SM, \mathbb{C})$ be

$$B_m g := \left[g - e^{m\int_0^{\tau(x,v)} A(\varphi_t(x,v))\,dt} (g \circ \alpha) \right] \Big|_{\partial_+ SM}.$$

In other words, with I_1 denoting the X-ray transform on 1-tensors, we have

$$Q_m w(x,v) = \begin{cases} w(x,v) & \text{if } (x,v) \in \partial_+ SM, \\ (e^{-mI_1(A)} w) \circ \alpha(x,v) & \text{if } (x,v) \in \partial_- SM, \end{cases}$$

and

$$B_m g = \left[g - e^{mI_1(A)} (g \circ \alpha) \right] \Big|_{\partial_+ SM}.$$

We define

$$P_{m,-} := B_m H_- Q_m.$$

The following result from Paternain et al. (2015b) describes the range of $\mathbf{I}_m$.

Theorem 10.3.1 *Assume that* (M, g) *is a simple surface. A function* $u \in C^\infty(\partial_+ SM, \mathbb{C})$ *belongs to the range of* $\mathbf{I}_m$ *if and only if* $u = \left(h^m|_{\partial_+ SM}\right) P_{m,-} w$ *for* $w \in \mathcal{S}^\infty_m(\partial_+ SM, \mathbb{C})$*, where this last space denotes the set of all smooth* w *such that* $Q_m w$ *is smooth.*

Suppose that F is a complex-valued symmetric tensor of order m and denote its restriction to SM by f. Recall from Proposition 6.3.5 that there is a one-to-one correspondence between complex-valued symmetric tensors of order m and functions in SM of the form $f = \sum_{k=-m}^{m} f_k$, where $f_k \in \Omega_k$ and $f_k = 0$ for all k odd (respectively even) if m is even (respectively odd).

Since

$$I(f) = \sum_{k=-m}^{m} \mathbf{I}_k(f_k),$$

we deduce directly from Theorem 10.3.1 the following.

Theorem 10.3.2 *Let* (M, g) *be a simple surface. If* $m = 2l$ *is even, a function* $u \in C^\infty(\partial_+ SM, \mathbb{C})$ *belongs to the range of the X-ray transform acting on complex-valued symmetric* m*-tensors if and only if there are* $w_{2k} \in \mathcal{S}^\infty_{2k}(\partial_+ SM, \mathbb{C})$ *such that*

$$u = \sum_{k=-l}^{l} \left(h^{2k}|_{\partial_+ SM}\right) P_{2k,-} w_{2k}.$$

Similarly, if $m = 2l + 1$ *is odd, a function* $u \in C^\infty(\partial_+ SM, \mathbb{C})$ *belongs to the range of the X-ray transform acting on complex-valued symmetric* m*-tensors if and only if there are* $w_{2k+1} \in \mathcal{S}^\infty_{2k+1}(\partial_+ SM, \mathbb{C})$ *such that*

$$u = \sum_{k=-l-1}^{l} \left(h^{2k+1}|_{\partial_+ SM}\right) P_{2k+1,-} w_{2k+1}.$$