Canad. Math. Bull. Vol. 19 (3), 1976

TANGENT CONES AND CONVEXITY

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Introduction and preliminaries. The study of general multiplier theorems (Kuhn–Tucker Conditions) for constrained optimization problems has led to extensions of the notion of a differentiable arc. Abadie [1], Varaiya [10], Guignard [5], Zlobec [11] and Massam [12] investigated the so called cone of tangent vectors to a point in a set for optimization purposes.

Among other things, these tangent cones have allowed a weakening of the classical convexity conditions, introduced by Kuhn and Tucker [7], which insure that first order necessary conditions become sufficient for constrained optima to exist. This has motivated the present authors to examine the relationship (in locally convex spaces) between convexity and various tangent cone properties.

We show that in a variety of situations, convexity and pseudoconvexity (see below) coincide; we also give limiting counter-examples.

One result of the work is a new characterization of convexity in finite dimensions. Another is that in some sense convexity is the weakest possible sufficiency condition that can be imposed globally on a constraint function.

1. **Definitions.** In the sequel X denotes a real Hausdorff locally convex space. If a_n and a belong to X, then $a_n \to a (a_n \rightharpoonup a)$ denotes convergence in the original (weak) topology of X; if A is a subset of X, then $\bar{A}(\bar{A}^{\sigma})$ denotes the closure of A in the original (weak) topology and $coA(\overline{co}A)$ denotes the (closed) convex hull of A. Notation and terms which we use without defining are standard and may be found in [3] or [9].

DEFINITION 1. Let $A \subset X$ be an arbitrary non-empty set and let $\bar{a} \in \bar{A}$. Then h is called a *tangent* to A at $\bar{a}$ if there is a sequence

$$t_n(a_n - \bar{a}) \to h \quad \text{where} \quad \{a_n\} \subset A,\, a_n \to \bar{a},\, t_n \geq 0.$$

The set of all such tangents is called the *tangent cone* to A at $\bar{a}$ and is denoted $T(A, \bar{a})$.

Received by the editors April 15, 1976.
* Supported in part by NRC grants A7751 and A7675.
† Supported by an I. W. Killam Postdoctoral Fellowship.

https://doi.org/10.4153/CMB-1976-040-x Published online by Cambridge University Press

DEFINITION 2. Similarly, h is called a *weak tangent* to A at $\bar{a}$ if there is a sequence $t_n(a_n-\bar{a})\rightharpoonup h$ *where*

$$\{a_n\}\subset A, \qquad a_n\to\bar{a}, \qquad t_n\geq 0.$$

The set of all such weak tangents is called the *weak tangent cone* to A at $\bar{a}$ and is denoted $wT(A,\bar{a})$. Definition 1 was introduced by Abadie [1] to generalize the concept of a differentiable arc for optimization purposes. Clearly both $T(A,\bar{a})$ and $wT(A,\bar{a})$ are non-empty cones for any $\bar{a}\in\bar{A}$. In a metrizable space $T(A,\bar{a})$ is actually closed [10]. In general, however, these cones are not convex. Guignard [5] introduced the following definition:

DEFINITION 3. The closed convex hull of $T(A,\bar{a})$ is called the *pseudotangent cone* to A at $\bar{a}$ and is denoted $P(A,\bar{a})$.

In studying the sufficiency of first order optimization conditions Guignard [5] has introduced the concept of pseudoconvexity of a set A at a point $\bar{a}$. It has also been studied by Zlobec and Massam [12], Borwein [2] and others.

DEFINITION 4. A set A is said to be *pseudoconvex* at $\bar{a}$ if $A-\bar{a}\subset P(A,\bar{a})$.

A set A is said to be *starshaped* at $\bar{a}$ if $a\in A$ implies that $ta+(1-t)\bar{a}\in A$ for $0\leq t\leq 1$. It is easy to see that when A is starshaped at $\bar{a}$, A is pseudoconvex at $\bar{a}$. It follows that a convex set is pseudoconvex at all its members. Our central results concern the converse to this observation.

DEFINITION 5. A closed set A will be called *pseudoconvex* if it is pseudoconvex at every $a\in A$.

REMARK. We consider only closed sets in the definition since there are trivially non-closed pseudoconvex sets that are not convex. In particular, an open set is pseudoconvex at all its members.

2. **Convexity and pseudoconvexity.**

PROPOSITION 1. (i) *Let E be a Hausdorff locally convex topological vector space and A a weakly compact subset of E. Then*

$$\bigcap_{a\in A}(\overline{wT(A,a)}^{\sigma}+a)\subset A.$$

(ii) *Let E be the dual of a normed space and A a weak* closed subset of E. Then*

$$\bigcap_{a\in A}(T(A,a)+a)\subset A.$$

Proof. (i) Suppose that y is not in A. Then there exists a weakly open convex neighborhood U of 0 such that $y+U$ is disjoint from A. By the

https://doi.org/10.4153/CMB-1976-040-x Published online by Cambridge University Press

compactness of A we can find a real number λ_0 with $(y+\lambda_0\bar{U})\cap A\neq\emptyset$ but with $y+\lambda_0 U$ disjoint from A. Fix $\bar{a}$ in $(y+\lambda_0\bar{U})\cap A$. We show that no member of $y+\lambda_0 U$ is in $wT(A, \bar{a})+\bar{a}$.

For suppose that some y' is. Then there exist $t_n>0$, and a_n converging to $\bar{a}$ such that $t_n(a_n-\bar{a})$ converges weakly to $y'-\bar{a}$. Thus $t_n(a_n-\bar{a})+\bar{a}$ converges weakly to y', and for all n larger than some n_0, we must have $t_n(a_n-\bar{a})+\bar{a}$ in $y+\lambda_0 U$. Since the t_n's are increasing to infinity, for n large enough $(1/t_n)[t_n(a_n-\bar{a})+\bar{a}]+(t_n-1/t_n)\bar{a}=a_n$ is a convex combination of a point in $y+\lambda_0 U$ and a point, $\bar{a}$, in $y+\lambda_0\bar{U}$. By the convexity of $y+\lambda_0 U$, we must have $a_n\in y+\lambda_0 U$ for such n. But this contradicts the choice of λ_0. Hence we have that if y is not in A, then for some $\bar{a}$ in A, y is not in $\overline{wT(A, \bar{a})}^{\sigma}$.

(ii) The proof proceeds in the same manner as the proof of (i) except that U is chosen to be an open ball in the norm topology and λ_0 is chosen using the weak* compactness of $\bar{U}$. Since in this case we are dealing with $T(A, \bar{a})$, we will have $t_n(a_n-\bar{a})+\bar{a}$ converging in the norm to y' and hence being in $y+\lambda_0 U$ eventually.

We now state our main theorem.

THEOREM 1. *Suppose that E is a normed linear space and A is a subset of E. If either* (i) *A is weakly compact or* (ii) *E is the dual of a normed space, A is weak* closed and E has an equivalent smooth norm, then $\bigcap_{a\in A}(P(A, a)+a)\subset A$. In particular, in either of these cases, if A is pseudoconvex then A is convex.*

Proof. In case (i) since A is weakly compact in E, it is weakly compact in the completion $\hat{E}$ of E. Hence by considering the closed space which A generates in $\hat{E}$, we assume that E is a weakly compactly generated Banach space. Since a Banach space which is weakly compactly generated has an equivalent smooth norm [3, p. 160], in both cases (i) and (ii) we can assume that A is a subset of a Banach space E which is smooth.

Now suppose y is not in A. Denote by $B(y, r)$ the open ball centered at y with radius r. If $A\cap\overline{B(y, r)}\neq\emptyset$ then in case (i) $A\cap\overline{B(y, r)}$ is weakly compact and in case (ii) it is weak* compact. In either case, by using the compactness, we can find an $r_0>0$ such that $\overline{B(y, r_0)}\cap A\neq\emptyset$ but $B(y, r_0)$ and A are disjoint. Fix $\bar{a}\in\overline{B(y, r_0)}\cap A$. By the Hahn–Banach separation theorem there exists a continuous linear functional f which separates $\bar{a}$ from $B(y, r_0)$; that is, $f(\bar{a})=\lambda>f(x)$ for all x in $B(y, r_0)$. We will show that $T(A, \bar{a})+\bar{a}$ is contained in $\{x: f(x)\geq\lambda\}$. Hence we will have $P(A, \bar{a})+\bar{a}$ contained in $\{x: f(x)\geq\lambda\}$, and since y is not in $\{x: f(x)\geq\lambda\}$, the proof of the first statement will be complete.

So suppose $h\in T(A, \bar{a})+\bar{a}$ and let $[\bar{a}, h]$ denote the line segment with endpoints $\bar{a}$ and h. By the same argument as that used in Proposition 1, we cannot have $[\bar{a}, h]\cap B(y, r_0)$ non-empty (since, because $[\bar{a}, h]$ is in $T(A, \bar{a})+\bar{a}$, this would imply that $A\cap B(y, r_0)$ is non-empty). Hence $[\bar{a}, h]$ and $B(y, r_0)$ are disjoint convex sets and can be separated by a continuous linear functional g.

Since $\bar{a}$ belongs to $[\bar{a}, h]$, g also separates $\bar{a}$ from $B(y, r_0)$. The smoothness of the norm now implies that $g = cf$ for some constant c and so $[\bar{a}, h]$ is contained in $\{x : f(x) \geq \lambda\}$. Thus $T(A, \bar{a}) + a$ is contained in $\{x : f(x) \geq \lambda\}$.

If A is also pseudoconvex, then $P(A, a) + a \supset A$ for all a in A. Hence we have $A = \bigcap_{a \in A} (P(A, a) + a)$ which is convex.

It is shown in [3, p. 160] that every separable Banach space and every reflexive Banach space has an equivalent smooth norm. Since in a reflexive Banach space the weak* and weak topologies agree, we have the following corollary.

Corollary 1. *In a reflexive Banach space every weakly closed pseudoconvex set is convex. Hence in a finite dimensional space a closed set is pseudoconvex if and only if it is convex.*

When X is a *superreflexive* space [3, p. 169] we can improve Theorem 1. Specifically:

Theorem 2. *A norm closed pseudoconvex set A in a superreflexive space is convex.*

Proof. Since X is superreflexive we may assume that the norm in X is simultaneously smooth and uniformly convex [3, p. 164]. Suppose $y \notin A$. There is a (smooth, uniformly convex) ball $B(y, \varepsilon)$ disjoint from A. By a result of Edelstein [4], some $\bar{y} \in B(y, \varepsilon/2)$ has a (unique) nearest point $\bar{a}$ in A. Consider $B(\bar{y}, \|\bar{y} - \bar{a}\|)$. As in Theorem 1, $\bar{a}$ is a support point of this ball and $P(A, \bar{a}) + \bar{a}$ is disjoint from $B(\bar{y}, \|\bar{y} - \bar{a}\|)$. Since $y \in B(\bar{y}, \varepsilon/2) \subset B(\bar{y}, \|\bar{y} - \bar{a}\|)$, $y \notin P(A, \bar{a}) + \bar{a}$. Proceeding as in Theorem 1 we see that A is convex.

This in particular implies that pseudoconvexity and convexity coincide for closed sets in Hilbert space or L_p spaces, $1 < p < \infty$. Note also that the proof can be adapted to any Banach space in which nearest points exist densely for closed sets.

Finally, we give a limiting example of a non-convex closed, bounded pseudoconvex set. The following proposition is central to this example.

Proposition 2. *If $A \subset X$ is pseudoconvex at $\bar{a}$, then either $\bar{a}$ is a support point of a, or $P(A, \bar{a}) = X$.*

Proof. Suppose A is pseudoconvex and $P(A, \bar{a}) \neq X$. Then there is some $x \in X$ with $x \notin P(A, \bar{a})$. By the separation theorem [3] we can find some non-zero continuous linear functional f such that $f(x) \leq f(h)$ for all h in $P(A, \bar{a})$. Since $P(A, \bar{a})$ is a cone, $f(h) \geq 0$ for all $h \in P(A, \bar{a})$. In particular $A - \bar{a} \subset P(A, \bar{a})$ so $f(a) \geq f(\bar{a})$ for all $a \in A$. Hence $\bar{a}$ is a support point of A.

Example. (i) There is a dense subspace E of $l_2(N)$ which contains a weakly closed, bounded disconnected set A which is pseudoconvex. In particular A is pseudoconvex but not convex.

(ii) There is a Frechet space F which contains a similar set A.

Proof. (i) Klee [6] has exhibited a dense subspace E of $1_2(N)$ which contains a closed bounded convex set S with no support points. Since S is bounded, we can pick a point x in E such that $(S+x)\cap S=\varnothing$. Let $A=(S+x)\cup S$. Then A is clearly bounded and disconnected, and since each component is closed and convex, it is weakly closed. To show pseudoconvexity, suppose $\bar{a}\in A$. We can assume $\bar{a}\in S$ without loss of generality. Then since S is (pseudo) convex with no support points, Proposition 2 implies that $P(S, \bar{a})=E$. Hence $P(A, \bar{a})=E$ for all $\bar{a}\in A$ and A is trivially pseudoconvex.

(ii) The construction is the same as in (i) using the existence (due to Peck [8]) of a Frechet space which contains a closed bounded convex set S with no support points.

Note that A so defined cannot be weakly compact by Proposition 1. The following questions remain unanswered.

(i) In a Banach space is every norm closed pseudoconvex set convex?

(ii) In a normed space is every norm closed relatively weakly compact pseudoconvex set convex?

(iii) In (i) or (ii) does it help to make the space reflexive?

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https://doi.org/10.4153/CMB-1976-040-x Published online by Cambridge University Press