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A PARTIAL GENERALIZATION OF MANN'S THEOREM CONCERNING ORTHOGONAL LATIN SQUARES

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ABSTRACT. Let $n = 4t + 2$, where the integer $t \geqq 2$. A necessary condition is given for a particular Latin square L of order n to have a complete set of $n - 2$ mutually orthogonal Latin squares, each orthogonal to L. This condition extends constraints due to Mann concerning the existence of a Latin square orthogonal to a given Latin square.

1. **Introduction.** The only positive integers n for which it is known that there is a class of $n - 1$ mutually orthogonal Latin squares of order n, or equivalently a projective plane of order n, are those which are powers of primes (see [3], pages 93-94 or [2]). Thus, there is an infinite number of examples of projective planes of order n in each of the cases, $n \equiv 0$, 1, or 3 modulo 4. There is no known example for which $n \equiv 2$ modulo 4 except $n = 2$ where the complete set of Latin squares is rather degenerately a single square. We will seek constraints on the existence of a complete set of $n - 1$ mutually orthogonal Latin squares of order n when $n = 4t + 2$.

This paper concerns only a restrictive class of Latin squares. However, very little is known about the existence of complete sets of orthogonal Latin squares when the order is neither a prime power nor ruled out by the Bruck-Ryser-Chowla theorem. This theorem states that if $n \equiv 1$ or 2 modulo 4 and the square-free part of n is divisible by a prime of the form $4m + 3$, then there does not exist a complete set of $n - 1$ mutually orthogonal Latin squares of order n. For another recent paper giving constraints on the existence of complete sets of Latin squares see Woodcock [4].

We will proceed by extending a theorem due to Mann giving conditions for a Latin square to have an orthogonal mate to a theorem giving conditions for a Latin square of order $4t + 2$ to have a complete set of $4t$ orthogonal mates which are mutually orthogonal. These theorems are given below as Theorems 1 and 2. Theorem 1, which is Mann's theorem, is specialized to the case $n = 4t + 2$. For Theorems 1 and 2, we assume $t \geqq 2$, since a Latin square

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of order 6 has no orthogonal mates. A proof of Theorem 1 is given in [1] page 194.

THEOREM 1 (Mann). *Let* $t \geqq 2$ *and let* L *be a Latin square of order* $4t + 2$ *with a square subarray consisting of* $2t + 1$ *rows and* $2t + 1$ *columns in which all entries are from a set of* $2t + 1$ *elements except for* t *or less of the cells. Then there is no Latin square orthogonal to* L.

THEOREM 2. *Let* $t \geqq 2$ *and let* L *be a Latin square of order* $4t + 2$ *with a square subarray of side* $2t + 1$ *in which all entries are from a set of* $2t + 1$ *elements except for less than* $t + u$ *of the cells, where* $u = 1$ *if* $t = 2$ *and* $u = ((t + 1)/8)^{1/2} + 1/2$ *otherwise. Then there does not exist a complete set of* $4t$ *mutually orthogonal Latin squares, each orthogonal to* L.

2. **Preliminaries.** Let L be a Latin square of order n based on the elements $1, 2, \ldots, n$. Then a *transversal* T of L is a set of n cells of L, one from each row, one from each column, and with distinct cells containing different elements. The existence of an orthogonal mate L' to L is equivalent to the existence of n mutually disjoint transversals $T_1, T_2, \ldots, T_n$ of L. To obtain L' we then map each cell in T_i into the same element i. By the orthogonality relationships, the existence of a complete set of $n - 2$ mutually orthogonal Latin squares, each orthogonal to L is equivalent to the existence of $n - 2$ sets of n mutually disjoint transversals of L for which a pair of transversals in distinct sets have exactly one cell in common.

Two cells of a Latin square are *connected* if they agree in their row, column, or element they contain. Given a pair of disconnected cells in a Latin square L of order n with a complete set of $n - 2$ mutually orthogonal mates, this pair is contained in exactly one of the $n(n - 2)$ transversals corresponding to these $n - 2$ orthogonal mates.

Let S be a square subarray of side $2t + 1$ of the Latin square L of order $4t + 2$, all of whose elements are from the set of $2t + 1$ elements $a_1, a_2, \ldots, a_{2t+1}$ except for k elements. Then S is said to have k *special* cells with respect to the elements $a_1, a_2, \ldots, a_{2t+1}$. Lemmas 1 and 2 give results concerning special cells. The proofs are straightforward and will be omitted.

LEMMA 1. *Let* L *be a Latin square of order* $4t + 2$ *based on the elements* $1, 2, \ldots, 4t + 2$. *Let* $a_1, a_2, \ldots, a_{2t+1}, b_1, b_2, \ldots, b_{2t+1}$ *be a permutation on the integers* $1, 2, \ldots, 4t + 2$. *Let* S_1 *be the square subarray of* L *consisting of rows* 1 *through* $2t + 1$ *and columns* 1 *through* $2t + 1$, S_2 be the square subarray consisting of rows 1 through $2t + 1$ *and columns* $2t + 2$ *through* $4t + 2$, S_3 *be the square subarray consisting of rows* $2t + 2$ *through* $4t + 2$ *and columns* 1 *through* $2t + 1$, *and* S_4 *be the square subarray consisting of rows* $2t + 2$ *through* $4t + 2$ *and columns* $2t + 2$ *through* $4t + 2$. *Suppose* S_1 *has* k *special cells*

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with respect to the elements $a_1, a_2, \ldots, a_{2t+1}$. *Then* S_2 *and* S_3 *both have* k *special cells with respect to the elements* $b_1, b_2, \ldots, b_{2t+1}$, *and* S_4 *has* k *special cells with respect to the elements* $a_1, a_2, \ldots, a_{2t+1}$.

LEMMA 2. *Let* L, S_1, S_2, S_3, *and* S_4 *be defined as in Lemma* 1. *Let* $a_1, a_2, \ldots, a_{2t+1}, b_1, b_2, \ldots, b_{2t+1}$ *be a permutation of the elements* $1, 2, \ldots, 4t + 2$. *Suppose* S_1 *and* S_4 *both have* k *special cells with respect to the elements* $a_1, a_2, \ldots, a_{2t+1}$, *and* S_2 *and* S_3 *both have* k *special cells with respect to the elements* $b_1, b_2, \ldots, b_{2t+1}$. *Let* T *be a transversal of* L. *Then* T *contains an odd number of special cells.*

3. **Proof of the Main Theorem.** We are now ready to prove Theorem 2:

PROOF OF THEOREM 2. Let L be a Latin square of order $4t + 2$ based on the elements $1, 2, 3, \ldots, 4t + 2$, where $t \geqq 2$. If $t = 2$, then the theorem follows by Theorem 1. Thus, we now assume $t \geqq 3$. Suppose L has a complete set of $4t$ mutually orthogonal mates $L_1, L_2, \ldots, L_{4t}$, each orthogonal to L. Let $a_1, a_2, \ldots, a_{2t+1}, b_1, b_2, \ldots, b_{2t+1}$ be a permutation of the elements $1, 2, \ldots, 4t + 2$. Suppose L has a square subarray S_1 of side $2t + 1$ with k special cells with respect to the elements $a_1, a_2, \ldots, a_{2t+1}$. Assume $k < t + ((t + 1)/8)^{1/2} + 1/2$.

By Theorem 1, $k \geqq t + 1$. By permuting the rows and columns of L and its $4t$ orthogonal mates, we can assume that S_1 consists of the rows $1, 2, \ldots, 2t + 1$ and the columns $1, 2, \ldots, 2t + 1$. Let the square subarrays S_1, S_2, S_3, and S_4 be defined as in Lemma 1. Then by Lemma 1, S_4 contains k special cells with respect to the elements $a_1, a_2, \ldots, a_{2t+1}$, while S_2 and S_3 both contain k special cells with respect to the elements $b_1, b_2, \ldots, b_{2t+1}$.

Let T_{ij} be the ith of the $4t + 2$ transversals of L corresponding to the jth orthogonal mate L, where $1 \leqq i \leqq 4t + 2$, $1 \leqq j \leqq 4t$. By Lemma 2, each transversal contains an odd number of the $4k$ special cells of L. By the discussion preceding Lemma 1, any pair of disconnected special cells appears on exactly one of the transversals T_{ij}. We will proceed by counting the maximum number of pairs of disconnected special cells available on transversals. We will obtain a contradiction by showing that the maximum number of pairs of disconnected special cells available is less than the minimum number of pairs of disconnected special cells required.

Let $1 \leqq j \leqq 4t$ be fixed and let T_{ij}, $1 \leqq i \leqq 4t + 2$ be the transversals corresponding to L_j. The arrangement that maximizes the number of pairs of special cells appearing on these $4t + 2$ transversals is the one in which $4t + 1$ of the transversals contain 1 special cell each and one transversal contains $4k - 4t - 1$ special cells. Thus, the maximum number of pairs of disconnected special cells appearing on all $(4t)(4t + 2)$ transversals is

(1) $(4t)(4k - 4t - 1)(4k - 4t - 2)/2 = 2t(4(k - t) - 1)(4(k - t) - 2)$.

We now find a lower bound for the number of pairs of disconnected special cells required. To do this, we first obtain an upper bound for the number of pairs of connected special cells. This upper bound will be attained if for each square subarray S_i, $1 \leqq i \leqq 4$, we maximize the number of pairs of connected special cells containing at least one special cell from S_i.

Let c_i, $1 \leqq i \leqq k$ denote the special cells appearing in S_1. Let x_i, y_i, and z_i be the number of special cells in S_1 connected to c_i by appearing in the same row, appearing in the same column, or containing the same element, respectively. Then

$$x_i + y_i + z_i \leqq k - 1.$$

Furthermore, counting by rows, c_i is connected to $x_i + 1$ special cells in S_2. Counting by columns, c_i is connected to $y_i + 1$ special cells in S_3. Finally, the total number of pairs of connected special cells with one cell contained in S_1 is maximized if for each c_i, c_i is connected to $z_i + 1$ special cells in S_4 containing the same element as c_i. Thus, an upper bound for the number of special cells connected to each cell c_i for $1 \leqq i \leqq k$ is

$$2x_i + 2y_i + 2z_i + 3 \leqq 2k + 1.$$

By considering all $4k$ special cells in S_1, S_2, S_3, and S_4 and proceeding as before, we see that an upper bound for the total number of pairs of connected special cells is

$$(1/2)(4k)(2k + 1) = 2k(2k + 1).$$

Thus, a lower bound for the total number of pairs of disconnected special cells is

(2) $$(4k)(4k - 1)/2 - 2k(2k + 1) = 2k(2k - 2).$$

By (1) and (2), we see that

(3) $$(k)(k - 1) \leqq t(4(k - t) - 1)(2(k - t) - 1).$$

Let $k = t + r$, where $r \geqq 1$. Suppose $r = 1$. Then, by (3) $t \leqq 2$, a case we have already considered. Thus, assume $r \geqq 2$. Solving for t in (3) by means of the quadratic formula and noting that t is a positive integer, we obtain

$$t \leqq 2(2r - 1)^2 - 1.$$

Solving for r, we get

$$r \geqq ((t + 1)/8)^{1/2} + 1/2.$$

Thus, $k \geqq t + ((t + 1)/8)^{1/2} + 1/2$, a contradiction, and the theorem is proved.

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