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A LITTLEWOOD AND PALEY-TYPE INEQUALITY ON THE BALL

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A multi-dimensional analogue of a well known inequality of Littlewood and Paley is obtained on the ball.

1. INTRODUCTION AND STATEMENT OF THE RESULT

A well known inequality of Littlewood and Paley states that (see [**2, 3**]) : *if* $2 \leqslant p < \infty$ *and* f *is a function, analytic in the unit disk,* D*, which belongs to the Hardy space* $H^p(D)$*, then there exists a positive constant* $C = C(p)$ *such that*

$$\int_D \left(1 - |z|^2\right)^{p-1} |f'(z)|^p \, dA(z) \leqslant C \|f\|^p_{H^p(D)}$$

where A *is the normalised area measure on* D.

Recently Luecking [**3**] gave a new proof of this result, which motivates our work. In this note we shall extend the result of Littlewood and Paley to the multi-dimensional case. To state our result, we need some notation and definitions.

Let B be the unit ball of the n-dimensional complex space $\mathbf{C}^n$. (Hereafter n will be fixed.) The letter V stands for the normalised Lebesgue measure on B so that $V(B) = 1$, while σ is the normalised surface area measure on its boundary S.

For $0 < p < \infty$, the Hardy space $H^p(B)$ is defined to be the set of all analytic functions f on B for which

$$\|f\|^p_{H^p(B)} = \sup_{0<r<1} \int_S |f(r\xi)|^p \, d\sigma(\xi) < \infty.$$

It is well known [4] that if $f \in H^p(B)$ for $0 < p < \infty$, then the radial limit $f^*(\xi) = \lim\limits_{r \to 1} f(r\xi)$ exists for $[\sigma]$-almost all $\xi \in S$ and $\|f\|^p_{H^p(B)} = \int_S |f^*(\xi)|^p \, d\sigma(\xi)$. We denote $f \in H^p(B)$ and its radial limit function f^* by the same latter f.

The result in this note is formulated as follows :

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Theorem. *Let $2 \leqslant p < \infty$. Then there exists a positive constant $C = C(p)$ such that*

$$\int_B \left(1-|z|^2\right)^{p/2-1} \left(|\nabla f(z)|^2 - |\mathcal{R}f(z)|^2\right)^{p/2} dV(z) \leqslant C \|f\|^p_{H^p(B)}$$

for all $f \in H^p(B)$.

Here and elsewhere, $\nabla f(z) = \left(\dfrac{\partial f}{\partial z_1}, \cdots, \dfrac{\partial f}{\partial z_n}\right)$ denotes the complex gradient of f and $\mathcal{R}f(z) = \sum_{j=1}^{n} z_j \dfrac{\partial f}{\partial z_j}$ denotes the radial derivative of f.

2. Preliminaries

2.1 Automorphisms of B. For $z, w \in \mathbf{C}^n$ let $\langle z, w\rangle = \sum_{j=1}^{n} z_j \overline{w}_j$ denote the complex inner product on $\mathbf{C}^n$ and $|z| = \langle z, z\rangle^{1/2}$. For $a, z \in B$, $(a \neq 0)$ define

$$\varphi_a(z) = \frac{a - P_a z - \left(1-|a|^2\right)^{1/2} Q_a z}{1 - \langle z, a\rangle}$$

where $P_a z = \langle z, a\rangle\, a/|a|^2$ and $Q_a z = z - P_a z$. For $a = 0$ we let $\varphi_0(z) = -z$. Then $\varphi_a(0) = a$ and $(\varphi_a \circ \varphi_a)(z) = z$ for all $z \in B$. Thus $\varphi_a \in \operatorname{Aut}(B)$, the group of all automorphsims of B. Furthermore each $\psi \in \operatorname{Aut}(B)$ has a unique representation $\psi = U \circ \varphi_a$ for some $a \in B$ and some unitary transformation U on $\mathbf{C}^n$. The following two properties of φ_a are found in Rudin [4, Section 2.2].

(1) $$1 - |\varphi_a(z)|^2 = \frac{\left(1-|a|^2\right)\left(1-|z|^2\right)}{|1-\langle z, a\rangle|^2}$$

(2) $$J_R\varphi_a(z) = \left(\frac{1-|a|^2}{|1-\langle z, a\rangle|^2}\right)^{n+1}$$

where $J_R\varphi_a$ is the real Jacobian of φ_a.

2.2 Invariant Laplacian. For $f \in C^2(B)$ and $a \in B$, we define

$$\left(\widetilde{\Delta} f\right)(a) = \frac{1}{n+1}\Delta(f \circ \varphi_a)(0)$$

where $\Delta = 4 \sum_{j=1}^{n} \dfrac{\partial^2}{\partial a_j \partial \overline{a}_j}$ is the ordinary Laplacian. The operator $\widetilde{\Delta}$ is invariant under $\operatorname{Aut}(B)$, that is, $\left(\widetilde{\Delta} f\right) \circ \psi = \widetilde{\Delta}(f \circ \psi)$ for all $\psi \in \operatorname{Aut}(B)$. For this reason, $\widetilde{\Delta}$ is called the invariant Laplacian. In terms of partial derivatives, the invariant Laplacian $\widetilde{\Delta}$ is computed as follows:

$$(3)\qquad \left(\widetilde{\Delta}f\right)(a) = \frac{4}{n+1}\left(1-|a|^2\right)\sum_{j,k=1}^{n}(\delta_{jk}-a_j\bar{a}_k)\frac{\partial^2 f(a)}{\partial a_j \partial \bar{a}_k},$$

where δ_{jk} is the Kroneker symbol. See [**4**, Section 4.1] for details.

2.3 Green's functions for $\widetilde{\Delta}$. For each $a \in B$, we define the Green's funtion $G(z,a)$ for $\widetilde{\Delta}$ on B by

$$G(z,a) = g(\varphi_a(z))$$

where $g(z) = (n+1)/(2n)\int_{|z|}^{1}\left(1-t^2\right)^{n-1}t^{-2n+1}\,dt$. See [**5**] for more information about Green's functions.

3. Proof of the theorem

We begin with the following Littlewood-Paley type identity whose proof is a slight modification of ones given in [**1**, Theorem A], and so we state it without proof.

Lemma 1. *If $2 \leqslant p < \infty$ and $h \in H^p(B)$, then*

$$(4)\qquad \int_S |h|^p\,d\sigma = |h(0)|^p + \int_B \left(\widetilde{\Delta}\,|h|^p\right)(z)\,g(z)\,d\tau(z)$$

where $g(z)$ is the function as defined in Section 2 and $d\tau(z) = \left(1-|z|^2\right)^{-(n+1)}dV(z)$.

Lemma 2. *Suppose $2 \leqslant p < \infty$ and $a \in B$. Then*

$$(5)\qquad \left(\widetilde{\Delta}\,|f|^p\right)(a) = \frac{p^2}{4}|f(a)|^{p-2}\left(\widetilde{\Delta}\,|f|^2\right)(a)$$

for every analytic function f on B.

Proof: For f analytic on B, using the property (3), we have

$$(6)\qquad \left(\widetilde{\Delta}\,|f|^2\right)(a) = \frac{4}{n+1}\left(1-|a|^2\right)\left(|\nabla f(a)|^2 - |\mathcal{R}f(a)|^2\right).$$

Replacing f by $f^{p/2}$ in (6) (here $f^{p/2} = \exp((p/2)\log f)$ is the principal branch), we get

$$(7)\qquad \left(\widetilde{\Delta}\,|f|^p\right)(a) = \frac{4}{n+1}\left(1-|a|^2\right)\left(\left|\nabla f^{p/2}(a)\right|^2 - \left|\mathcal{R}f^{p/2}(a)\right|^2\right).$$

A little computation shows that

$$\left|\nabla f^{p/2}(a)\right|^2 = \frac{p^2}{4}\,|f(a)|^{p-2}\,|\nabla f(a)|^2$$

and

$$\left|\mathcal{R} f^{\frac{p}{2}}(a)\right|^2 = \frac{p^2}{4}\,|f(a)|^{p-2}\,|\mathcal{R} f(a)|^2.$$

By inserting these into (7) we obtain

$$\left(\widetilde{\Delta}\,|f|^p\right)(a) = \frac{p^2}{n+1}\left(1-|a|^2\right)|f(a)|^{p-2}\left(|\nabla f(a)|^2 - |\mathcal{R} f(a)|^2\right),$$

which gives the desired conclusion. □

Lemma 3. *If $2 \leqslant p < \infty$ and $f \in H^p(B)$, then there exists a positive constant $C = C(p)$ such that*

$$|\nabla f(0)|^p \leqslant C\left(\|f\|^p_{H^p(B)} - |f(0)|^p\right). \tag{8}$$

Proof: Suppose $f \in H^p(B)$ with $2 \leqslant p < \infty$. Then by the reproducing property we have

$$f(z) - f(0) = \int_S \frac{f(\xi) - f(0)}{(1-\langle z, \xi\rangle)^n}\,d\sigma(\xi).$$

Differentiation under the integral sign gives

$$\frac{\partial f}{\partial z_j}(0) = n\int_S \xi_j\,(f(\xi) - f(0))\,d\sigma(\xi),\ 1 \leqslant j \leqslant n.$$

It follows that

$$\begin{aligned}|\nabla f(0)|^2 &\leqslant n^2 \int_S |f(\xi) - f(0)|^2\,d\sigma(\xi)\\ &= n^2\left(\|f\|^2_{H^2(B)} - |f(0)|^2\right).\end{aligned}$$

Without loss of generality, we may assume that $f(0) = 0$ or $|f(0)| = 1$.

In the first case, we clearly obtain

$$|\nabla f(0)|^p \leqslant n^p\left(\|f\|^p_{H^p(B)} - |f(0)|^p\right)$$

because $\|f\|_{H^2(B)} \leqslant \|f\|_{H^p(B)}$ for $p \geqslant 2$.

We next consider the case $|f(0)| = 1$. A simple calculation shows that

$$(x-1)^{p/2} \leqslant x^{p/2} - 1 \text{ for } x \geqslant 1. \tag{9}$$

Replacing x by $\|f\|^2_{H^p(B)}$ and 1 by $|f(0)|^2$ in (9), we deduce that

$$\begin{aligned} |\nabla f(0)|^p &\leqslant n^p \left(\|f\|^2_{H^p(B)} - |f(0)|^2 \right)^{p/2} \\ &\leqslant n^p \left(\|f\|^p_{H^p(B)} - |f(0)|^p \right) \end{aligned}$$

which shows Lemma 3. □

In the proof of the rest of this paper, we use the same letter C to denote a positive constant which may change with each occurrence.

Lemma 4. *Let $G(z,a)$ and $g(z)$ be as defined in Section 2. Then there exists a positive constant C such that*

$$\int_B \frac{G(z,a)}{1-|a|^2}\, dV(a) \leqslant C g(z). \tag{10}$$

Proof: Since $|\varphi_a(z)| = |\varphi_z(a)|$, we have

$$\int_B \frac{G(z,a)}{1-|a|^2}\, dV(a) = \int_B \frac{g(\varphi_a(z))}{1-|a|^2}\, dV(a) = \int_B \frac{g(\varphi_z(a))}{1-|a|^2}\, dV(a).$$

By making the change of variable $\varphi_z(a) = w$ and using formulas (1) and (2), we see that the last integral becomes

$$\left(1-|z|^2\right)^n \int_B \frac{g(w)}{\left(1-|w|^2\right)|1-\langle w,z\rangle|^{2n}}\, dV(w) = \left(1-|z|^2\right)^n (\mathrm{I}+\mathrm{II}), \tag{11}$$

where I is the integral over $|w| \geqslant 1/2$ and II is the integral over $|w| < 1/2$. It remains to show that each integral of I and II in (11) is bounded.

On $|w| \geqslant 1/2$, we clearly have $g(w) \leqslant C\left(1-|w|^2\right)^n$. Thus

$$\mathrm{I} \leqslant C \int_{|w|\geqslant 1/2} \frac{\left(1-|w|^2\right)^{n-1}}{|1-\langle w,z\rangle|^{2n}}\, dV(w) \leqslant C \int_B \frac{\left(1-|w|^2\right)^{n-1}}{|1-\langle w,z\rangle|^{2n}}\, dV(w).$$

Therefore the first integral I is bounded by Proposition 1.4.10 in [4].

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For the second integral, it is easy to verify that

$$\mathrm{II} \leqslant C \int_{|w|<1/2} g(w)\, dV(w) \leqslant C \int_{|w|<1/2} \frac{1}{|w|^{2n-1}}\, dV(w)$$

and integrating in polar coordinates the last integral shows that the second integral II is bounded . This finishes the proof. □

We now turn to the proof of the main result of this paper.

PROOF OF THEOREM: Suppose f is analytic on B. By successive applications of (8), (4) and (5) in this order, we have

$$\begin{aligned}\left[\widetilde{\Delta}\,|f|^2(0)\right]^{p/2} = |\nabla f(0)|^p &\leqslant C \int_S |f(\xi)|^p\, d\sigma - |f(0)|^p \\ &\leqslant C \int_B \left(\widetilde{\Delta}\,|f|^p\right)(w) g(w)\, d\tau(w) \\ &= C \int_B |f(w)|^{p-2} \left(\widetilde{\Delta}\,|f|^2\right)(w)\, g(w)\, d\tau(w).\end{aligned}$$

If f is replaced by $f \circ \varphi_a$, we obtain by a simple change of variables

$$\begin{aligned}\left[\widetilde{\Delta}\,|f|^2(a)\right]^{p/2} &\leqslant C \int_B |f \circ \varphi_a(w)|^{p-2} \left(\widetilde{\Delta}\,|f|^2\right)(\varphi_a(w)) g(w)\, d\tau(w) \\ &= C \int_B |f(z)|^{p-2} \left(\widetilde{\Delta}\,|f|^2\right)(z)\, G(z,a)\, d\tau(z). \qquad (12)\end{aligned}$$

If we now integrate both sides of (12) with respect to the measure $dV(a)/\left(1-|a|^2\right)$ and use (4), (5) and (10), we obtain

$$\begin{aligned}\int_B \left[\widetilde{\Delta}\,|f|^2(a)\right]^{p/2} \frac{dV(a)}{1-|a|^2} &\leqslant C \int_B |f(z)|^{p-2} \left(\widetilde{\Delta}\,|f|^2\right)(z) \left\{\int_B \frac{G(z,a)}{1-|a|^2} dV(a)\right\} d\tau(z) \\ &\leqslant C \int_B |f(z)|^{p-2} \left(\widetilde{\Delta}\,|f|^2\right)(z)\, g(z)\, d\tau(z) \\ &\leqslant C\,\|f\|^p_{H^p(B)}.\end{aligned}$$

Therefore our conclusion comes easily from (6). □

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