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# CONGRUENCE-PRESERVING ISOMORPHISMS OF THE TRANSLATION GROUP ASSOCIATED WITH A TRANSLATION PLANE

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Let $\Pi$, $\Pi'$ be projective translation planes, $\mathscr{P}$, $\mathscr{P}'$ their sets of points, $l_\infty$, $l_\infty'$ the improper lines, and $\mathsf{T}$, $\mathsf{T}'$ the corresponding translation groups. $\mathsf{T}$ is an Abelian group, simply transitive on $\mathscr{P}\backslash l_\infty$. The set of the subgroups $\mathsf{T}_S = \{\tau | \tau \in \mathsf{T}, \text{ cen } \tau = S\}$ for all $S \in l_\infty$ is called the *congruence of* $\Pi$ (cen $\tau$ = centre of $\tau$). An injective map $\varphi: \mathscr{A} \to \mathscr{P}'$, where $\mathscr{A} \subset \mathscr{P}$, is said to be a *collineation of* $\mathscr{A}$ when $\varphi(l_\infty \cap \mathscr{A}) \subset l_\infty'$ and three points in $\mathscr{A}$ are collinear if and only if their images are collinear; the set of these $\varphi$ is denoted by $\Phi(\mathscr{A}, \mathscr{P}')$ and for $O \in \mathscr{A}$, $O' \in \mathscr{P}'$ we write

$$\Phi_{O,O'}(\mathscr{A}, \mathscr{P}') = \{\varphi | \varphi \in \Phi(\mathscr{A}, \mathscr{P}'), \varphi(O) = O'\}.$$

An injective map $\omega: \mathsf{T}_0 \to \mathsf{T}' (\mathsf{T}_0 \subset \mathsf{T})$ is called a *congruence-preserving isomorphism of* $\mathsf{T}_0$ if $\tau_1, \tau_2, \tau_2\tau_1 \in \mathsf{T}_0 \Rightarrow \omega(\tau_2\tau_1) = \omega(\tau_2)\omega(\tau_1)$ and $\text{cen } \tau_1 = \text{cen } \tau_2 \Leftrightarrow \text{cen } \omega(\tau_1) = \text{cen } \omega(\tau_2)$; we denote the set of all these $\omega$ by $\Omega(\mathsf{T}_0, \mathsf{T}')$.

For $\mathscr{A} = \mathscr{P}$ it is known [**4**; **7**] that: (1) If $\varphi \in \Phi(\mathscr{P}, \mathscr{P}')$, then the map $\bar{\varphi}: \mathsf{T} \to \mathsf{T}'$ defined by $\bar{\varphi}(\tau) = \varphi\tau\varphi^{-1}$ is a congruence-preserving isomorphism of $\mathsf{T}$. (2) $\varphi \mapsto \bar{\varphi}$ defines a bijection of $\Phi_{O,O'}(\mathscr{P}, \mathscr{P}')$ onto $\Omega(\mathsf{T}, \mathsf{T}')$. These results may be thought of as the core of the theorem on the representation of collineations of Desarguesian planes by semilinear maps.

It was realized that collineations of certain subsets $\mathscr{A}$ of a Desarguesian projective plane also induced semilinear maps and could be embedded in collineations of the entire plane. This was proved in case $\mathscr{A}$ consists of four non-concurrent lines [**1**; **2**; **3**], in case $\mathscr{A}$ consists of three non-concurrent lines and at least one more point [**8**], and for a more general case (requiring only two full lines to be in $\mathscr{A}$ and certain additional conditions) [**6**]. When $\mathscr{A}$ contained four non-concurrent lines, any collineation of $\mathscr{A}$ could be embedded into a collineation of the whole plane under much more general assumptions as to the projective plane [**5**; **9**].

Our aim in the present paper is to investigate what connections exist between the collineations of certain subsets in a translation plane $\Pi$ and the congruence-preserving isomorphisms of corresponding complexes of the translation group $\mathsf{T}$. We will also note subsets $\mathscr{A}$ such that any collineation of $\mathscr{A}$ is embeddable in a collineation of $\Pi$.

*Definition* 1. We call a set $\mathscr{A} \subset \mathscr{P}$ containing at least two proper points a *semi-anchor* if for any translation plane $\Pi'$, and $\varphi \in \Phi(\mathscr{A}, \mathscr{P}')$, and

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$\tau \in \mathsf{T}, \mathscr{A} \cap \tau^{-1}\mathscr{A} \not\subset l_\infty$ implies that there exists $\tau' \in \mathsf{T}'$ such that for any $M \in \mathscr{A} \cap \tau^{-1}\mathscr{A}$ we have $\tau'\varphi(M) = \varphi(\tau M)$. In other words, we require a translation $\tau' \in \mathsf{T}'$ such that the diagram

$$\begin{array}{ccc} \mathscr{A} \cap \tau^{-1}\mathscr{A} & \xrightarrow{\varphi} & \mathscr{P}' \\ \downarrow \tau & & \downarrow \tau' \\ \tau\mathscr{A} \cap \mathscr{A} & \xrightarrow{\varphi} & \mathscr{P}' \end{array}$$

commutes.

Note that the condition $\mathscr{A} \cap \tau^{-1}\mathscr{A} \not\subset l_\infty$ ensures the existence of a proper point $M$ such that both $M$ and $\tau M$ belong to $\mathscr{A}$.

For a given semi-anchor $\mathscr{A}$ we denote the uniquely determined $\tau' \in \mathsf{T}'$ by $\bar{\varphi}(\tau)$. We write $\mathsf{T}(\mathscr{A}) = \{\tau \mid \tau \in \mathsf{T}, \mathscr{A} \cap \tau^{-1}\mathscr{A} \not\subset l_\infty\}$ and have

(1) $\quad \varphi(\tau M) = \bar{\varphi}(\tau)\varphi(M) \quad$ for any $\tau \in \mathsf{T}(\mathscr{A})$ and $M \in \mathscr{A} \cap \tau^{-1}\mathscr{A}$.

Since we shall be discussing only one collineation $\varphi$ at any given time, we shall generally write $\varphi(M) = M'$, $\bar{\varphi}(\tau) = \tau'$. We then see that for all $\tau \in \mathsf{T}(\mathscr{A})$ there exists $\tau' \in \mathsf{T}'$ such that for any $M \in \mathscr{A} \cap \tau^{-1}\mathscr{A}$,

$$(1') \qquad (\tau M)' = \tau' M'.$$

*Definition* 2. If $\mathscr{A}$ is a semi-anchor and $\bar{\varphi}: \mathsf{T}(\mathscr{A}) \to \mathsf{T}'$ defined by (1) is a congruence-preserving isomorphism for every $\varphi \in \Phi(\mathscr{A}, \mathscr{P}')$, then the set $\mathscr{A}$ is called a *near-anchor*.

*Definition* 3. A near-anchor $\mathscr{A}$ is said to be an *anchor* if $\mathsf{T}(\mathscr{A}) = \mathsf{T}$.

THEOREM 1. *A semi-anchor containing the distinct lines $l$ and $l_\infty$ is a near-anchor.*

The main part of the proof of this theorem consists in showing that $(\tau_2\tau_1)' = \tau_2'\tau_1'$ whenever $\tau_1, \tau_2, \tau_2\tau_1 \in \mathsf{T}(\mathscr{A})$. If there exists a proper point $M$ such that $M \in \mathscr{A}$, $\tau_1 M \in \mathscr{A}$, and $\tau_2\tau_1 M \in \mathscr{A}$, then using (1′) we have

$$(\tau_2\tau_1)'M' = (\tau_2\tau_1 M)' = \tau_2'(\tau_1 M)' = \tau_2'\tau_1'M',$$

so that

$$(\tau_2\tau_1)' = \tau_2'\tau_1'.$$

However, it is not difficult to construct an $\mathscr{A}$, even in the real projective plane, satisfying the conditions of the theorem, for which such a choice of $M$ is not always possible (see for instance example (2) below). We therefore introduce the notion of *projections*.

Let $A = l \cap l_\infty$, and let $S$ be any point of $l_\infty$ distinct from $A$. The map from $\mathscr{P}\backslash\{S\}$ onto $l$ defined by $P \mapsto SP \cap l$ $(P \in \mathscr{P}\backslash\{S\})$ is called *the projection onto $l$ from $S$*. We shall denote $SP \cap l$ by $P^*$, and shall call $P^*$ *the projection of $P$ onto $l$ from $S$*.

Given any $\tau \in \mathsf{T}$, there exists $\tau^* \in \mathsf{T}_A$ such that

(2) $$\tau^* P^* = (\tau P)^* \quad \text{for all } P \in \mathscr{P}\backslash\{S\}.$$

Indeed, let $B$ be a fixed point in $\mathscr{P}\backslash l_\infty$ and determine $\tau^* \in \mathsf{T}_A$ by

$$\tau^* B^* = (\tau B)^*.$$

Denoting the line $SB$ by $b$, we have $\tau b = \tau^* b$, hence $\tau^{-1}\tau^* b = b$. Therefore the centre of the translation $\tau^{-1}\tau^*$ is $S$. Then, for every point $P$ different from $S$, $\tau^{-1}\tau^*(SP) = SP$, i.e. $\tau^*(SP) = \tau(SP)$. It follows that $\tau^* P^*$, $\tau P$, and $S$ are on a line, hence $\tau^* P^* = (\tau P)^*$. The map $\tau \mapsto \tau^*$ is also called a *projection* and $\tau^*$ is *the projection of* $\tau$ *onto* $l$ *from* $S$. This map $\tau \mapsto \tau^*$ is a homomorphism of $\mathsf{T}$ onto $\mathsf{T}_A$, for, using (2) we have

$$(\tau_2\tau_1)^* P^* = (\tau_2\tau_1 P)^* = \tau_2^*(\tau_1 P)^* = \tau_2^*\tau_1^* P^*,$$

and hence

(3) $$(\tau_2\tau_1)^* = \tau_2^*\tau_1^*.$$

We introduce the symbol $[P_1, Q_1; P_2, Q_2]$ to show that the points $P_1, Q_1, P_2, Q_2$ are on a line and that there exists a translation $\tau$ such that $Q_1 = \tau P_1$, $Q_2 = \tau P_2$. Assuming $[P_1, Q_1; P_2, Q_2]$ we have by (2) $Q_i^* = (\tau P_i)^* = \tau^* P_i^*$ $(i = 1, 2)$, hence

(4) $$[P_1, Q_1; P_2, Q_2] \Rightarrow [P_1^*, Q_1^*; P_2^*, Q_2^*].$$

*Proof of Theorem* 1. Let $\mathscr{A}$ be a semi-anchor containing the lines $l$ and $l_\infty$. Let $\varphi$ be an arbitrary collineation of $\mathscr{A}$ into any translation plane $\mathscr{P}'$ and let $\bar{\varphi}$ be the map induced by (1).

Since $\varphi$ sends collinear points onto collinear points, and since $l_\infty \subset \mathscr{A}$, we may conclude that

(5) $$\operatorname{cen} \bar{\varphi}(\tau) = \varphi(\operatorname{cen} \tau) \quad \text{or} \quad \operatorname{cen} \tau' = (\operatorname{cen} \tau)' \quad \text{for all } \tau \in \mathsf{T}(\mathscr{A}).$$

Using (5) and the injectivity of $\varphi$ we can easily deduce that the map $\bar{\varphi}$: $\mathsf{T}(\mathscr{A}) \to \mathsf{T}'$ is congruence-preserving.

Let $S$ denote any point of $l_\infty\backslash\{A\}$ (where $A = l_\infty \cap l$ as before) and let $\tau \in \mathsf{T}(\mathscr{A})$, $P \in \mathscr{A}\backslash\{S\}$. Denote the projections of $P$, $\tau$ onto $l$ from $S$ by $P^*$, $\tau^*$ and denote the projections of $P'$, $\tau'$ onto $l'$ from $S'$ by $P'^*$, $\tau'^*$. Since $\varphi$ is a collineation,

(6) $$P^{*\prime} = P'^* \quad \text{for all } P \in \mathscr{A}\backslash\{S\}.$$

Hence, using (2) (applied in $\Pi'$), (1′), (6), (2), (1′), (6) in this order, we deduce that for any proper point $M \in \mathscr{A} \cap \tau^{-1}\mathscr{A}$,

$$\tau'^*(M'^*) = (\tau' M')^* = (\tau M)'^* = (\tau M)^{*\prime} = (\tau^* M^*)' = \tau^{*\prime} M^{*\prime} = \tau^{*\prime}(M'^*).$$

Since $\mathsf{T}$ is simply transitive, we deduce that

(7) $$\tau'^* = \tau^{*\prime} \quad \text{for all } \tau \in \mathsf{T}(\mathscr{A}).$$

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Next we show that

$$(8)\qquad (\tau_2\tau_1)' = \tau_2'\tau_1' \quad \text{provided } \tau_1, \tau_2, \tau_2\tau_1 \in \mathbf{T}(\mathscr{A}).$$

If $M$ is any proper point of $l$, then $M$, $\tau_1^*M$, $\tau_2^*\tau_1^*M$ all belong to $l$, and therefore to $\mathscr{A}$. Hence from (1′) we have

$$(\tau_2^*\tau_1^*)'M' = (\tau_2^*\tau_1^*M)' = \tau_2^{*\prime}(\tau_1^*M)' = \tau_2^{*\prime}\tau_1^{*\prime}M',$$

hence

$$(\tau_2^*\tau_1^*)' = \tau_2^{*\prime}\tau_1^{*\prime}.$$

Using (3) and (7) we have

$$(\tau_2\tau_1)'^* = (\tau_2\tau_1)^{*\prime} = (\tau_2^*\tau_1^*)' = \tau_2^{*\prime}\tau_1^{*\prime} = \tau_2'^*\tau_1'^* = (\tau_2'\tau_1')^*.$$

The translations $(\tau_2\tau_1)'$ and $\tau_2'\tau_1'$ have the same projection onto $l'$ and this is true regardless of the position of $S$ on $l_\infty$. Considering two positions of $S$ we conclude that $(\tau_2\tau_1)' = \tau_2'\tau_1'$.

All that remains is to see whether the map $\bar{\varphi}: \mathbf{T}(\mathscr{A}) \to \mathbf{T}'$ is injective. This is clear for the restriction $\bar{\varphi}|\mathbf{T}_A$. Suppose now that $\tau_1' = \tau_2'$ for some $\tau_1, \tau_2 \in \mathbf{T}(\mathscr{A})$. This and (7) imply that $\tau_1^{*\prime} = \tau_2^{*\prime}$; since $\tau_1^*$, $\tau_2^*$ belong to $\mathbf{T}_A$ we obtain $\tau_1^* = \tau_2^*$. Two different projections yield, as above, $\tau_1 = \tau_2$.

THEOREM 2. *If $\mathscr{A}$ is a semi-anchor containing the proper line $l$, then every point-set $\mathscr{B}$ containing $\mathscr{A} \cup l_\infty$ is also a semi-anchor.*

*Proof.* Consider $\varphi \in \Phi(\mathscr{B}, \mathscr{P}')$, write again $\varphi(P) = P'$, select $\tau \in \mathbf{T}(\mathscr{B})$, $M \in \mathscr{B} \cap \tau^{-1}\mathscr{B}$ ($M \notin l, l_\infty$), and determine $\tau' \in \mathbf{T}'$ such that $\tau'M' = (\tau M)'$. We have to show that

$$\tau'P' = (\tau P)' \qquad \text{for all } P \in \mathscr{B} \cap \tau^{-1}\mathscr{B}.$$

We may admit that $\tau \neq 1$. Let $N = \tau M$, $Q = \tau P$. If $P$ is not on the line $MN$, we can construct the point $Q$ by intersecting $l_\infty$ with $MN$, $MP$ in $U$, $V$, respectively, and then intersecting $UP$ with $VN$ in $Q$. The points $M$, $N$, $P$, $Q$, $U$, $V$ belong to $\mathscr{B}$ and their images under $\varphi$ share just the same collinearity properties. Hence $N' = \tau'M'$ yield $Q' = \tau'P'$, i.e. $\tau'P' = (\tau P)'$. We call this argument the "quadrilateral argument".

If $P$ is on the line $MN$, we have $[M, N; P, Q]$. Choose $S \in l_\infty$, $S \notin l, MN$, and intersect $l$ with $SM$, $SN$, $SP$, $SQ$ in $M^*$, $N^*$, $P^*$, $Q^*$, respectively. Because of (4) we also have $[M^*, N^*; P^*, Q^*]$. Since $l$ belongs to the semi-anchor $\mathscr{A}$, it is easily seen from (1′) that $[M^{*\prime}, N^{*\prime}; P^{*\prime}, Q^{*\prime}]$ is also valid. Using projection onto $M'N'$ from $S'$ we deduce $[M', N'; P', Q']$. Hence $N' = \tau'M'$ yields $Q' = \tau'P'$; that is, $\tau'P' = (\tau P)'$. This completes the proof of Theorem 2.

COROLLARY. *Consider $l, l_\infty \subset \mathscr{A} \subset \mathscr{B} \subset \mathscr{P}$ ($l$ is a proper line). If $\mathscr{A}$ is a near-anchor, then $\mathscr{B}$ is also a near-anchor. If $\mathscr{A}$ is an anchor, then $\mathscr{B}$ is also an anchor.*

Indeed, in both cases $\mathscr{B}$ is a semi-anchor by Theorem 2, but then it is a near-anchor by Theorem 1. If $\mathscr{A}$ is an anchor, then obviously $\mathsf{T}(\mathscr{B}) = \mathsf{T}$.

*Examples.* (1) *If the characteristic of the translation plane* $\Pi$ *is different from* 2 *and if the point-set* $\mathscr{A}$ *contains* $l_\infty$ *and no more than two proper points on any proper line, then* $\mathscr{A}$ *is a semi-anchor.* Since two pairs of proper points corresponding in a translation will never occur on the same line, it is sufficient to apply the quadrilateral argument.

(2) *If* $\mathscr{A}$ *contains the concurrent distinct lines* $l_\infty, l_1, l_2$, *it is a near-anchor.* Because of the corollary we have to consider only the case $\mathscr{A} = l_\infty \cup l_1 \cup l_2$ and apply the quadrilateral argument.

A similar reasoning leads to the following generalization.

(3) Let $\mathscr{A}$ contain the distinct lines $l_\infty, l$ and suppose that $A, B, C, D \in l$ and $[A, B; C, D]$ imply the existence of proper points $E, F$ not on $l$ such that $EF, l, l_\infty$ are concurrent and $AE, BF, l_\infty$ are concurrent. *Then* $\mathscr{A}$ *is a near-anchor.*

(4) *A point-set consisting of the non-concurrent lines* $l_\infty, l_1, l_2$ *and at least one point more is an anchor.* Pick out a point $E$ of the given set $\mathscr{A}$ which does not belong to any of the lines $l_\infty, l_1, l_2$. Denote $l_1 \cap l_2 = O$, $l_\infty \cap l_1 = O_1$, $l_\infty \cap l_2 = O_2$, $O_2E \cap l_1 = E_1$, $O_1E \cap l_2 = E_2$, and $\tau_0 \in \mathsf{T}$, $\tau_0(E_2) = E$. Relation (1′) holds for $\tau = \tau_0$ by the quadrilateral argument. Next we see that (1′) holds for any $\tau \in \mathsf{T}_{O_2}$ because one can project $\tau_0$ on $l_2$ from every point of $l_\infty$. Using the projections of the translations belonging to $\mathsf{T}_{O_2}$ one concludes that relation (1′) is valid for any $\tau \in \mathsf{T}(\mathscr{A})$, that is $\mathscr{A}$ is a semi-anchor. By Theorem 1 it is also a near-anchor. What we still have to do is to show that $\tau \in \mathbf{T} \Rightarrow \tau \in \mathbf{T}(\mathscr{A})$. For $\tau \in \mathsf{T}_{O_1}$ this is evident; thus suppose that

$$S = \operatorname{cen} \tau \neq O_1.$$

We intersect $l_2$ with the image of $l_1$ under $\tau$, join this point $M_2$ with $S$, and cut $SM_2$ with $l_1$ in $M_1$. Then $\tau(M_1) = M_2$ implies $\tau \in \mathsf{T}(\mathscr{A})$.

THEOREM 3. *Let* $\mathscr{P}, \mathscr{P}'$ *be the point-sets of two translation planes,* $\mathscr{A}$ *a near anchor in* $\mathscr{P}$, *such that for every* $S \in \mathscr{A} \cap l_\infty$ *there are two proper points of* $\mathscr{A}$ *collinear with* $S$ *and let* $O$ *be a proper point of* $\mathscr{A}$ *and* $O' \in \mathscr{P}'$. *The map* $\psi: \varphi \mapsto \bar{\varphi}$ *determined by relation* (1) *is a bijection of* $\Phi_{O,O'}(\mathscr{A}, \mathscr{P}')$ *onto* $\Omega(\mathsf{T}(\mathscr{A}), \mathsf{T}')$.

*Proof.* Let $\varphi$ be an element of $\Phi_{O,O'}(\mathscr{A}, \mathscr{P}')$. Using the notation $\tau_{AB}$ for $\tau \in \mathsf{T}$, $\tau(A) = B$, we have, by (1),

$$\varphi(\tau_{OM}O) = \bar{\varphi}(\tau_{OM})\varphi(O) \quad \text{for any } M \in \mathscr{A}_0,$$

where $\mathscr{A}_0$ is the set of the proper points in $\mathscr{A}$, hence

$$\varphi(M) = \bar{\varphi}(\tau_{OM})(O') \quad \text{for any } M \in \mathscr{A}_0. \tag{9}$$

If $M \in \mathscr{A} \cap l_\infty$, consider distinct points $A, B \in \mathscr{A}_0$, collinear with $M$. Since

$M = \text{cen}\,\tau_{AB}$ and since $M, A, B$ are mapped in collinear points $M', A', B'$, respectively, we have $M' = \text{cen}\,\tau_{A'B'} = \text{cen}\,\bar{\varphi}(\tau_{AB})$ or

$$(10)\quad \varphi(M) = \text{cen}\,\bar{\varphi}(\tau_{AB}) \quad \text{for any } M \in \mathscr{A} \cap l_\infty, A, B \in \mathscr{A}_0, A \neq B, \; M, A, B \text{ collinear.}$$

Formulae (9) and (10) show that $\psi$ is injective, for

$$\bar{\varphi}_1 = \bar{\varphi}_2, \qquad \varphi_1, \varphi_2 \in \Phi_{O,O'}(\mathscr{A}, \mathscr{P}')$$

implies that $\varphi_1(M) = \varphi_2(M)$ for any $M \in \mathscr{A}$.

We take now an element $\omega$ of $\Omega(\mathbf{T}(\mathscr{A}), \mathbf{T}')$ and look for an element $\varphi$ of $\Phi_{O,O'}(\mathscr{A}, \mathscr{P}')$ which is sent into $\omega$ by $\psi$. Formulae (9) and (10) show that if such a $\varphi$ does exist, it must be given by

$$(11)\quad \varphi(M) = \begin{cases} \omega(\tau C_M)(O') & \text{for any } M \in \mathscr{A}_0, \\ \text{cen}\,\omega(\tau_{AB}) & \text{for any } M \in \mathscr{A} \cap l_\infty, A, B \in \mathscr{A}_0, A \neq B, \\ & M, A, B \text{ collinear.} \end{cases}$$

Consequently, we consider the map $\varphi: \mathscr{A} \to \mathscr{P}'$ defined by (11) and will prove that it is a collineation sending $O$ into $O'$ and that $\bar{\varphi} = \omega$. But first we have to see that in the case of the second line of (11), $\varphi(M)$ does not depend on the choice of $A, B$; indeed, if $C, D \in \mathscr{A}_0$ are also collinear with $M$, then $\text{cen}\,\tau_{AB} = \text{cen}\,\tau_{CD}$ and since $\omega$ is congruence-preserving, $\text{cen}\,\omega(\tau_{AB}) = \text{cen}\,\omega(\tau_{CD})$.

Suppose now that $\varphi(M) = \varphi(M_1)$ for $M, M_1 \in \mathscr{A}$. Then either both $M$ and $M_1$ belong to $\mathscr{A}_0$ or both belong to $\mathscr{A} \cap l_\infty$. In the first case, writing the first row of (11) for $M$ and $M_1$, we see that the translations $\omega(\tau_{OM})$ and $\omega(\tau_{OM_1})$ have the same effect on $O'$, hence they are equal; $\omega$ being injective, $\tau_{OM} = \tau_{OM_1}$, therefore $M = M_1$. In the second case, we have $\text{cen}\,\omega(\tau_{AB}) = \text{cen}\,\omega(\tau_{CD})$ with $A, B, C, D \in \mathscr{A}_0$, $M, A, B$ and $M_1, C, D$ collinear, which implies that $\text{cen}\,\tau_{AB} = \text{cen}\,\tau_{CD}$, that is $M = M_1$. Thus $\varphi: \mathscr{A} \to \mathscr{P}'$ is an injective map.

Now, take any points $P, Q, R$ in $\mathscr{A}_0$ and write

$$Q^\varphi = \varphi(Q) = \omega(\tau_{OQ})(O') = \omega(\tau_{PQ}\tau_{OP})(O');$$

since $\omega$ is an isomorphism and since $\tau_{OP}, \tau_{PQ}, \tau_{OQ} \in \mathbf{T}(\mathscr{A})$, we have

$$Q^\varphi = \omega(\tau_{PQ})\omega(\tau_{OP})(O'),$$

or, by (11),

$$(12)\qquad Q^\varphi = \omega(\tau_{PQ})(P^\varphi),$$

and similarly

$$(12')\qquad R^\varphi = \omega(\tau_{PR})(P^\varphi).$$

If $P, Q, R$ are collinear, then $\text{cen}\,\tau_{PQ} = \text{cen}\,\tau_{PR}$ and since $\omega$ is congruence-preserving, $\text{cen}\,\omega(\tau_{PQ}) = \text{cen}\,\omega(\tau_{PR})$. It follows from (12) and (12′) that $P^\varphi, Q^\varphi, R^\varphi$ are on a line. If $P, Q, R$ are not on a line, then $\text{cen}\,\tau_{PQ} \neq \text{cen}\,\tau_{PR}$, hence $\text{cen}\,\omega(\tau_{PQ}) \neq \text{cen}\,\omega(\tau_{PR})$ and $P^\varphi, Q^\varphi, R^\varphi$ are not collinear. Consider now

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$P \in \mathscr{A} \cap l_\infty$, $Q$, $R \in \mathscr{A}_0$ and denote the images of these points, as before, by $P^\varphi$, $Q^\varphi$, $R^\varphi$. If $P$, $Q$, $R$ are collinear, then $P = \text{cen}\, \tau_{QR}$; using (11) we have $P^\varphi = \text{cen}\, \omega(\tau_{QR}) = \text{cen}\, \tau_{Q^\varphi R^\varphi}$, thus $P^\varphi$, $Q^\varphi$, $R^\varphi$ are collinear. If $P$, $Q$, $R$ are not collinear, then $P \neq \text{cen}\, \tau_{QR}$. Choosing $S$, $T \in \mathscr{A}_0$ such that $P = \text{cen}\, \tau_{ST}$ we conclude that $\text{cen}\, \omega(\tau_{QR}) \neq \text{cen}\, \omega(\tau_{ST})$ or $\text{cen}\, \tau_{Q^\varphi R^\varphi} \neq \text{cen}\, \tau_{S^\varphi T^\varphi} = P^\varphi$; hence $P^\varphi$, $Q^\varphi$, $R^\varphi$ are not on a line. Thus $\varphi$ is a collineation.

$\varphi(O) = O'$ is a consequence of (11) and of the fact that the unit element $\tau_{OO}$ of $\mathbf{T}$ is sent by the isomorphism $\omega$ into the unit element of $\mathbf{T}'$.

The comparison of (9) and (11) shows that the translations $\bar{\varphi}(\tau_{OM})$ and $\omega(\tau_{OM})$ have the same effect on $O'$, hence they are equal:

$$\bar{\varphi}(\tau_{OM}) = \omega(\tau_{OM}) \quad \text{for any } M \in \mathscr{A}_0. \tag{13}$$

An arbitrary $\tau \in \mathbf{T}(\mathscr{A})$ may be written as $\tau_{PQ}$ with $P, Q \in \mathscr{A}_0$, and we have

$$\begin{aligned}
\bar{\varphi}(\tau_{PQ})\bar{\varphi}(\tau_{OP}) &= \bar{\varphi}(\tau_{PQ}\tau_{OP}) \text{ since } \bar{\varphi} \text{ is an isomorphism } (\mathscr{A} \text{ being a near-anchor})\\
&= \bar{\varphi}(\tau_{OQ}) = \omega(\tau_{OQ}) \text{ by } (13)\\
&= \omega(\tau_{PQ}\tau_{OP}) = \omega(\tau_{PQ})\omega(\tau_{OP}) \text{ since } \omega \text{ is an isomorphism}\\
&= \omega(\tau_{PQ})\bar{\varphi}(\tau_{OP}) \text{ by } (13),
\end{aligned}$$

therefore $\bar{\varphi}(\tau_{PQ}) = \omega(\tau_{PQ})$, i.e. $\bar{\varphi} = \omega$ and the proof of Theorem 3 is complete.

THEOREM 4. *Every collineation defined on an anchor of a translation plane can be embedded into a collineation of the whole plane.*

*Proof.* Let $\mathscr{A} \subset \mathscr{P}$ be a given anchor and let $\varphi \in \Phi(\mathscr{A}, \mathscr{P}')$. Since $\mathbf{T}(\mathscr{A}) = \mathbf{T}$, we have $\bar{\varphi} \in \Omega(\mathbf{T}, \mathbf{T}')$. Consider the map

$$M \to \tilde{\varphi}(M) = \begin{cases} \bar{\varphi}(\tau_{OM})(O') & \text{for any } M \in \mathscr{P}\backslash l_\infty,\\ \text{cen}\, \bar{\varphi}(\tau_{AB}) & \text{for any } M \in l_\infty, A, B \in \mathscr{A}_0, A \neq B,\\ & \qquad M, A, B \text{ collinear,} \end{cases}$$

where $O$ is a fixed point in $\mathscr{A}_0$ and $O'$ is its image under $\varphi$. If $M \in \mathscr{A}$, then $\tilde{\varphi}(M) = \varphi(M)$ by (9) and (10). On the other hand, $\tilde{\varphi}$: $\mathscr{P} \to \mathscr{P}'$ is a collineation of $\mathscr{P}$ by a proof similar to that used to show that formula (11) defines a collineation.

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