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A CANONICAL DECOMPOSITION IN MIXED EXTERIOR ALGEBRA

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1. Introduction. Let E be a vector space of dimension $n \in \mathbf{N}$ over a field Γ, of characteristic 0. Choose E^*, dual to E, and form

$$\Lambda E = \bigoplus_{p=0}^{n} \Lambda^p E,$$

the space of exterior powers of E, as well as ΛE^*. Finally, let

$$\Lambda(E^*, E) = \Lambda E^* \otimes \Lambda E.$$

Although $\Lambda(E^*, E)$ is constructed by the most basic vector space operations, it is rich in algebraic structure:

(i) as a vector space over Γ it is bigraded,

$$\Lambda(E^*, E) = \bigoplus_{p,q=0}^{n} (\Lambda^p E^* \otimes \Lambda^q E),$$

and has dimension 2^{2n};

(ii) it has a canonical inner product, $\langle , \rangle$, induced from the duality of E^* and E, with respect to which,

$$\langle \Lambda^p E^* \otimes \Lambda^q E, \Lambda^r E^* \otimes \Lambda^s E \rangle = 0,$$

unless $p = s$.

(iii) since both ΛE^* and ΛE are (exterior) algebras, their tensor product is an algebra; we denote its product by a dot and call $\Lambda(E^*, E)$, with this product, the *mixed exterior algebra* over E;

(iv) since $\Lambda^p E^* \otimes \Lambda^q E$ is isomorphic to the space of linear maps from $\Lambda^p E$ to $\Lambda^q E$ and the latter may be "composed", $\Lambda(E^*, E)$ also has a composition product, which we denote by "$\circ$".

The inner product and both algebra structures restrict to the diagonal space

$$\Delta(E) = \bigoplus_{p=0}^{n} \Delta_p(E), \quad \Delta_p(E) = \Lambda^p E^* \otimes \Lambda^p E,$$

and the resulting "dot" algebra is commutative. We call it the diagonal subalgebra. Henceforth, we shall only be concerned with $\Delta(E)$.

Many results about the structure of $\Delta(E)$ and its applications to classical linear algebra are to be found in [**1**]. Others have been announced

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in [4]. One of these has been fully proved in [5]. The purpose of this paper is to treat the other main result of [4].

2. Preliminaries. In this section we gather together the results of [5] and [1] which will be needed in the sequel.

For any $z \in \Delta_1(E)$, we adopt the convention:

$$(1) \qquad z^0 = 1,\ z^p = \frac{1}{p!}\underset{(p \text{ factors})}{(z \ldots . z)},\ p = 1, 2, \ldots, n, \text{ and } z^p = 0, \text{ otherwise.}$$

The unit tensor $t \in \Delta_1(E)$ corresponds to the identity map of E under the isomorphism of $\Delta_1(E)$ with the space of linear transformations of E and hence satisfies

$$(2) \qquad t \circ z = z \circ t = z, \quad z \in \Delta_1(E).$$

For each $u \in \Delta(E)$, we obtain a linear transformation of $\Delta(E)$, given by $v \mapsto u \cdot v = v \cdot u$; it will be denoted by $\mu(u)$. Its dual, $\mu(u)^*$, is written $i(u)$; i.e.,

$$\langle \mu(u)v, w\rangle = \langle v, i(u)w\rangle, \quad u, v, w \in \Delta(E).$$

The Poincaré map, $D:\Delta(E) \rightarrow \Delta(E)$, is defined by

$$(3) \qquad Du = i(u)t^n, \quad u \in \Delta(E).$$

It is an involutory isometry

$$(4) \qquad D^2 = \iota,\ \langle Du, Dv\rangle = \langle u, v\rangle,\ u, v \in \Delta(E),$$

and it satisfies

$$(5) \quad i(u) \circ D = D \circ \mu(u), \quad u \in \Delta(E).$$

One of the key identities of the subject, proved in [1], is

$$(6) \quad i(z)(z_1 \ldots z_p) = \sum_{q=1}^{p} z_1 \ldots \langle z, z_q\rangle \ldots z_p - \sum_{1 \leqq q < r \leqq p} (z_q \circ z \circ z_r + z_r \circ z \circ z_q) \cdot z_1 \ldots \hat{z}_q \ldots \hat{z}_r \ldots z_p,$$

where $z, z_1, \ldots, z_p \in \Delta_1(E)$. The reason that this plays a basic rôle is that it relates the two algebraic structures on $\Delta(E)$. For example, one may deduce from (6), by induction on q, that

$$(7) \quad i(t^q)t^p = \binom{n-p+q}{q} t^{p-q}, \quad 0 \leqq q \leqq p \leqq n.$$

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Since the dot algebra $\Delta(E)$ is generated by $\Delta_0(E) = \Gamma$ and $\Delta_1(E)$, there are unique derivations $\lambda_z, \rho_z, z \in \Delta_1(E)$, of $\Delta(E)$ such that $\lambda_z(z_1) = z \circ z_1$, $\rho_z(z_1) = z_1 \circ z$, $z_1 \in \Delta_1(E)$. In particular,

$$(8) \qquad \lambda_z(z_1 \dots z_p) = \sum_{q=1}^{p} z_1 \dots (z \circ z_q) \dots z_p,$$

$$\rho_z(z_1 \dots z_p) = \sum_{q=1}^{p} z_1 \dots (z_q \circ z) \dots z_p,$$

where $z, z_1, \dots, z_p \in \Delta_1(E)$. Thus, it follows from (2) that

$$(9) \qquad \lambda_t(u) = \rho_t(u) = pu, \quad u \in \Delta_p(E).$$

Furthermore, the identity (6) may be written as

$$[i(z) \circ \mu(z_1) - \mu(z_1) \circ i(z)](z_2 \dots z_p)$$
$$= [\langle z, z_1 \rangle \iota - \lambda_{z_1 \circ z} - \rho_{z \circ z_1}](z_2 \dots z_p).$$

We conclude that

$$(10) \qquad [i(z), \mu(z_1)] = \Gamma(z, z_1), \quad z, z_1 \in \Delta_1(E),$$

where

$$(11) \qquad \Gamma(z, z_1) = \langle z, z_1 \rangle \iota - \lambda_{z_1 \circ z} - \rho_{z \circ z_1}$$

and [,] denotes the commutator of linear transformations of $\Delta(E)$.

The formula (10) was the main ingredient in the proof of the principal result of [**5**]; namely,

$$(12) \quad i(u)t^{2p+q} = (-1)^p \sum_{r \in \mathbf{Z}} (-1)^r \mu(t^{q+r}) i(t^r) u,$$
$$p, q \in \mathbf{Z}, u \in \Delta_p(E).$$

3. A basic identity. The purpose of this section is to prove another consequence of the formula (6).

LEMMA. *For any* $z_1, z \in \Delta(E)$ *and* $p \in \boldsymbol{Z}$,

$$[i(z_1), \mu(z^{p+1})] = \mu(z^p) \circ \Gamma(z_1, z) - \mu(z^{p-1}) \circ \mu(z \circ z_1 \circ z).$$

Proof. For $p = 0$, the formula is equivalent to (10).

Because λ_z and ρ_z are derivations of $\Delta(E)$, we have

$$[\lambda_{z_1}, \mu(z)] = \mu(z_1 \circ z) \quad \text{and} \quad [\rho_{z_1}, \mu(z)] = \mu(z \circ z_1).$$

It follows from (11) that

$$(13) \quad [\Gamma(z_1, z), \mu(z)] = -2\mu(z \circ z_1 \circ z) \quad z, z_1 \in \Delta_1(E).$$

Now assume that the lemma holds for p; i.e., by (1),

$$[i(z_1), \mu(z)^{p+1}] = (p + 1)\mu(z)^p \circ \Gamma(z_1, z) - p(p + 1)\mu(z)^{p-1} \circ \mu(z \circ z_1 \circ z).$$

Then, by (13),

$$\begin{aligned}[i(z_1), \mu(z)^{p+2}] &= [i(z_1), \mu(z)^{p+1}] \circ \mu(z) + \mu(z)^{p+1} \circ [i(z_1), \mu(z)]\\ &= [(p + 1)\mu(z)^p \circ \Gamma(z_1, z) - p(p+1)\mu(z)^{p-1}\\ &\qquad \circ \mu(z \circ z_1 \circ z) \circ \mu(z) + \mu(z)^{p+1} \circ \Gamma(z_1, z)\\ &= (p + 1)\mu(z)^p \circ [\mu(z) \circ \Gamma(z_1, z)\\ &\qquad - 2\mu(z \circ z_1 \circ z)]\\ &\quad - p(p + 1)\mu(z)^{p-1} \circ \mu(z) \circ \mu(z \circ z_1 \circ z)\\ &\qquad + \mu(z)^{p+1} \circ \Gamma(z_1, z)\\ &= (p + 2)\mu(z)^{p+1} \circ \Gamma(z_1, z)\\ &\qquad - (p + 1)(p + 2)\mu(z)^p \circ \mu(z \circ z_1 \circ z);\end{aligned}$$

i.e.,

$$[i(z_1), \mu(z^{p+2})] = \mu(z^{p+1}) \circ \Gamma(z_1, z) - \mu(z^p) \circ \mu(z \circ z_1 \circ z).$$

This closes the induction and hence proves the lemma for $p \in \mathbf{N}$. It is clearly true for $p < 0$ (both sizes are zero) and hence for $p \in \mathbf{Z}$.

COROLLARY 1.

$$i(z_1)z^{p+1} = \langle z_1, z\rangle z^p - (z \circ z_1 \circ z) \cdot z^{p-1}, \quad z_1, z \in \Delta_1(E), \quad p \in \mathbf{Z}.$$

Proof. Let both sides of the lemma act on $1 \in \Delta_0(E)$ and note that derivations of $\Delta(E)$ map $\Delta_0(E)$ to 0.

COROLLARY 2.

$$[i(t), \mu(t^{p+1})] = \mu(t^p) \circ [(n - p)\iota - \lambda_t - \rho_t], \quad p \in \mathbf{Z}.$$

Proof. When $z_1 = z = t$, the right hand side of the lemma is

$$\mu(t^p) \circ \Gamma(t, t) - \mu(t^{p-1}) \circ \mu(t),$$

by (2). Since $\langle t, t\rangle = i(t)t = n$, by (7), formula (11) yields

$$\Gamma(t, t) = n\iota - \lambda_t - \rho_t.$$

Finally,

$$\mu(t^{p-1}) \circ \mu(t) = \mu(t \cdot t^{p-1}) = p\mu(t^p),$$

by (1).

Remarks. (i) In view of (9), Corollary 2 implies that

(14) $[i(t), \mu(t^{p+1})]v = (n - p - 2q)\mu(t^p)v, \quad v \in \Delta_q(E), p, q \in \mathbf{Z}.$

(ii) Suppose $v \in \ker i(t) \cap \Delta_q(E)$. Then (14) reads

$$i(t)\mu(t^{p+1})v = (n - p - 2q)\mu(t^p)v$$

and hence, by induction, we obtain

(15) $$i(t^r)\mu(t^p)v = \frac{(m + 1)(m + 2)\dots(m + r)}{r!}\mu(t^{p-r})v,$$

where $v \in \ker i(t) \cap \Delta_q(E)$, $p, q \in \mathbf{Z}$, $r \in \mathbf{N}$ and $m = n - p - 2q$.

4. The subspaces F_p, G_p of $\Delta_p(E)$. We define subspaces of $\Delta_p(E)$ by

$$F_p = \ker i(t) \cap \Delta_p(E), \quad G_p = \ker\mu(t) \cap \Delta_p(E), \quad p \in \mathbf{Z}.$$

If $p < 0$ or $p > n$ both F_p and G_p are zero. Also it is clear that $F_0 = \Delta_0(E) = \Gamma$ and $G_n = \Delta_n(E)$, which is isomorphic to Γ, since t^n is a basis. From (5), we see that D maps F_p isomorphically onto G_{n-p}. A more precise version of this result follows from the

LEMMA.

$$i(t^q)Du = D\mu(t^q)u = (-1)^p\mu(t^{n-2p-q})u, \quad u \in F_p, p, q \in \mathbf{Z}.$$

Proof. We have

$$i(t^q)Du = D(t^q \cdot u) = i(u)Dt^q = i(u)t^{n-q},$$

by (5), (3) and (7). The lemma then follows from (12), since $u \in F_p$.

COROLLARY 1.

$$D\mu(t^q)F_p = \mu(t^{n-2p-q})F_p = i(t^q)G_{n-p}, \quad p, q \in \mathbf{Z}.$$

Proof. The first equality follows from the lemma which also shows that

$$D\mu(t^q)F_p = i(t^q)DF_p.$$

Since $DF_p = G_{n-p}$, as we have remarked above, the second equality holds as well.

COROLLARY 2. *$\mu(t^q)F_p \neq 0$ if and only if p, q satisfy $0 \leqq q \leqq n - 2p$.*

Proof. The left hand side of the formula in the lemma vanishes unless $q \geqq 0$, while the right hand side vanishes unless $n - 2p - q \geqq 0$. Since D is an isomorphism, we conclude that $\mu(t^q)F_p = 0$ unless $0 \leqq q \leqq n - 2p$.

We now prove that, if $0 \leqq q \leqq n - 2p$, then $\mu(t^q)F_p \neq 0$. First, note that it is sufficient to prove that $\mu(t^{n-2p})F_p \neq 0$, since $\mu(t^q)F_p = 0$ implies that $\mu(t^r)F_p = 0$ for $r \geqq q$.

Let $e_1, \ldots, e_n$ be a basis for E and let $e^{*1}, \ldots, e^{*n}$ be its dual. Define $z \in \Delta_1(E)$ by

$$z = \sum_{i=1}^{p} e^{*i} \otimes e_{p+i}.$$

It is easily seen that

$$z^p = e^{*1}\Lambda \ldots \Lambda e^{*p} \otimes e_{p+1}\Lambda \ldots \Lambda e_{2p}$$

and hence $z^p \neq 0$, since $2p \leqq n$.

On the other hand, Corollary 1 of Section 3 shows that

$$i(t)z^p = \langle t, z\rangle z^{p-1} - (z \circ z) \cdot z^{p-2}.$$

But

$$\langle t, z\rangle = \sum_{i=i}^{p} \langle e^{*i}, e_{p+i}\rangle = 0$$

and

$$z \circ z = \sum_{i,j=1}^{p} \langle e^{*i}, e_{p+j}\rangle e^{*j} \otimes e_{p+i} = 0.$$

Hence $z^p \in F_p$.

Next, note that, since $t = \sum_{i=1}^{n} e^{*i} \otimes e_i$, we have

$$t^q = \frac{1}{q!} \sum e^{*i_1}\Lambda \ldots \Lambda e^{*i_q} \otimes e_{i_1} \Lambda \ldots \Lambda e_{i_q},$$

where $1 \leqq i_1, \ldots, i_q \leqq n$. It follows that

$$t^{n-2p} \cdot z^p = e^{*2p+1}\Lambda \ldots \Lambda e^{*n}\Lambda e^{*1}\Lambda \ldots \Lambda e^{*p} \otimes$$
$$e_{2p+1}\Lambda \ldots \Lambda e_n \Lambda e_{p+1}\Lambda \ldots \Lambda e_{2p},$$

since the only non-zero terms in the product are those for which $(i_1, \ldots, i_{n-2p})$ is a permutation of $(2p + 1, \ldots, n)$.

Finally, then, $t^{n-2p} \cdot z^p \neq 0$ since $2p \leqq n$.

5. The orthogonal projections π_p. In this section we will construct orthogonal projections

$$\pi_p : \Delta_p(E) \to \Delta_p(E)$$

whose image is F_p.

LEMMA. *Let $m = n - 2p + 1$, where $2p \leqq n$. Let $i(t)_p$ denote the restriction of $i(t)$ to $\Delta_p(E)$. Put*

$$\pi_p = \sum_{r=0}^{p} \frac{(-1)^r}{\binom{m+r}{m}} \mu(t^r) \circ i(t^r)_p.$$

Then π_p is the orthogonal projection of $\Delta_p(E)$ onto F_p.

Proof. Suppose that $u \in \Delta_p(E)$. By (14) we have

$$[i(t), \mu(t^r)](i(t^r)u) = [n - 2(p - r) - (r - 1)]\mu(t^{r-1})i(t^r)u,$$

which may be written as

$$\begin{aligned} i(t)\mu(t^r)i(t^r)u &= \mu(t^r)i(t)i(t^r)u + (n - 2p + 1 + r)\mu(t^{r-1})i(t^r)u \\ &= (r + 1)\mu(t^r)i(t^{r+1})u + (m + r)\mu(t^{r-1})i(t^r)u, \end{aligned}$$

or

$$\begin{aligned} \frac{r!}{(m + r)!} i(t)\mu(t^r)i(t^r)u &= \frac{(r + 1)!}{(m + r)!} \mu(t^r)i(t^{r+1})u \\ &\quad + \frac{r!}{(m + r - 1)!} \mu(t^{r-1})i(t^r)u \\ &= v_{r+1} + v_r, \end{aligned}$$

where

$$v_r = \frac{r!}{(m + r - 1)!}\mu(t^{r-1})i(t^r)u.$$

It follows from the definition that $v_0 = 0$, $v_{p+1} = 0$ and hence that

$$\begin{aligned} i(t)\pi_p(u) &= m! \sum_{r=0}^{p} (-1)^r(v_{r+1} + v_r) \\ &= m![v_0 + (-1)^p v_{p+1}] = 0. \end{aligned}$$

Since $u \in \Delta_p(E)$ is arbitrary, we conclude that

(16) $\quad i(t) \circ \pi_p = 0.$

Therefore

$$\begin{aligned} \pi_p^2 &= \sum_{r=0}^{p} \frac{(-1)^r}{\binom{m+r}{m}}\mu(t^r) \circ i(t^r) \circ \pi_p \\ &= \frac{(-1)^0}{\binom{m+0}{m}} \mu(t^0) \circ i(t^0) \circ \pi_p = \pi_p. \end{aligned}$$

This proves that π_p is a projection and, since $\mu(t)^* = i(t)$, it follows that $\pi_p^* = \pi_p$ and hence that π_p is orthogonal.

Finally, if $u \in F_p$, then $\pi_p u = u$, from the definition of π_p and, conversely, if $\pi_p u = u$, then

$$i(t)u = i(t)\pi_p u = 0,$$

from (16). Thus the image of π_p is F_p.

COROLLARY 1.

$$\ker \pi_p = \operatorname{Im}\mu(t)_{p-1}.$$

Proof. Since

$$\pi_p = \iota_p + \sum_{r=1}^{p} \frac{(-1)^r}{\binom{m+r}{r}} \mu(t^r) \circ i(t^r)_p,$$

it follows that $\pi_p u = 0$ if and only if $u \in \operatorname{Im}\mu(t)_{p-1}$.

COROLLARY 2.

$$\tilde{\pi}_{n-p} = \sum_{r=0}^{p} \frac{(-1)^r}{\binom{n - 2p + 1 + r}{r}} i(t^r) \circ \mu(t^r)_{n-p}$$

is the orthogonal projection of $\Delta_{n-p}(E)$ whose image is G_{n-p} and whose kernel is $\operatorname{Im} i(t)_{n-p+1}$.

Proof. From (5) it is easy to check that

$$D \circ \pi_p = \tilde{\pi}_{n-p} \circ D_p.$$

The fact that $\tilde{\pi}_{n-p}$ is an orthogonal projection then follows from (4). The rest of the corollary follows from the fact that D maps F_p isomorphically (in fact, isometrically) onto G_{n-p} (cf. Corollary 1, Section 4).

6. The direct sum decomposition of $\Delta(E)$. In this section we will show that $\Delta(E)$ is the direct sum of spaces of the form $\mu(t^q)F_p$, $p, q \in \mathbf{Z}$. Note that by Corollary 2 of Section 4, the only such spaces which are different from 0 are those for which $0 \leqq q \leqq n - 2p$.

LEMMA. *If*

$$u = \sum_{r=0}^{p} \mu(t^{p-r})u_r,$$

where $u_r \in F_r$, $r = 0, 1, \ldots, p$ and $2p \leqq n$, then the u_r are uniquely determined by $i(t^q)u$, $q = 0, 1, \ldots, p$.

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Proof. By (15), we have

$$i(t^q)u = \sum_{r=0}^{p} \binom{n-p-r+q}{q} \mu(t^{p-q-r})u_r, \quad q = 0, \ldots, p.$$

The coefficient of u_{p-q} in this equation is $\binom{n-2p+2q}{q}$ and hence the determinant of the system is

$$\prod_{q=0}^{p} \binom{n-2p+2q}{q} > 0,$$

since $2p \leqq n$.

COROLLARY. *If*

$$\sum_{r=0}^{p} \mu(t^{p-r})u_r = 0, \quad \textit{for } u_r \in F_r, r = 0, \ldots, p, 2p \leqq n,$$

then $u_r = 0$, *for* $r = 0, \ldots, p$.

We are now in a position to prove the

THEOREM.

$$\Delta(E) = \bigoplus_{p=0}^{n} \Delta_p(E),$$

where

$$\begin{aligned}
\Delta_0(E) &= F_0 \\
\Delta_1(E) &= \mu(t)F_0 \oplus F_1 \\
&\ \ldots \\
\Delta_p(E) &= \mu(t^p)F_0 \oplus \mu(t^{p-1})F_1 \oplus \ldots \oplus F_p \\
&\ \ldots \qquad\qquad\qquad\qquad\qquad\qquad (2p \leqq n) \\
\Delta_{n-p}(E) &= \mu(t^{n-p})F_0 \oplus \mu(t^{n-p-1})F_1 \oplus \ldots \oplus \mu(t^{n-2p})F_p \\
&\ \ldots \\
\Delta_{n-1}(E) &= \mu(t^{n-1})F_0 \oplus \mu(t^{n-2})F_1 \\
\Delta_n(E) &= \mu(t^n)F_0.
\end{aligned}$$

Proof. The results of Section 5 show that for $2p \leqq n$,

$$\Delta_p(E) = \operatorname{Im}\pi_p \oplus \ker\pi_p = F_p \oplus \operatorname{Im}\mu(t)_{p-1}.$$

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The corollary of the above lemma, together with an induction on p show that

$$\mathrm{Im}\mu(t)_{p-1} = \mu(t)F_{p-1} \oplus \mu(t^2)F_{p-2} \oplus \ldots \oplus \mu(t^p)F_0.$$

This proves the formulae when $2p \leqq n$. The remaining formulae follow from these by virtue of Corollary 1 of the lemma in Section 4.

COROLLARY 1. *For $2p \leqq n$, we have*

$$\dim F_p = \dim G_{n-p} = \binom{n}{p}^2 - \binom{n}{p-1}^2.$$

Proof. $\dim \Delta_p(E) = \binom{n}{p}^2$.

COROLLARY 2.

$$\Delta_0(E) = i(t^n)G_n$$

$$\Delta_1(E) = i(t^{n-1})G_n \oplus i(t^{n-2})G_{n-1}$$

$$\ldots$$

$$\Delta_p(E) = i(t^{n-p})G_n \oplus i(t^{n-p-1})G_{n-1} \oplus \ldots \oplus i(t^{n-2p})G_{n-p}$$

$$\ldots \qquad (2p \leqq n)$$

$$\Delta_{n-p}(E) = i(t^p)G_n \oplus i(t^{p-1})G_{n-1} \oplus \ldots \oplus G_{n-p}$$

$$\ldots$$

$$\Delta_{n-1}(E) = i(t)G_n \oplus G_{n-1}$$

$$\Delta_n(E) = G_n.$$

Proof. Apply D to the decomposition in the theorem and use Corollary 1 of the lemma of Section 4.

7. Concluding remarks.

1. The decomposition of Section 6 is, in fact, orthogonal. To see this, assume that $2p \leqq n$ and consider

$$\begin{aligned}\langle \mu(t^q)u_{p-q}, \mu(t^r)u_{p-r}\rangle &= \langle i(t^r)\mu(t^q)u_{p-q}, u_{p-r}\rangle \\ &= \binom{n-2p+q+r}{r}\langle \mu(t^{q-r})u_{p-q}, u_{p-r}\rangle \\ &= \binom{n-2p+q+r}{r}\langle u_{p-q}, i(t^{q-r})u_{p-r}\rangle \\ &= 0,\end{aligned}$$

if $0 \leqq r < q \leqq p$ (cf. (15)).

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The other orthogonalities follow from that of $\Delta_p(E)$, $\Delta_q(E)$, $p \neq q$, and the fact that D is an isometry.

2. Now that we know that $u \in \Delta_p(E)$ $(2p \leqq n)$ can be uniquely written in terms of $u_q \in F_q$ $(0 \leqq q \leqq p)$ the lemma of Section 6 provides an algorithm for computing these u_q in terms of u; in fact, the system

$$i(t^p)u = \binom{n}{p}u_0$$

$$i(t^{p-1})u = \binom{n-1}{p-1}\mu(t)u_0 + \binom{n-2}{p-1}u_1$$

$$\ldots$$

$$\begin{aligned} i(t^q)u = &\binom{n-p+q}{q}\mu(t^{p-q})u_0 + \ldots \\ &+ \binom{n-p+q-r}{q}\mu(t^{p-q-r})\, u_r + \ldots \\ &+ \binom{n-2p+2q}{q}u_{p-q} \end{aligned}$$

$$\ldots$$

$$\begin{aligned} i(t^0)u = &\binom{n-p}{0}\mu(t^p)u_0 + \ldots \\ &+ \binom{n-p-r}{0}\mu(t^{p-r})u_r + \ldots + \binom{n-2p}{0}u_p, \end{aligned}$$

may be solved, successively, for $u_0, u_1, \ldots, u_p$.

For example, if $n \geqq 4$ and $p = 2$, we find

$$u_0 = \frac{2}{n(n-1)}\langle t^2, u\rangle,\ u_1 = \frac{1}{n-2}\left[i(t)u - \frac{2}{n}\langle t^2, u\rangle\, t\right]$$

and

$$u_2 = u - \frac{1}{n-2}\left[i(t)u - \frac{2}{n}\langle t^2, u\rangle t\right]\cdot t - \frac{2}{n(n-1)}\langle t^2, u\rangle t^2.$$

3. The eigenvectors of D and inner products of elements of $\Delta(E)$ can be computed in terms of the decomposition of Section 6.

For example, if $2p \leqq n$, and we write

$$u = \sum_{q=0}^{p} \mu(t^{p-q})u_q,$$

where $u_q \in F_q (q = 0, \ldots, p)$, then

$$Du = \sum_{q=0}^{p} (-1)^q \mu(t^{n-p-q}) u_q,$$

by the lemma of Section 4.

Thus $u \in \Delta_p(E)$ is an eigenvector of D only if $2p = n$. In this case, the eigenspace of D corresponding to the eigenvalue $+1$ is

$$\mu(t^p)F_0 \oplus \mu(t^{p-2})F_2 \oplus \dots,$$

while that corresponding to -1 is

$$\mu(t^{p-1})F_1 \oplus \mu(t^{p-3})F_3 \oplus \dots.$$

If we write

$$v = \sum_{q=0}^{p} \mu(t^{p-q}) v_q \in \Delta_p(E),$$

where $v_q \in F_q (q = 0, \dots, p)$, then

$$\langle u, v \rangle = \sum_{q=0}^{p} \binom{n-2q}{p-q} \langle u_q, v_q \rangle,$$

since

$$i(t^q)\mu(t^q)u = \binom{n-2q}{q} u, \ u \in F_p, \ 0 \leqq 2p \leqq n, \quad 0 \leqq q \leqq n - 2p,$$

by (15).

Similarly, we conclude that, if $n = 2p$, then

$$\langle u, Dv \rangle = \sum_{q=0}^{p} (-1)^q \binom{2p-2q}{p-q} \langle u_q, v_q \rangle.$$

4. The case $u \in \Delta_2(E)$ is of special interest because the curvature tensor of a pseudo-Riemannian manifold, when regarded as a tensor of type (2, 2), has the symmetries of $\Delta_2(E)$. In this case $i(t)u$ corresponds to the Ricci tensor, while $\langle t^2, u \rangle$ corresponds to the scalar of curvature.

The terms in the decomposition of Section 6 also have geometric significance. According to the example of Remark 2, above, u_0 corresponds to the scalar of curvature, u_1 corresponds to the trace-free Ricci tensor and u_2 corresponds to the Weyl conformal curvature tensor. In particular, if $u_2 = 0$, the manifold is conformally flat (if $n \geqq 4$) and if $u_1 = 0$, then we have an Einstein manifold.

Decompositions of the above type (using the Bianchi symmetries as well as those of $\Delta_2(E)$) have been employed in **[3]** to obtain inequalities between the signature of a four-dimensional Einstein manifold and its Euler Characteristic.

The specific decomposition of Remark 2 was used in [**2**] to show that Pontrjagin classes of a manifold depend only on the Weyl tensor and hence are conformal invariants.

It is hoped that the generality of the above results, with respect to both the field Γ and the inner product $\langle , \rangle$, will lead to further applications.

References

1. W. H. Greub, *Multilinear algebra* (2nd Ed.) (Springer-Verlag, New York, 1978), Chapters 6 and 7.

2. ——— *Pontrjagin classes and the Weyl tensor*, C. R. Math. Rep. Acad. Sci. Canada *3* (1981), 177-182.

3. A. Palombo, *Nombres caracteristiques d'une variété riemannienne de dimension* 4, J. Differential Geometry *13* (1978), 145-162.

4. J. R. Vanstone, *Some new identities in mixed exterior algebra*, C. R. Math. Rep. Acad. Sci. Canada *2* (1980), 269-271.

5. ——— *The Poincaré map in mixed exterior algebra*, Bull. Can. Math. Soc., to appear.

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