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Abstract

We prove that after inverting the Planck constant h, the Bezrukavnikov–Kaledin quantization $(X, \mathcal{O}_h)$ of symplectic variety X in characteristic p with $H^2(X, \mathcal{O}_X) = 0$ is Morita equivalent to a certain central reduction of the algebra of differential operators on X.

Contents

1. Introduction

1.1 Frobenius-constant quantizations

For the duration of this paper, let k be a perfect field of characteristic $p > 2$. Given a scheme X over k we denote by X' the Frobenius twist of X and by $F : X \to X'$ the k-linear Frobenius morphism. Since F is a homeomorphism on the underlying topological spaces, we shall identify the categories of sheaves on X and X'.

Let X be a smooth variety over k equipped with a symplectic 2-form ω. Recall, that a quantization $(X, \mathcal{O}_h)$ of (X, ω) is a sheaf $\mathcal{O}_h$ on the Zariski site of X of flat $k[[h]]$-algebras complete with respect to the h-adic topology together with an isomorphism of k-algebras

$$\mathcal{O}_h/h \xrightarrow{\sim} \mathcal{O}_X$$

such that, for any two local sections $\tilde{f}$, $\tilde{g}$ of $\mathcal{O}_h$, one has

$$\{f, g\} \equiv \frac{\tilde{f}\tilde{g} - \tilde{g}\tilde{f}}{h} \mod h.$$

Here f and g stand for the images in $\mathcal{O}_X$ of $\tilde{f}$ and $\tilde{g}$ respectively and $\{,\}$ for the Poisson bracket $\mathcal{O}_X$ induced by the symplectic structure. Note that if X is affine then giving a quantization

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$(X, \mathcal{O}_h)$ of (X, ω) is equivalent to giving a quantization $\mathcal{O}_h(X)$ of the Poisson algebra $\mathcal{O}_X(X)$ (see e.g. [BK04a, Remark 1.6]).

A feature special to characteristic p is that the Poisson algebra $\mathcal{O}_X$ of a symplectic variety has a large center consisting of pth powers of functions. We are going to identify it with the sheaf $\mathcal{O}_{X'}$ using the Frobenius morphism

$$F^* : \mathcal{O}_{X'} \xrightarrow{\sim} \mathcal{O}_X^p \subset \mathcal{O}_X.$$

Given a quantization $(X, \mathcal{O}_h)$ of (X, ω) we have k-linear homomorphisms

$$\mathcal{Z}_h \twoheadrightarrow \mathcal{Z}_h/h \hookrightarrow \mathcal{O}_{X'}. \tag{1.1}$$

from the center $\mathcal{Z}_h$ of the quantization $\mathcal{O}_h$ to the Poisson center. Following [BK08], a quantization is called central if the composition (1.1) is surjective. A Frobenius-constant quantization of (X, ω) is a pair consisting of a quantization $(X, \mathcal{O}_h)$ of the symplectic variety X together with a $k[[h]]$-algebra isomorphism

$$s : \mathcal{O}_{X'}[[h]] \xrightarrow{\sim} \mathcal{Z}_h \tag{1.2}$$

such that, for any local section $f^p \in \mathcal{O}_X^p = \mathcal{O}_{X'} \subset \mathcal{O}_{X'}[[h]]$ and a lift $\tilde{f} \in \mathcal{O}_h$ of $f \in \mathcal{O}_X$, one has that[1]

$$s(f^p) = \tilde{f}^p \mod h^{p-1}.$$

It is clear that a quantization that admits a Frobenius-constant structure is central.

A Frobenius-constant structure on $(X, \mathcal{O}_h)$ makes $\mathcal{O}_h$ into a sheaf of algebras over $\mathcal{O}_{X'}[[h]]$. It was shown in [BK08] that $\mathcal{O}_h$ is locally free of rank $p^{\dim X}$ as an $\mathcal{O}_{X'}[[h]]$-module for the Zariski topology on X'.

Frobenius-constant quantizations of symplectic varieties have been first introduced by Bezrukavnikov and Kaledin as a tool for proving the categorical McKay correspondence for symplectic resolutions of singularities (see [BK04b]). Most of the foundational results have been obtained in [BK08]. The technique introduced in [BK04b] has found some other applications in geometric representations theory (see e.g. [BF14, BL21]). A key to all these applications is *the Azumaya property* of the algebra $\mathcal{O}_h(h^{-1})$ obtained from $\mathcal{O}_h$ by inverting h: it was shown in [BK08] that for any Frobenius-constant quantization on $(X, \mathcal{O}_h)$, the algebra $\mathcal{O}_h(h^{-1})$ is isomorphic, that is locally, for the *fppf* topology on X', to a matrix algebra over $\mathcal{O}_{X'}((h))$. Since the algebra $\mathcal{O}_h$ has no zero divisors $\mathcal{O}_h(h^{-1})$ does not split even locally for the étale topology on X' (unless $\dim X = 0$). In [BK08, Proposition 1.24] a formula for the class of this Azumaya algebra in an appropriate Brauer group was proposed. However, it has been observed in [Mol17] that the formula in [BK08] is not correct as stated. The immediate goal of this paper is to correct it. The technique introduced along its proof (in particular, the Basic Lemma from § 1.8) plays an essential role in a sequel paper joint with Kubrak and Travkin [BKTV22], where we prove that the category of quasi-coherent sheaves on any restricted symplectic variety admits a canonical Frobenius-constant quantization.

1.2 Differential operators as a Frobenius-constant quantization

A basic example of a Frobenius-constant quantization is as follows. Let Y be a smooth variety over k, $X := \mathbb{T}^*_Y$ the cotangent bundle to Y equipped with the canonical symplectic structure ω.

[1] Recall that for any associative algebra A over a field of characteristic p and every elements $x, y \in A$, the element $(x+y)^p - x^p - y^p$ can be written as a homogeneous Lie polynomial in x and y of degree p. Applying this to $A = \mathcal{O}_h$ we infer that $\tilde{f}^p \mod h^p$ depends only on $f \in \mathcal{O}_h/h$ and not on the choice of a lifting $\tilde{f} \in \mathcal{O}_h$ of f. In addition, using that, for any $x \in A$, one has that $\mathrm{ad}_{x^p} = (\mathrm{ad}_x)^p : A \to A$, it follows that $\tilde{f}^p$ is a central element of $\mathcal{O}_h/h^p$.

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Denote by D_Y the sheaf of differential operators on Y. This comes with a filtration given by the order of a differential operator. Applying the Rees construction to the filtered algebra D_Y we obtain a sheaf of algebras $D_{Y,h}$ flat over $k[h]$ whose fiber over $h=1$ is D_Y and whose fiber over $h=0$ is the symmetric algebra $S^{\cdot}T_Y$. Explicitly, $D_{Y,h}$ is the subalgebra of $D_Y[h]$ generated by h, $\mathcal{O}_Y$, and hT_Y. The p-curvature homomorphism

$$S^{\cdot}T_{Y'} \to D_{Y,h}$$

sending a function $f \in \mathcal{O}_Y$ to f^p and a vector field $\theta \in T_Y$ to $(h\theta)^p - h^{p-1}(h\theta^{[p]})$ induces an isomorphism between the algebra $S^{\cdot}T_{Y'}[h]$ and the center of $D_{Y,h}$. In particular, $D_{Y,h}$ can be viewed as a quasi-coherent sheaf on $\mathbb{T}^*_{Y'}$. The canonical Frobenius-constant quantization of $(\mathbb{T}^*_Y, \omega)$ is obtained from $D_{Y,h}$ by h-completion. We shall denote this canonical Frobenius-constant quantization of $(\mathbb{T}^*_Y, \omega)$ by $(\mathbb{T}^*_Y, \mathcal{D}_{Y,h})$.

1.3 Restricted Poisson structures

There is a local obstruction to the existence of a central quantization of a symplectic variety (X, ω). It was observed in [BK08] that if $f^p \in \mathcal{O}_{X'} \xrightarrow{\sim} \mathcal{O}_X^p$ is in the image (1.1), then the restricted power $H_f^{[p]}$ of the Hamiltonian vector field H_f is again Hamiltonian: $H_f^{[p]} = H_{f^{[p]}}$ for some $f^{[p]} \in \mathcal{O}_X$. For example, it follows that the torus $(\mathbb{G}_m \times \mathbb{G}_m, \omega = dx/x \wedge dy/y)$ does not admit central quantizations.

A Frobenius-constant structure on $(X, \mathcal{O}_h)$ provides a canonical Hamiltonian for $H_f^{[p]}$. In fact, given a Frobenius-constant quantization $(X, \mathcal{O}_h, s)$ the formula

$$f^{[p]} = \frac{1}{h^{p-1}}(\tilde{f}^p - s(f)) \mod h \tag{1.3}$$

defines a restricted structure on the Poisson algebra $\mathcal{O}_X$, that is, the structure of a restricted Lie algebra on $\mathcal{O}_X$ such that $(f^2)^{[p]} = 2f^{[p]}f^p$ and $H_f^{[p]} = H_{f^{[p]}}$.[2]

It was shown in [BK08] that, for every symplectic variety (X, ω), giving a restricted Poisson structure on $\mathcal{O}_X$ is equivalent to giving a class

$$[\eta] \in H^0_{\mathrm{Zar}}(X, \mathrm{coker}(\mathcal{O}_X \xrightarrow{d} \Omega^1_X))$$

such that

$$d([\eta]) = \omega.$$

In one direction, if $\eta \in \Omega^1_X$, $d\eta = \omega$, then the formula

$$f^{[p]} = L_{H_f}^{p-1}\iota_{H_f}\eta - \iota_{H_f^{[p]}}\eta$$

defines a restricted structure on $\mathcal{O}_X$. In particular, if (X, ω) admits a restricted structure, then ω is exact locally for Zariski topology on X.

1.4 Classification of Frobenius-constant quantizations

Fix a symplectic variety X with a restricted Poisson structure $[\eta]$. Denote by $Q(X, [\eta])$ the set of isomorphism classes of Frobenius-constant quantizations $(X, \mathcal{O}_h, s)$ compatible with $[\eta]$.

[2] Whereas the notion of restricted Lie algebra goes back to Jacobson in 1937, the concept of restricted Poisson algebra is an invention of Bezrukavnikov and Kaledin [BK08, Definition 1.9]. Note that using the identity $fg = \frac{1}{4}((f+g)^2 - (f-g)^2)$, one has that $(fg)^{[p]} = f^p g^{[p]} + f^{[p]}g^p + P(f,g)$, where $P(f,g)$ is an element of a free Poisson algebra on f and g. In [BK08], the authors construct $P(f,g)$ explicitly in any characteristic which makes it possible to define the notion of restricted Poisson algebra even in characteristic 2.

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In [BK08], Bezrukavnikov and Kaledin constructed a map of sets

$$\rho : Q(X, [\eta]) \to H^1_{\text{et}}(X', \mathcal{O}^*_{X'}/\mathcal{O}^{*p}_{X'}) \tag{1.4}$$

and showed that if $H^1_{\text{Zar}}(X', \mathcal{O}_{X'}/\mathcal{O}^p_{X'}) = 0$, then ρ is injective, and if $H^2_{\text{Zar}}(X', \mathcal{O}_{X'}/\mathcal{O}^p_{X'}) = 0$, then ρ is surjective. Consequently, if both cohomology groups vanish the map ρ is a bijection and there is a canonical Frobenius-constant quantization of $(X, [\eta])$ corresponding to $0 \in H^1_{\text{et}}(X', \mathcal{O}^*_{X'}/\mathcal{O}^{*p}_{X'})$. This quantization $(X, \mathcal{O}_h, s)$ is uniquely characterized as the one that admits a $\mathbb{Z}/2$-equivariant structure: an isomorphism $\mathcal{O}_{X'}[[h]]$-algebras

$$\alpha : \mathcal{O}^{\text{op}}_{-h} \xrightarrow{\sim} \mathcal{O}_h \tag{1.5}$$

identical modulo h and such that $\alpha \circ \alpha = \text{Id}$. We review the construction of ρ in § 2.

1.5 A central reduction of the algebra of differential operators

A class

$$[\eta] \in H^0_{\text{Zar}}\Big(X, \text{coker}\left(\mathcal{O}_X \xrightarrow{d} \Omega^1_X\right)\Big)$$

gives rise to a certain central reduction $\mathcal{D}_{X,[\eta],h}$ of the algebra $\mathcal{D}_{X,h}$. We first construct this reduction locally and then glue. For any open subset U together with a 1-form $\eta \in \Omega^1(U)$ representing $[\eta]$, consider the graph $\Gamma_\eta : U' \to \mathbb{T}^*_{U'}$ of $\eta \otimes 1 \in \Omega^1(U')$. Let $\mathcal{D}_{U,h}$ be the quantization of $\mathbb{T}^*_U$ defined above regarded as a locally free sheaf of modules over $S^\cdot T_{U'}[[h]]$ on U. Set

$$\Gamma^*_{\eta_1} \mathcal{D}_{U,h} = \mathcal{D}_{U,h}/I_{\Gamma_\eta}\mathcal{D}_{U,h}. \tag{1.6}$$

Here $I_{\Gamma_\eta} \subset S^\cdot T_{U'}$ is sheaf of ideals defined by the closed embedding Γ_η. Note that $\Gamma^*_{\eta_1}\mathcal{D}_{U,h}$ is a sheaf of algebras over $S^\cdot T_{U'}/I_{\Gamma_\eta}[[h]] \xrightarrow{\sim} \mathcal{O}_{U'}[[h]]$.

Suppose we are given two forms η_1, η_2 on U representing the class $[\eta]$. Let us construct a canonical isomorphism between the algebras $\Gamma^*_{\eta_1}\mathcal{D}_{X,h}$ and $\Gamma^*_{\eta_2}\mathcal{D}_{X,h}$. Set $\mu = \eta_1 - \eta_2$. Define the automorphism ϕ_μ of $\mathcal{D}_{X,h}$ by setting $\phi_\mu(f) = f$ and $\phi_\mu(h\theta) = h\theta + \iota_\theta\mu$, for any function f and vector field θ.[3] Let t_μ be the translation by μ on $S^\cdot T_{U'}$ i.e. an automorphism sending a vector field θ to $\theta + \iota_\theta\mu$. Then using the Katz formula [Kat72, § 7.22] (and the exactness of μ) the following diagram is commutative.

$$\begin{array}{ccccc}
\mathcal{O}_{U'} & \xleftarrow{\Gamma^*_{\eta_1}} & S^\cdot T_{U'} & \xrightarrow{s} & \mathcal{D}_{U,h} \\
 & \nwarrow^{\Gamma^*_{\eta_2}} & \downarrow{\scriptstyle t_\mu} & & \downarrow{\scriptstyle \phi_\mu} \\
 & & S^\cdot T_{U'} & \xrightarrow{s} & \mathcal{D}_{U,h}
\end{array}$$

The desired isomorphism is given by the formula

$$\Gamma^*_{\eta_1}\mathcal{D}_{U,h} = \mathcal{D}_{U,h} \otimes_{S^\cdot T_{U'}} S^\cdot T_{U'}/\mathcal{I}_{\Gamma_{\eta_1}} \xrightarrow{\phi_\mu \otimes t_\mu} \mathcal{D}_{U,h} \otimes_{S^\cdot T_{U'}} S^\cdot T_{U'}/\mathcal{I}_{\Gamma_{\eta_2}} = \Gamma^*_{\eta_2}\mathcal{D}_{U,h}.$$

Given three 1-forms η_1, η_2, and η_3 representing the class $[\eta]$ one has

$$(\phi_{\eta_1-\eta_2} \otimes t_{\eta_1-\eta_2}) \circ (\phi_{\eta_2-\eta_3} \otimes t_{\eta_2-\eta_3}) = \phi_{\eta_1-\eta_3} \otimes t_{\eta_1-\eta_3}.$$

The sheaf of algebras $\mathcal{D}_{X,[\eta],h}$ is obtained by gluing $\Gamma^*_\eta\mathcal{D}_{U,h}$ along the above isomorphisms.

The sheaf $\mathcal{D}_{X,[\eta],h}$ of $\mathcal{O}_{X'}[[h]]$-algebras is locally free as a $\mathcal{O}_{X'}[[h]]$-module of rank $p^{2\dim X}$. The commutative algebra $\mathcal{D}_{X,[\eta],h}/h$ is isomorphic to the algebra of functions on the Frobenius neighborhood of the zero section $X \hookrightarrow \mathbb{T}^*_X$ with the Poisson structure given by the symplectic form $\omega_{\text{can}} + \text{pr}^*\omega$ on $\mathbb{T}^*_X$. Here ω_{can} is the canonical symplectic form on the cotangent bundle, and $\text{pr} : \mathbb{T}^*_X \to X$ is the projection.

[3] Informally, this isomorphism is the conjugation by $e^{(1/h)\int \mu}$.

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Remark 1.1. The sheaf $\mathcal{D}_{X,[\eta],h}$ is the restriction of a certain canonical locally free $\mathcal{O}_{X'\times\mathbb{P}^1}$-algebra over $X'\times\mathbb{P}^1$ to the formal completion of $X'\times\{0\}\hookrightarrow X'\times\mathbb{P}^1$ (see [BKTV22, § 3.3]).

1.6 Main result

Denote by $\mathrm{Br}(X'[[h]])$ the Brauer group of the formal scheme $(X',\mathcal{O}_{X'}[[h]])$ obtained from $X'\times\operatorname{Spec}k[h]$ by completion along the closed subscheme cut by the equation $h=0$. We have homomorphisms:

$$\delta\colon H^1_{\mathrm{et}}(X',\mathcal{O}^*_{X'}/\mathcal{O}^{*p}_{X'})\to H^2_{\mathrm{et}}(X',\mathcal{O}^*_{X'})\cong\mathrm{Br}(X')\hookrightarrow\mathrm{Br}(X'[[h]]). \tag{1.7}$$

The first map in (1.7) is the boundary morphism associated to the short exact sequence of sheaves for the étale topology

$$0\to\mathcal{O}^*_{X'}\xrightarrow{p}\mathcal{O}^*_{X'}\longrightarrow\mathcal{O}^*_{X'}/\mathcal{O}^{*p}_{X'}\to 0.$$

The right arrow in (1.7) is the pullback homomorphism which is a split injection because its composition with the restriction homomorphism

$$i^*\colon\ \mathrm{Br}(X'[[h]])\to\mathrm{Br}(X')$$

is the identity. Given a class $\gamma\in H^1_{\mathrm{et}}(X',\mathcal{O}^*_{X'}/\mathcal{O}^{*p}_{X'})$ we denote by $\delta(\gamma)\in\mathrm{Br}(X'[[h]])$ the image of γ under the composition (1.7). Finally, we can state the main result of this paper.

THEOREM 1. *Let (X,ω) be a smooth symplectic variety of dimension $2n$ over an algebraically closed field k of characteristic $p>2$, and let $(X,\mathcal{O}_h,s)$ be a Frobenius-constant quantization of (X,ω). Denote by $[\eta]\in H^0_{\mathrm{Zar}}(X,\operatorname{coker}(\mathcal{O}_X\xrightarrow{d}\Omega^1_X))$ the restricted Poisson structure corresponding to $(X,\mathcal{O}_h,s)$ and by $\gamma=\rho(X,\mathcal{O}_h,s)\in H^1_{\mathrm{et}}(X',\mathcal{O}^*_{X'}/\mathcal{O}^{*p}_{X'})$ the image of $(X,\mathcal{O}_h,s)$ under (1.4). Then there exists an Azumaya algebra $\mathcal{O}^\sharp_h$ over the formal scheme $(X',\mathcal{O}_{X'}[[h]])$ with the following properties.*

(i) *There exists an isomorphism of $\mathcal{O}_{X'}((h))$-algebras*

$$(\mathcal{O}_h\otimes_{\mathcal{O}_{X'}[[h]]}\mathcal{D}^{\mathrm{op}}_{X,[\eta],h})(h^{-1})\xrightarrow{\sim}\mathcal{O}^\sharp_h(h^{-1}). \tag{1.8}$$

(ii) *We have that*

$$i^*[\mathcal{O}^\sharp_h]=i^*(\delta(\gamma)). \tag{1.9}$$

In particular, if $H^2(X,\mathcal{O}_X)=0$ then $[\mathcal{O}^\sharp_h]=\delta(\gamma)$.

For example, let $(X,\mathcal{O}_h,s)$ be a Frobenius-constant quantization that admits $\mathbb{Z}/2\mathbb{Z}$-equivariant structure (1.5). Assume that $H^2(X,\mathcal{O}_X)=0$. Then by Theorem 1 $\mathcal{O}^\sharp_h$ is a split Azumaya algebra, that is there exists a locally free $\mathcal{O}_{X'}[[h]]$-module E of finite rank and an isomorphism of $\mathcal{O}_{X'}[[h]]$-algebras

$$\mathcal{O}^\sharp_h\xrightarrow{\sim}\mathrm{End}_{\mathcal{O}_{X'}[[h]]}(E).$$

Using (1.8) and the Azumaya property of $\mathcal{O}_h(h^{-1})$ and $\mathcal{D}_{X,[\eta],h}(h^{-1})$ we observe an equivalence of categories

$$\mathrm{Mod}(\mathcal{D}_{X,[\eta],h}(h^{-1}))\xrightarrow{\sim}\mathrm{Mod}(\mathcal{O}_h(h^{-1})) \tag{1.10}$$

between the category of $\mathcal{O}_h(h^{-1})$-modules and the category of $\mathcal{D}_{X,[\eta],h}(h^{-1})$-modules. The functor from left to right carries a $\mathcal{D}_{X,[\eta],h}(h^{-1})$-module M to $E\otimes_{\mathcal{D}_{X,[\eta],h}}M$; the quasi-inverse functor takes an $\mathcal{O}_h(h^{-1})$-module N to $\mathcal{H}om_{\mathcal{O}_h}(E,N)$.

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Also note that if $H^1_{\mathrm{Zar}}(X', \mathcal{O}_{X'}/\mathcal{O}^p_{X'}) = 0$, then the map

$$H^0(X, \Omega^1_X) \to H^0_{\mathrm{Zar}}(X, \mathrm{coker}(\mathcal{O}_X \xrightarrow{d} \Omega^1_X))$$

is surjective and, thus, any restricted Poisson structure arises from a global 1-form η. In this case objects of $\mathrm{Mod}(\mathcal{D}_{X,[\eta],h}(h^{-1}))$ can be viewed as $\mathcal{D}_{X,h}(h^{-1})$-modules whose p-curvature equals η.

1.7 $\mathbb{G}_m$-equivariant quantizations

Let (X, ω) be a symplectic variety equipped with an action

$$\lambda\colon \mathbb{G}_m \times X \to X \tag{1.11}$$

of the multiplicative group such that ω has a positive weight m with respect to this action. Moreover, we shall assume that m invertible in k. Denote by θ the Euler vector field on X corresponding to the $\mathbb{G}_m$-action. Then the formula $\eta = (1/m)\iota_\theta\omega$ defines a restricted structure on X. Define a $\mathbb{G}_m$-action on X' twisting (1.11) by the pth power map $\mathbb{G}_m \xrightarrow{F} \mathbb{G}_m$. Also let $\mathbb{G}_m$ act on $X'[h] := X' \times \mathrm{Spec}\, k[h]$ as above on the first factor and by $z * h = z^m h$ on the second one.

A $\mathbb{G}_m$-equivariant Frobenius-constant quantization of X is a $\mathbb{G}_m$-equivariant sheaf O_h of associative $\mathcal{O}_{X'[h]}$-algebras on $X'[h]$, locally free as an $\mathcal{O}_{X'[h]}$-module, such that the restriction $\mathcal{O}_h$ of O_h to the formal completion of $X'[h]$ along the divisor $h = 0$ is a Frobenius-constant quantization of X compatible with the restricted structure $[\eta]$. Examples of $\mathbb{G}_m$-equivariant quantizations arise in geometric representation theory (see e.g. [BK04b, BF14, BL21, KT19]).

Assume that morphism (1.11) extends to a morphism

$$\tilde{\lambda} : \mathbb{A}^1 \times X \to X. \tag{1.12}$$

Then the restriction of O_h to the open subscheme $X'[h, h^{-1}] \hookrightarrow X'[h]$ is an Azumaya algebra. As an application of Theorem 1 we prove in § 5 a conjecture of Kubrak and Travkin concerning the class of this algebra in the Brauer group. Namely, we show that, for every $\mathbb{G}_m$-equivariant Frobenius-constant quantization O_h, the following equality in $\mathrm{Br}(X')$ holds:

$$[O_{h=1}] = \left[\frac{1}{m}\eta\right] + \tilde{\lambda}_0^*[\rho(O_h)].$$

Here $[\eta]$ denotes the image of η under the canonical map $\Gamma(X', \Omega^1_{X'}) \to \mathrm{Br}(X')$, $\rho(O_h) \in H^1_{\mathrm{et}}(X', \mathcal{O}^*_{X'}/\mathcal{O}^{*p}_{X'})$ for the class associated to the formal quantization via (1.4), and $[\rho(O_h)]$ for its image in the Brauer group.

1.8 Plan of the proof

Using the language of formal geometry we reduce the theorem to a group-theoretic statement. We shall start by explaining the latter.

Let (V, ω_V) be a finite-dimensional symplectic vector space over k, and let A_h be the algebra over $k[[h]]$ generated by the dual vector space V^* subject to the relations

$$fg - gf = \omega_V^{-1}(f, g)h, \quad f^p = 0$$

for any $f, g \in V^*$. We refer to A_h as the restricted *Weyl algebra*. This is a flat $k[[h]]$-algebra whose reduction modulo h is the finite-dimensional commutative algebra of functions on the Frobenius neighborhood of the origin in the affine space $\mathrm{Spec}(S^\cdot V^*) := \mathbf{V}$. Explicitly, $A_0 := S^\cdot V^*/J_V$ where

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J_V is the ideal generated by f^p for all $f \in V^*$. The quantization A_h of A_0 specifies a restricted Poisson structure $[\eta_V] \in \operatorname{coker}(A_0 \xrightarrow{d} \Omega^1_{A_0})$.[4]

Denote by G the group scheme $\underline{\mathrm{Aut}}(A_h)$ of $k[[h]]$-linear automorphisms of the algebra A_h, by $G^{\geq 1}$ the subgroup of automorphisms identical modulo h, and by G_0 the quotient of G by $G^{\geq 1}$. As shown in [BK08], G_0 is the group scheme of automorphisms of A_0 preserving the class $[\eta_V]$.

A pair (W, W^*) of transversal Lagrangian subspaces of V defines an isomorphism between the algebra $A_h(h^{-1})$ and the matrix algebra $\mathrm{End}_k(S^{\cdot}W^*/J_W)((h))$ which, in turn, gives an embedding $G \hookrightarrow L\,\mathrm{PGL}(p^n)$, where $L\,\mathrm{PGL}(p^n)$ is the loop group of $\mathrm{PGL}(p^n)$ (viewed as a sheaf for the *fpqc* topology). Then the extension

$$1 \to \mathbb{G}_m \to \mathrm{GL}(p^n) \to \mathrm{PGL}(p^n) \to 1$$

gives rise to[5]

$$1 \to L\mathbb{G}_m \to \tilde{G} \to G \to 1. \tag{1.13}$$

In [BK08] it is proved that $G^{\geq 1}$ is the subgroup of inner automorphisms. Hence, we have a subextension of (1.13)

$$1 \to L^+\mathbb{G}_m \to \underline{A_h^*} \to G^{\geq 1} \to 1, \tag{1.14}$$

where $L^+\mathbb{G}_m$ is the positive loop group of $\mathbb{G}_m$. Then passing to the quotient we get a central extension by the affine grassmannian[6]

$$1 \to \mathrm{Gr}_{\mathbb{G}_m} \to \tilde{G}_0 \to G_0 \to 1. \tag{1.15}$$

Let $i : V \hookrightarrow V^\flat$ be a morphism of symplectic vector spaces such that the restriction to V of the symplectic form on $V^\flat$ is ω_V. Let $\widetilde{G}_0 \to G_0$ and $\widetilde{G}_0^\flat \to G_0^\flat$ be the corresponding extensions. We emphasize that $\widetilde{G}_0$ and $\widetilde{G}_0^\flat$ depend on a choice of Lagrangian pairs in V and $V^\flat$.

Finally, denote by $G_0^\sharp \subset G_0^\flat$ the group subscheme that consists of automorphisms preserving the kernel of the homomorphism $i^* : A_0^\flat \to A_0$. We have a natural homomorphism $G_0^\sharp \to G_0$. In § 4 we prove the following assertion.

BASIC LEMMA. *The homomorphism $G_0^\sharp \to G_0$ lifts uniquely to a homomorphism of central extensions*

$$\widetilde{G}_0^\flat \times_{G_0^\flat} G_0^\sharp \to \widetilde{G}_0.$$

Our proof of the Basic Lemma, that occupies almost the half of the paper, is based on a new construction of (1.15) that makes this functoriality property obvious. Namely, consider two subgroups $\alpha \subset G_0 \supset G_0^0$, where G_0^0 is the subgroup of automorphisms preserving the origin in $\mathbf{V}$ (which by a result of Bezrukavnikov and Kaledin coincides with the reduced subgroup of G_0) and $\alpha = \operatorname{Spec} A_0 (\cong \alpha_p^{\dim V})$ is the finite group scheme of translations. Then the product map $\alpha \times G_0^0 \to G_0$ induces an isomorphism of the underlying schemes. Let $\widetilde{\alpha}$ be the restriction of the central extension (1.15) to α. This is a version of the Heisenberg group. We show that the extension (1.15) splits uniquely over the reduced subgroup G_0^0.[7] Thus, we can view G_0^0 as a

[4] Explicitly $[\eta_V]$ is characterized as a unique homogeneous class such that $d[\eta_V] = \omega_V$.

[5] We do not know whether the morphism of *fpqc* sheaves $L\,\mathrm{GL}(p^n) \to L\,\mathrm{PGL}(p^n)$ is surjective. However, we check in Proposition A.5 that its pullback to any group subscheme $G \subset L\,\mathrm{PGL}(p^n)$ satisfying some finiteness assumptions is surjective even for the Zariski topology on G.

[6] Recall from [Con94] that $\mathrm{Gr}_{\mathbb{G}_m}$ is isomorphic to the direct product $\hat{\mathbb{W}} \times \underline{\mathbb{Z}}$, where $\hat{\mathbb{W}}$ a group ind-scheme whose points with values in a k-algebra R is the subgroup of $R[h^{-1}]^*$ consisting of invertible polynomials with zero constant term. In particular, if R is reduced and connected, then $\mathrm{Gr}_{\mathbb{G}_m}(R) = \mathbb{Z}$.

[7] *A posteriori*, this is a corollary of the Basic Lemma applied to the embedding $0 \hookrightarrow V$.

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subgroup of $\widetilde{G_0}$, and the quotient $\widetilde{G_0}/G_0^0$ is identified (as an ind-scheme) with $\widetilde{\alpha}$. The left action of $\widetilde{G_0}$ on $\widetilde{G_0}/G_0^0$ defines an embedding of $\widetilde{G_0}$ into the group of automorphisms of $\widetilde{\alpha}$ viewed as a space with an action of $\mathrm{Gr}_{\mathbb{G}_m}$. We prove in Theorem 3 that the image of this embedding is precisely the group of automorphisms that preserve a unique $\mathrm{Sp}_{2n} \ltimes \alpha_p^{2n}$-invariant connection on $\mathrm{Gr}_{\mathbb{G}_m}$-torsor $\widetilde{\alpha}$.

To derive the Basic Lemma from the above, we classify all central extensions of α by $\mathrm{Gr}_{\mathbb{G}_m}$ in § 4.2. In particular, we show that extensions of α by $\mathrm{Gr}_{\mathbb{G}_m}$ that split over every α_p factor are classified by $\mathrm{Lie}(\mathrm{Gr}_{\mathbb{G}_m})$-valued skew-symmetric 2-forms on $\mathrm{Lie}(\alpha)$. Then it follows that the morphism $\alpha \to \alpha^\flat$ induced by i lifts uniquely to a morphism of extensions $\widetilde{\alpha} \to \widetilde{\alpha}^\flat$ respecting the connections. Since $\widetilde{G}_0^\flat \times_{G_0^\flat} G_0^\sharp$ is the group of automorphisms of $\widetilde{\alpha}^\flat$ that preserve the connection and the subspace $\widetilde{\alpha} \hookrightarrow \widetilde{\alpha}^\flat$, by restriction we get the desired lifting $\widetilde{G}_0^\flat \times_{G_0^\flat} G_0^\sharp \to \tilde{G}_0$.

Let us explain how the Basic Lemma implies the theorem. The Bezrukavnikov–Kaledin construction of Frobenius-constant quantizations is based on a characteristic-p version of the Gelfand–Kazhdan formal geometry. Namely, it is shown in [BK04b] that any Frobenius-constant quantization is locally for the *fpqc* topology on X' isomorphic to the constant quantization $\mathcal{O}_{X'}[[h]] \otimes_{k[[h]]} A_h$ for a fixed finite-dimensional space V of dimension $2n = \dim X$. It follows that a Frobenius-constant quantization $(X, \mathcal{O}_h, s)$ gives rise to a torsor $\mathcal{M}_{X,\mathcal{O}_h,s}$ over G. Conversely, the algebra $\mathcal{O}_h$ is the twist of $\mathcal{O}_{X'}[[h]] \otimes_{k[[h]]} A_h$ by the torsor $\mathcal{M}_{X,\mathcal{O}_h,s}$, i.e.

$$\mathcal{M}_{X,\mathcal{O}_h,s} \times^G (\mathcal{O}_{X'}[[h]] \otimes_{k[[h]]} A_h) \xrightarrow{\sim} \mathcal{O}_h.$$

The reduction of differential operators $\mathcal{D}_{X,[\eta],h}$ also can be constructed using formal geometry. Namely, choosing a homogeneous form η_V in the class $[\eta_V]$ on V consider its graph

$$\mathbf{V} \hookrightarrow \mathbb{T}^*_{\mathbf{V}},$$

and let $i : V \to V \oplus V^* = V^\flat$ be the corresponding linear map of vector spaces. Let $G_0^{\sharp,f}$ be the subgroup of $G_0^\sharp \subset G_0^\flat$ of automorphisms g of $\alpha^\flat$ respecting the fibers of the projection $\pi : \alpha^\flat \to \alpha$, that is fitting in the following diagram.

$$\begin{array}{ccc} \alpha^\flat & \xrightarrow{g} & \alpha^\flat \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ \alpha & \xrightarrow{\bar{g}} & \alpha \end{array}$$

Then the restriction of the natural map $G_0^\sharp \to G_0$ to $G_0^{\sharp,f}$ is an isomorphism. This yields a homomorphism $\psi_0 : G_0 \hookrightarrow G_0^\sharp \subset G_0^\flat$. In § 3.1 we construct a lifting $\psi : G \to G^\flat$ of ψ_0 that makes $\mathcal{D}_{X,[\eta],h}$ a twist of $\mathcal{O}_{X'}[[h]] \otimes_{k[[h]]} A_h^\flat$ by $\mathcal{M}_{X,\mathcal{O}_h,s}$.

Then we consider the following diagram.

$$\begin{array}{ccccc} & & & & \mathrm{Gr}_{\mathbb{G}_m} \\ & & & & \downarrow \\ & & & & L\,\mathrm{GL}(p^{3n})/L^+\mathbb{G}_m \\ & & & \nearrow & \downarrow \\ G & \longrightarrow & \underline{\mathrm{Aut}}(A_h \otimes A_h^{\flat,op}) & \longrightarrow & L\,\mathrm{PGL}(p^{3n}) \end{array}$$

Here the loop group $L\,\mathrm{PGL}$ is the group ind-scheme of projective automorphisms of a certain vector space U over $k((h))$. Suppose we can construct a G-invariant $k[[h]]$-lattice Λ in U. Then

$$\mathcal{O}_h^\sharp := \mathcal{M}_{X,\mathcal{O}_h,s} \times^G (\mathcal{O}_{X'}[[h]] \otimes_{k[[h]]} \mathrm{End}_{k[[h]]}(\Lambda))$$

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does the job for the first part of the theorem. By a general result proven in the Appendix (Proposition A.5) the existence of an invariant lattice is equivalent to the existence of the dashed arrow making the diagram above commutative. This is where we use the Basic Lemma. It follows from the latter that

$$\widetilde{G}_0^\flat \times_{G_0^\flat} G_0^{\sharp,f} \cong \widetilde{G}_0.$$

We infer (Proposition 3.3) that ψ_0 lifts to $\widetilde{\psi}_0 : \widetilde{G}_0 \to \widetilde{G}_0^\flat$, which implies the existence of the dashed arrow.

The proof of the second part of the theorem amounts to unveiling the Bezrukavnikov–Kaledin construction of the map ρ.

1.9 Further directions

In this subsection we briefly discuss some applications of Theorem 1 obtained in a sequel paper joint with Dmitry Kubrak and Roman Travkin [BKTV22].

According to the Bezrukavnikov–Kaledin theorem from § 1.4 every smooth affine restricted symplectic variety admits a unique up to a non-canonical isomorphism Frobenius-constant quantization O_h with $\rho[O_h] = 0$. The formation of O_h is not functorial in $(X, [\eta])$. However, we show in [BKTV22] that the assignment $(X, [\eta]) \mapsto \mathrm{Mod}(O_h(X))$ extends to a contravariant functor from the category of smooth affine restricted symplectic varieties and open embeddings to the category of abelian categories. Applying the right Kan extension this yields a functorial quantization QCoh_h of the category of quasi-coherent sheaves of any smooth restricted symplectic variety. Moreover, using Remark 1.1 and equivalence (1.10) we extend the range of quantum parameter h from being a formal variable to a genuine coordinate on $\mathbb{P}^1$. The construction of QCoh_h uses in an essential way Corollary 3.5 of the Basic Lemma.

Let $Y \hookrightarrow X$ be a smooth Lagrangian subvariety such that $[\eta]_{|Y} = 0$ in $H^0_{\mathrm{Zar}}\big(Y, \Omega^1_Y/d\mathcal{O}_Y\big)$. Using results from [Mun22] we show in § 6.1 of [BKTV22] that every such Y determines a canonical object in QCoh_h, which is a quantization of the line bundle $(\Omega^n_Y)^{(1-p)/2}$ viewed as a quasi-coherent sheaf on X.

1.10 Plan of the paper

In § 2 we review the Bezrukavnikov–Kaledin construction of Frobenius-constant quantizations which is based on a characteristic p version of the Gelfand–Kazhdan formal geometry. In § 3 we recast the construction of $\mathcal{D}_{X,h}$ using the language of formal geometry and reduce Theorem 1 to a certain statement on central extensions of the group of automorphisms of the restricted Weyl algebra. In § 4 we prove this statement. In § 5 we study $\mathbb{G}_m$-equivariant quantizations and prove a conjecture of Kubrak and Travkin. Finally, in the Appendix we prove some results (probably known to experts) on loop groups that are used in the main body of the paper.

2. Review of the Bezrukavnikov–Kaledin construction

For reader's convenience we review the Bezrukavnikov–Kaledin construction of quantizations. We also introduce some notation to be used later. Nothing in this section is an invention of the authors.

2.1 Darboux lemma in characteristic p

Our proof of Theorem 1, as well as the Bezrukavnikov–Kaledin construction of quantizations, is based on a version of the Gelfand–Kazhdan formal geometry that makes it possible to localize the problem and ultimately reduce it to a statement in group theory. The main idea is as follows.

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For a symplectic variety X, the Poisson bracket on $\mathcal{O}_X$ is $\mathcal{O}_{X'}$-linear. Therefore, we can view X as a Poisson scheme over X'. For any restricted structure on $\mathcal{O}_X$, one has $\mathcal{O}_{X'}^{[p]} = 0$. Therefore, a symplectic variety X with a restricted structure can be viewed as a restricted Poisson scheme over X'. Consider the constant restricted Poisson scheme over X':

$$X' \times \operatorname{Spec} A_0 \to X', \tag{2.1}$$

where

$$A_0 = k[x_1, y_1, \dots, x_n, y_n]/(x_1^p, y_1^p, \dots, x_n^p, y_n^p),$$

$2n = \dim X$, the morphism (2.1) is the projection to the first factor, the Poisson structure is given by symplectic form $\sum_i dy_i \wedge dx_i$, and the restricted structure is determined by $x_i^{[p]} = y_i^{[p]} = 0$. A key insight of Bezrukavnikov and Kaledin is that any smooth symplectic variety X with a restricted structure, viewed as a restricted Poisson scheme over X', is locally for the *fpqc* topology on X' isomorphic to the constant restricted Poisson scheme $X' \times \operatorname{Spec} A_0 \to X'$. This is an analogue of the Darboux lemma.

2.2 Quantum Darboux lemma

There is also a quantum version of the Darboux lemma proven in [BK08]: for any Frobenius-constant quantization $(X, \mathcal{O}_h, s)$, the sheaf of associative $\mathcal{O}_{X'}[[h]]$-algebras $\mathcal{O}_h$ is isomorphic locally for the *fpqc* topology on X' to the h-completed tensor product $\mathcal{O}_{X'} \otimes_k A_h$, where A_h is *the reduced Weyl algebra* that is the $k[[h]]$-algebra generated by variable x_i, y_i $(1 \leq i, j, \leq n)$, subject to the relations

$$y_j x_i - x_i y_j = \delta_{ij} h, \quad x_i^p = y_i^p = 0. \tag{2.2}$$

2.3 Formal geometry

Let $\underline{\mathrm{Aut}}(A_0)$ be the group scheme of automorphisms of the algebra A_0. For any smooth scheme X over k of dimension $2n$, assigning to a scheme Z over X' the set $\mathcal{M}_X(Z)$ of isomorphisms

$$Z \times \operatorname{Spec} A_0 \xrightarrow{\sim} Z \times_{X'} X$$

of schemes over Z, we get a $\underline{\mathrm{Aut}}(A_0)$-torsor over X'. Next, let $G_0 \subset \underline{\mathrm{Aut}}(A_0)$ be the group subscheme consisting of automorphisms of A_0 that preserve the restricted Poisson structure on A_0. Then the Darboux lemma above implies that, for every symplectic variety $(X, [\eta])$ with a restricted structure of dimension $2n$ the functor assigning to a scheme Z over X' the set $\mathcal{M}_{X,[\eta]}(Z)$ of isomorphisms

$$Z \times \operatorname{Spec} A_0 \xrightarrow{\sim} Z \times_{X'} X$$

of restricted Poisson schemes over Z is a G_0-torsor over X'. Using the faithfully flat descent one gets a bijection between the set of nondegenerate (that is, arising from a symplectic form) restricted Poisson structures $[\eta]$ on X and the set of G_0-torsors over X' equipped with an isomorphism of $\mathrm{Aut}(A_0)$-torsors

$$\underline{\mathrm{Aut}}(A_0) \times^{G_0} \mathcal{M}_{X,[\eta]} \xrightarrow{\sim} \mathcal{M}_X.$$

Lastly, the set of all Frobenius-constant quantizations $(X, \mathcal{O}_h, s)$ of X such that the induced Poisson structure on X is nondegenerate is in bijection with the set of torsors $\mathcal{M}_{X,\mathcal{O}_h,s}$ over the group scheme $G := \underline{\mathrm{Aut}}(A_h)$ of automorphisms of $k[[h]]$-algebra A_h (that is a group scheme whose group of points with values in a k-algebra R is the group of $R[[h]]$-algebra automorphisms of the

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h-adically completed tensor product $A_h\hat{\otimes}R$) together with an isomorphism $\underline{\mathrm{Aut}}(A_0)$-torsors

$$\underline{\mathrm{Aut}}(A_0) \times^G \mathcal{M}_{X,\mathcal{O}_h,s} \xrightarrow{\sim} \mathcal{M}_X. \tag{2.3}$$

In particular, for a symplectic variety with a restricted structure $(X,[\eta])$, giving a Frobenius constant quantization of $(X,[\eta])$ is equivalent to lifting a G_0-torsor $\mathcal{M}_{X,[\eta]}$ to a G-torsor $\mathcal{M}_{X,\mathcal{O}_h,s}$ along the group scheme homomorphism

$$G \to G_0. \tag{2.4}$$

2.4 Automorphisms of the reduced Weyl algebra

It was shown in [BK08] that homomorphism (2.4) is surjective and its kernel $G^{\geq 1}$ consists of inner automorphisms. We have the exact sequence

$$1 \to L^+\mathbb{G}_m \to \underline{A}_h^* \to G \to G_0 \to 1. \tag{2.5}$$

Here $\underline{A}_h^*$ (respectively, $L^+\mathbb{G}_m$) is the group scheme over k whose group of R-points is $(R\otimes A_h)^*$ (respectively, $R[[h]]^*$). Letting $G^{\geq n}\subset G$ $(n\geq 0)$, be the group subscheme consisting of automorphisms that are identical modulo h^n, we have that $G^{\geq n}/G^{\geq n+1} \xrightarrow{\sim} \underline{A}_0/\mathbb{G}_a$ for every $n>1$ and $G^{\geq 1}/G^{\geq 2} \xrightarrow{\sim} \underline{A}_0^*/\mathbb{G}_m$, $G^{\geq 0}/G^{\geq 1} \xrightarrow{\sim} G_0$.

Consider the isomorphism of $k[[h]]$-algebras

$$\alpha : A_{-h}^{\mathrm{op}} \xrightarrow{\sim} A_h \tag{2.6}$$

sending x_i to x_i and y_j to y_j. The conjugation by α defines an involution $\tau : G\to G$ preserving the subgroups $G^{\geq n}$, $(n\geq 0)$, such that the induced action on $G^{\geq n}/G^{\geq n+1}$ takes an element g to $g^{(-1)^n}$. In particular, it follows that the extension

$$1 \to G^{\geq 1}/G^{\geq 2} \to G/G^{\geq 2} \to G_0 \to 1$$

has a unique $\mathbb{Z}/2\mathbb{Z}$-equivariant splitting

$$G/G^{\geq 2} \xrightarrow{\sim} G_0 \ltimes \underline{A}_0^*/\mathbb{G}_m.$$

2.5 $\mathbb{Z}/2\mathbb{Z}$-equivariant structures

Any $\mathbb{Z}/2\mathbb{Z}$-equivariant Frobenius-constant quantization $(X,\mathcal{O}_h,s,\alpha)$ is isomorphic locally for the *fpqc* topology on X' to the h-completed tensor product $\mathcal{O}_{X'}\hat{\otimes}_k A_h$ equipped with the equivariant structure (2.6). Indeed, consider the action of $\mathbb{Z}/2\mathbb{Z}$ on $G^{\geq 1}$ given by τ. Then $H^1(\mathbb{Z}/2\mathbb{Z}, G^{\geq 1})=0$ as $G^{\geq 1}$ has a filtration $G^{\geq n}$ with uniquely 2-divisible quotients. It follows that every two $\mathbb{Z}/2\mathbb{Z}$-equivariant structures on $\mathcal{O}_{X'}\hat{\otimes}_k A_h$ are locally isomorphic. Now the claim follows from the quantum Darboux lemma.

Consequently, giving a $\mathbb{Z}/2\mathbb{Z}$-equivariant Frobenius constant quantization of $(X,[\eta])$ is equivalent to lifting a G_0-torsor $\mathcal{M}_{X,[\eta]}$ to a torsor $\mathcal{M}_{X,\mathcal{O}_h,s,\alpha}$ over the subgroup $G^{\tau=1}\subset G$ of τ-invariants.

2.6 Construction of quantizations

Bezrukavnikov and Kaledin use the standard obstruction theory to classify liftings of a given G_0-torsor to a G-torsor. Namely, given a G_0-torsor $\mathcal{M}_{X,[\eta]}$ the set of isomorphism classes of its liftings to a torsor over $G/G^{\geq 2} \xrightarrow{\sim} G_0 \ltimes \underline{A}_0^*/\mathbb{G}_m$ is identified with the set of isomorphism classes of torsors over the smooth group scheme $\underline{A}_0^*/\mathbb{G}_m \times_{G_0} \mathcal{M}_{X,[\eta]}$ over X'. The latter group scheme is identified with the quotient of the group scheme of invertible elements in the sheaf of $\mathcal{O}_{X'}$-algebras $F_*\mathcal{O}_X$ by constant group scheme $\mathbb{G}_m\times X'$. Using smoothness of $\underline{A}_0^*/\mathbb{G}_m \times_{G_0} \mathcal{M}_{X,[\eta]}$ every torsor over this group scheme is locally trivial for the étale topology on X'. Hence, the set

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of isomorphism classes of $\underline{A}_0^*/\mathbb{G}_m \times_{G_0} \mathcal{M}_{X,[\eta]}$-torsors is in bijection with $H^1_{\text{et}}(X', \mathcal{O}^*_{X'}/\mathcal{O}^{*p}_{X'})$. This defines a map of sets

$$\rho : Q(X, [\eta]) \to H^1_{\text{et}}(X', \mathcal{O}^*_{X'}/\mathcal{O}^{*p}_{X'}) \tag{2.7}$$

from the set $Q(X, [\eta])$ of isomorphisms classes of Frobenius-constant quantizations $(X, \mathcal{O}_h, s)$ compatible with $[\eta]$ to the étale cohomology group classifying torsors over $G_0 \ltimes \underline{A}_0^*/\mathbb{G}_m$ lifting the G_0-torsor $\mathcal{M}_{X,[\eta]}$. Note that under this identification the trivial cohomology class corresponds to the lifting obtained from $\mathcal{M}_{X,[\eta]}$ via the natural group homomorphism $G_0 \to G_0 \ltimes \underline{A}_0^*/\mathbb{G}_m$.

Next, the obstruction class to lifting of a $G/G^{\geq n}$-torsor, with $n > 1$, to a $G/G^{\geq n+1}$-torsor lies in $H^2(X', \mathcal{O}_{X'}/\mathcal{O}^p_{X'})$. If the obstruction class vanishes, then the set of isomorphism classes of the liftings is a torsor over $H^1(X', \mathcal{O}_{X'}/\mathcal{O}^p_{X'})$. Hence, if $H^1_{\text{Zar}}(X', \mathcal{O}_{X'}/\mathcal{O}^p_{X'}) = 0$, then ρ is injective, and if $H^2_{\text{Zar}}(X', \mathcal{O}_{X'}/\mathcal{O}^p_{X'}) = 0$, then ρ is surjective. In particular, if the two cohomology groups vanish ρ is a bijection. The trivial cohomology class corresponds to a quantization that admits (a unique) $\mathbb{Z}/2\mathbb{Z}$-equivariant structure.

3. Reduction of the main theorem to a group-theoretic statement

In this section we recast the construction of $\mathcal{D}_{X,[\eta],h}$ using the language of formal geometry, and reduce Theorem 1 to a certain statement, Proposition 3.3, on central extensions of the group of automorphisms of the restricted Weyl algebra.

3.1 $\mathcal{D}_{X,[\eta],h}$ via formal geometry

Let $A_h^\flat$ be the reduced Weyl algebra in $4n$ variables, that is, the $k[[h]]$-algebra generated by variables x_i, y_i, v_i, u_i $(1 \leq i, j, \leq n)$, subject to the relations

$$\begin{gathered} v_i x_j - x_j v_i = u_i y_j - y_j u_i = \delta_{ij} h, \\ v_i y_j - y_j v_i = u_i x_j - x_j u_i = v_i u_j - u_j v_i = y_i x_j - x_j y_i = 0, \\ x_i^p = y_i^p = v_i^p = u_i^p = 0. \end{gathered} \tag{3.1}$$

We shall identify $A_h^\flat$ with the central reduction $D_{\text{Spec}\, A_0, \eta_A, h} (= D_{\text{Spec}\, A_0, 0, h})$, $\eta_A = \eta = \sum y_i\, dx_i$, of the algebra $D_{\text{Spec}\, A_0, h} \subset D_{\text{Spec}\, A_0}[[h]]$ spanned by A_0 and $hT_{\text{Spec}\, A_0}$. In particular, the group scheme $\underline{\text{Aut}}(A_0)$ acts on $A_h^\flat$:

$$\psi_{\text{can}} : \underline{\text{Aut}}(A_0) \to \underline{\text{Aut}}_{k[[h]]}(A_h^\flat) =: G^\flat, \quad g \mapsto \psi_{\text{can},g}. \tag{3.2}$$

We define a homomorphism

$$\psi : G_0 \to G^\flat, \tag{3.3}$$

to be the restriction of ψ_{can} to $G_0 \subset \underline{\text{Aut}}(A_0)$ twisted by a 1-cocycle

$$G_0 \to G^\flat, \quad g \mapsto \phi_{g^*\eta - \eta}.$$

Namely, for any k-algebra R, an R-point of G_0 is an automorphism g of the R-algebra $A_0 \otimes R$ such that the 1-form $\mu := g^*\eta - \eta \in \Omega^1_{A_0 \otimes R/R}$, $\eta = \sum y_i\, dx_i$, is exact. Let $\phi_\mu : A_h^\flat \hat{\otimes} R \to A_h^\flat \hat{\otimes} R$

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be the $R[[h]]$-algebra automorphism given by the formulas[8]

$$\phi_\mu(x_i) = x_i, \quad \phi_\mu(y_i) = y_i,$$
$$\phi_\mu(v_i) = v_i + \iota_{\partial/\partial x_i}\mu, \quad \phi_\mu(u_i) = u_i + \iota_{\partial/\partial y_i}\mu.$$

Define (3.3) by the formula

$$\psi_g = \phi_{g^*\eta-\eta} \circ \psi_{\mathrm{can},g}.$$

We claim that (3.3) is a homomorphism. Indeed, one has that

$$\psi_{\mathrm{can},g} \circ \phi_\mu \circ \psi^{-1}_{\mathrm{can},g} = \phi_{g^*\mu}.$$

Using this formula we find

$$\phi_{g_1^*\eta-\eta} \circ \psi_{\mathrm{can},g_1} \circ \phi_{g_2^*\eta-\eta} \circ \psi_{\mathrm{can},g_2} = \phi_{g_1^*\eta-\eta} \circ \phi_{g_1^*g_2^*\eta-g_1^*\eta} \circ \psi_{\mathrm{can},g_1} \circ \psi_{\mathrm{can},g_2}$$

and the claim follows.

The key assertion of this subsection is the following.

LEMMA 3.1. *Let (X,ω) be a symplectic variety with a restricted Poisson structure $[\eta]$. Then one has an isomorphism of $\mathcal{O}_{X'}[[h]]$-algebras:*

$$\mathcal{M}_{X,[\eta]} \hat{\times}^{G_0} A_h^\flat \xrightarrow{\sim} \mathcal{D}_{X,[\eta],h},$$

where the action of G_0 on $A_h^\flat$ is given by (3.3).

Proof. Let $\pi : \mathcal{M}_{X,[\eta]} \to X'$ be the projection. For a morphism $u\colon T \to X'$, we shall denote by $u^*\mathcal{D}_{X,h,[\eta]}$ the pullback $\mathcal{D}_{X,h,[\eta]}$, viewed as a coherent sheaf on the formal scheme $X'[[h]]$, along the morphism $T[[h]] \to X'[[h]]$ induced by u. It suffices to check that for every S-point f of $\mathcal{M}_{X,[\eta]}$ there is an isomorphism

$$\alpha_f : \mathcal{O}_S \hat{\otimes} A_h^\flat \cong (\pi \circ f)^*\mathcal{D}_{X,h,[\eta]}$$

such that the following diagram is commutative for every $g \in G_0(S)$.

$$\begin{array}{ccc} \mathcal{O}_S\hat{\otimes}A_h^\flat & \xrightarrow{\alpha_f} & (\pi\circ f)^*\mathcal{D}_{X,h,[\eta]} \\ \downarrow{\scriptstyle \psi(g)} & & \| \\ \mathcal{O}_S\hat{\otimes}A_h^\flat & \xrightarrow{\alpha_{gf}} & (\pi\circ gf)^*\mathcal{D}_{X,h,[\eta]} \end{array}$$

Construct α_f as follows. By definition of $\mathcal{M}_{X,[\eta]}$ the point f determines an isomorphism $S \times \mathrm{Spec}(A_0) \xrightarrow{\sim} S \times_{X'} X$ also denoted by f fitting into the following commutative diagram.

$$\begin{array}{ccccc} S\times\mathrm{Spec}(A_0) & \xrightarrow{\cong f} & S\times_{X'}X & \longrightarrow & X \\ & \searrow & \downarrow{\scriptstyle pr_S} & & \downarrow{\scriptstyle F} \\ & & S & \xrightarrow{\pi\circ f} & X' \end{array}$$

[8] Let us verify that ϕ_μ is an algebra automorphism. The fact that the formulas above define an automorphism of $\mathcal{D}_{\mathrm{Spec}\,A_0,h}\hat{\otimes}R$ is clear because μ is closed. To check that this automorphism descends to $\mathcal{D}_{\mathrm{Spec}\,A_0,[0],h}\hat{\otimes}R$ we need to show that the following identities hold in $\mathcal{D}_{\mathrm{Spec}\,A_0,[0],h}\hat{\otimes}R$:

$$(v_i + \iota_{\partial/\partial x_i}\mu)^p = (u_i + \iota_{\partial/\partial y_i}\mu)^p = 0.$$

Using the Katz formula [Kat72, § 7.22] and the exactness of μ we find that

$$(v_i + \iota_{\partial/\partial x_i}\mu)^p = v_i^p + (\iota_{\partial/\partial x_i}\mu)^p = (\iota_{\partial/\partial x_i}g^*\eta)^p - (\iota_{\partial/\partial x_i}\eta)^p = 0,$$

because η vanishes at the origin. The second relation is proven similarly.

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This induces an isomorphism of the corresponding algebras of differential operators:

$$\mathcal{O}_S \otimes D_{\mathrm{Spec}(A_0)} \cong D_{S\times \mathrm{Spec}(A_0)/S} \cong (pr_S)_* D_{S\times_{X'} X/S} \cong (\pi\circ f)^* F_* D_{X/X'} \cong (\pi\circ f)^* F_* D_X.$$

Applying the Artin–Rees construction we get

$$f_* : \mathcal{O}_S \otimes D_{A_0,h} \xrightarrow{\sim} (\pi\circ f)^* D_{X,h}.$$

First, we assume that the class $[\eta]$ is represented by a global 1-form η. Then the sheaf $\mathcal{D}_{X,h,[\eta]}$ is obtained from $D_{X,h}$ as the h-completion of the quotient $D_{X,h}/I_{\Gamma_\eta} D_{X,h}$ (see formula (1.6)). The algebra $A_h^\flat$ is the quotient $D_{A_0,h}/I_{\Gamma_{\eta_A}} D_{A_0,h}$. The desired isomorphism α_f is defined from the following commutative diagram.

$$\begin{array}{ccc} \mathcal{O}_S \otimes D_{A_0,h} & \xrightarrow{f_*\circ\phi_{\eta_A} - f^*\eta} & (\pi\circ f)^* D_{X,h} \\ \downarrow & & \downarrow \\ \mathcal{O}_S \hat{\otimes} A_h^\flat & \xrightarrow{\alpha_f} & \mathcal{D}_{X,h,[\eta]} \end{array}$$

One checks that α_f is independent of the choice of representative η for $[\eta]$. Therefore, covering X' by open subsets where $[\eta]$ is represented by a 1-form, we can patch α_f from local pieces. The compatibility with the action of G_0 is straightforward. □

3.2 Central extensions of G

Consider the action of the $k[[h]]$-algebra A_h on the free $k[[h]]$-module $k[x_1,\dots,x_n]/(x_1^p,\dots,x_n^p)[[h]]$ given by the formulas

$$x_i \mapsto \text{multiplication by } x_i, \quad y_i \mapsto h\frac{\partial}{\partial x_i}.$$

It is well known (see e.g. [BMR02, Lemma 2.2.1]) and easy to verify that this action defines an isomorphism of $k((h))$-algebras

$$A_h(h^{-1}) \xrightarrow{\sim} \mathrm{Mat}_{p^n}(k((h))). \tag{3.4}$$

For any k-algebra R, isomorphism (3.4) gives rise to a natural homomorphism

$$G(R) = \mathrm{Aut}_{R[[h]]}(A_h \hat{\otimes} R) \hookrightarrow \mathrm{Aut}_{R((h))}(\mathrm{Mat}_{p^n}(R((h)))) \xrightarrow{\sim} \mathrm{PGL}(p^n, R((h))).$$

This defines an embedding

$$G \hookrightarrow L\,\mathrm{PGL}(p^n), \tag{3.5}$$

where $L\,\mathrm{PGL}(p^n)$ is the loop group of $\mathrm{PGL}(p^n)$, that is a sheaf of groups on the category of affine schemes of over k equipped with the *fpqc* topology sending k-scheme $\mathrm{Spec}\, R$ to $\mathrm{PGL}(p^n, R((h)))$. The natural morphism of algebraic groups $\mathrm{GL}(p^n) \to \mathrm{PGL}(p^n)$ gives rise to a morphism of the loop groups $L\,\mathrm{GL}(p^n) \to L\,\mathrm{PGL}(p^n)$. By part (i) of Proposition A.5 in the pullback diagram of *fpqc* sheaves

$$\begin{array}{ccc} \widetilde{G} := G \times_{L\,\mathrm{PGL}(p^n)} L\,\mathrm{GL}(p^n) & \to & L\,\mathrm{GL}(p^n) \\ \downarrow & & \downarrow \\ G & \hookrightarrow & L\,\mathrm{PGL}(p^n) \end{array}$$

the left vertical arrow is surjective. Thus, we have a central extension of *fpqc* sheaves

$$1 \to L\mathbb{G}_m \to \widetilde{G} \to G \to 1. \tag{3.6}$$

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Recall from § 2.4 that subgroup $G^{\geq 1} \subset G$ of automorphisms identical modulo h consists of inner automorphisms. Therefore, extension (3.6) fits into a commutative diagram

$$\begin{array}{ccccccccc} 1 & \to & L^+\mathbb{G}_m & \to & \underline{A_h^*} & \to & G^{\geq 1} & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & L\mathbb{G}_m & \to & \widetilde{G} & \to & G & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & \mathrm{Gr}_{\mathbb{G}_m} & \to & \widetilde{G}_0 & \to & G_0 & \to & 1 \end{array}$$

where $\widetilde{G}_0 = \widetilde{G}/\underline{A_h^*}$ and $\mathrm{Gr}_{\mathbb{G}_m} = L\mathbb{G}_m/L^+\mathbb{G}_m$ is the affine grassmannian for $\mathbb{G}_m$.

Remark 3.2. Consider the central extension of the Lie algebras

$$0 \to \mathrm{Lie}\,\mathrm{Gr}_{\mathbb{G}_m} \to \mathrm{Lie}\,\widetilde{G}_0 \to \mathrm{Lie}\,G_0 \to 0$$

corresponding to the bottom line in the diagram above. Identify the Lie algebra of the affine grassmannian with the vector space $h^{-1}k[h^{-1}]$ of polynomial vanishing at the origin equipped with the trivial Lie bracket. It is shown in [BK08] that the Lie algebra of G_0 consists of Hamiltonian vector fields on A_0, that is, $\mathrm{Lie}\,G_0 = A_0/k$, where the Lie bracket is induced by the Poisson bracket on A_0. Then $\mathrm{Lie}\,\widetilde{G}_0$ is isomorphic to the direct sum of Lie algebras $A_0 \oplus h^{-2}k[h^{-1}]$ with map to $\mathrm{Lie}\,G_0$ given by the projection to the first summand followed by $A_0 \to A_0/k$.

Proof. It suffices to construct a morphism of extensions as follows.

$$\begin{array}{ccccccccc} 0 & \to & k & \to & A_0 & \to & A_0/k & \to & 0 \\ & & \downarrow{\scriptstyle h^{-1}} & & \downarrow & & \downarrow{\scriptstyle \mathrm{Id}} & & \\ 0 & \to & \mathrm{Lie}\,\mathrm{Gr}_{\mathbb{G}_m} & \to & \mathrm{Lie}\,\widetilde{G}_0 & \to & \mathrm{Lie}\,G_0 & \to & 0 \end{array}$$

Define $A_0 \to \mathrm{Lie}(\widetilde{G}_0)$ that coincides with the map $A_0/k \to \mathrm{Lie}(G_0)$ sending $a \in A_0$ to $\mathrm{Ad}_{1+\epsilon\tilde{a}/h} \in \widetilde{G}(k[\epsilon]/\epsilon^2) \to \widetilde{G}_0(k[\epsilon]/\epsilon^2)$, where $\tilde{a} \in A_h$ is any lifting of a. □

Applying the same construction to the algebra $A_h^\flat$ and to its representation on $k[x_1, \ldots, x_n, y_1, \ldots, y_n]/(x_1^p, \ldots, x_n^p, y_1^p, \ldots, y_n^p)[[h]]$ we construct the following commutative diagram.

$$\begin{array}{ccccccccc} 1 & \to & L^+\mathbb{G}_m & \to & \underline{A_h^{\flat *}} & \to & G^{\geq 1\flat} & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & L\mathbb{G}_m & \to & \widetilde{G}^\flat & \to & G^\flat & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & \mathrm{Gr}_{\mathbb{G}_m} & \to & \widetilde{G}_0^\flat & \to & G_0^\flat & \to & 1 \end{array}$$

Recall from (3.3) the homomorphism $\psi : G_0 \to G^\flat$. Denote by $\psi_0 : G_0 \to G_0^\flat$ its composition with the projection $G^\flat \to G_0^\flat$. The key step in the proof of our main theorem is the following result.

PROPOSITION 3.3. *There is a unique homomorphism $\tilde{\psi}_0$ making the following diagram commutative.*

$$\begin{array}{ccccccccc} 1 & \to & \mathrm{Gr}_{\mathbb{G}_m} & \to & \widetilde{G}_0 & \to & G_0 & \to & 1 \\ & & \downarrow{\scriptstyle \mathrm{Id}} & & \downarrow{\scriptstyle \tilde{\psi}_0} & & \downarrow{\scriptstyle \psi_0} & & \\ 1 & \to & \mathrm{Gr}_{\mathbb{G}_m} & \to & \widetilde{G}_0^\flat & \to & G_0^\flat & \to & 1 \end{array}$$

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Remark 3.4. Let us describe the morphism of the Lie algebras induced by $\tilde{\psi}_0$. Namely, define $d(\tilde{\psi}_0) : f \mapsto f + \eta(H_f) + \mathcal{H}_f$. Here H_f denotes the Hamiltonian vector field on A_0, while $\mathcal{H}_f$ is the same vector field viewed as a function on $A_0^\flat$.

We end this subsection with a reformulation of Proposition 3.3 that will be used in forthcoming paper [BKTV22]. Set $B_h = A_h \otimes_{k[[h]]} A_h^{\flat,\mathrm{op}}$. Let

$$G^\sharp \subset \underline{\mathrm{Aut}}(B_h) \to \underline{\mathrm{Aut}}(B_0)$$

be the preimage of $\Gamma_{\psi_0} : G_0 \hookrightarrow \underline{\mathrm{Aut}}(B_0) = \underline{\mathrm{Aut}}(A_0 \otimes A_0^\flat)$, $\Gamma_{\psi_0}(g) = g \otimes \psi_0(g)$. Proposition 3.3 implies that the canonical extension of $\underline{\mathrm{Aut}}(B_h)$ by $L\mathbb{G}_m$ (cf. (3.6)) restricted to $G^\sharp$ admits a unique reduction to $L^+\mathbb{G}_m$:

$$1 \to L^+\mathbb{G}_m \to \widehat{G}^\sharp \to G^\sharp \to 1. \tag{3.7}$$

Corollary 3.5. *There exists a unique (up to a unique isomorphism) triple $(\widehat{G}^\sharp, \alpha, i)$ displayed in the diagram*

$$\begin{array}{ccccccccc} & & & & \underline{B_h(h^{-1})}^* & & & & \\ & & & & \uparrow{\scriptstyle i} & & & & \\ 1 & \longrightarrow & \underline{B_h^*} & \longrightarrow & \widehat{G}^\sharp & \longrightarrow & G_0 & \longrightarrow & 1 \\ & & & {\scriptstyle Ad}\searrow & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \Gamma_{\psi_0}} & & \\ & & & & \underline{\mathrm{Aut}}(B_h) & \longrightarrow & \underline{\mathrm{Aut}}(B_0) & & \end{array} \tag{3.8}$$

where the north east arrow is the natural inclusion, i is a monomorphism and $\alpha(g) = \mathrm{Ad}_{i(g)}$. In addition, if W is an irreducible representation of $B_h(h^{-1})$, $B_h(h^{-1}) \xrightarrow{\sim} \mathrm{End}_{k((h))}(W)$, there exists a $k[[h]]$-lattice $\Lambda \subset W$, invariant under the B_h-action on W and under the action of $\widehat{G}^\sharp$:

$$i : \widehat{G}^\sharp \hookrightarrow L^+\,\mathrm{GL}(\Lambda) \subset L\,\mathrm{GL}(W) = \underline{B_h(h^{-1})}^*.$$

Proof. The diagram is merely a rearrangement of (3.7). The existence of a lattice Λ stable under $\widehat{G}^\sharp$ follows Proposition A.2. Since $\underline{B_h^*} \subset \widehat{G}^\sharp$ the lattice Λ is B_h-invariant. □

3.3 Proposition 3.3 implies the main theorem

In this subsection we prove Theorem 1 assuming Proposition 3.3.

For the first part, let us start by reinterpreting the construction of the algebra $\mathcal{O}_h \otimes_{\mathcal{O}_{X'}[[h]]} \mathcal{D}^{\mathrm{op}}_{X,[\eta],h}$. Consider the homomorphism

$$G \to \mathrm{Aut}_{k[[h]]}(A_h) \times \mathrm{Aut}_{k[[h]]}(A_h^{\flat,\mathrm{op}}) \hookrightarrow \mathrm{Aut}_{k[[h]]}(A_h \otimes_{k[[h]]} A_h^{\flat,\mathrm{op}}) \tag{3.9}$$

whose first component is the identity map and whose second component is the composition $G \longrightarrow G_0 \xrightarrow{\psi} \mathrm{Aut}_{k[[h]]}(A_h^\flat) = \mathrm{Aut}_{k[[h]]}(A_h^{\flat,\mathrm{op}})$.[9] Homomorphism (3.9) defines a sheaf of $\mathcal{O}_{X'}[[h]]$-algebras $\mathcal{M}_{X,\mathcal{O}_h,s} \times_G (A_h \otimes_{k[[h]]} A_h^{\flat,\mathrm{op}})$. By Lemma 3.1, we have an isomorphism

$$\mathcal{M}_{X,\mathcal{O}_h,s} \times_G (A_h \otimes_{k[[h]]} A_h^{\flat,\mathrm{op}}) \xrightarrow{\sim} \mathcal{O}_h \otimes_{\mathcal{O}_{X'}[[h]]} \mathcal{D}^{\mathrm{op}}_{X,[\eta],h}.$$

Next, the $k((h))$-algebra $(A_h \otimes_{k[[h]]} A_h^{\flat,\mathrm{op}})(h^{-1})$ is isomorphic to the matrix algebra $\mathrm{End}_{k((h))}(V)$, for some vector space over $k((h))$ of dimension p^{3n}:

$$(A_h \otimes_{k[[h]]} A_h^{\flat,\mathrm{op}})(h^{-1}) \xrightarrow{\sim} \mathrm{End}_{k((h))}(V). \tag{3.10}$$

[9] Note that $\mathrm{Aut}_{k[[h]]}(A_h^{\flat,\mathrm{op}})$ is equal, as a subgroup of the group of automorphisms of the $k[[h]]$-module $A_h^\flat$, to $\mathrm{Aut}_{k[[h]]}(A_h^\flat)$.

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Isomorphisms (3.10) and (3.9) give rise to a homomorphism

$$G \to L\,\mathrm{PGL}(p^{3n}) \tag{3.11}$$

and, consequently, to an extension of G by $L\mathbb{G}_m$. Proposition 3.3 asserts that this extension admits a unique reduction to $L^+\mathbb{G}_m$.

$$1 \to L^+\mathbb{G}_m \to \widehat{G} \to G \to 1. \tag{3.12}$$

Thus, by part (ii) of Proposition A.5, it follows that homomorphism (3.11), possibly after conjugation by an element of $\mathrm{PGL}(p^{3n}, k((h)))$, factors through $L^+\,\mathrm{PGL}(p^{3n}) \subset L\,\mathrm{PGL}(p^{3n})$. In the other words, there exists a $k[[h]]$-lattice $\Lambda \subset V$ such that the action of G on $(A_h \otimes_{k[[h]]} A_h^{\flat,\mathrm{op}})(h^{-1})$ preserves $\mathrm{End}_{k[[h]]}(\Lambda)$:

$$A_h \otimes_{k[[h]]} A_h^{\flat,\mathrm{op}} \subset (A_h \otimes_{k[[h]]} A_h^{\flat,\mathrm{op}})(h^{-1}) \xrightarrow{\sim} \mathrm{End}_{k((h))}(V) \supset \mathrm{End}_{k[[h]]}(\Lambda). \tag{3.13}$$

The homomorphism

$$G \to \mathrm{Aut}(\mathrm{End}_{k[[h]]}(\Lambda)) \xrightarrow{\sim} L^+\,\mathrm{PGL}(p^{3n})$$

and the G-torsor $\mathcal{M}_{X,\mathcal{O}_h,s}$ give rise to an Azumaya algebra

$$\mathcal{O}_h^\sharp := \mathcal{M}_{X,\mathcal{O}_h,s} \times_G \mathrm{End}_{k[[h]]}(\Lambda),$$

which, by construction, coincides with $\mathcal{O}_h \otimes_{\mathcal{O}_{X'}[[h]]} \mathcal{D}^{\mathrm{op}}_{X,[\eta],h}$ after inverting h. This proves part (i) of the Theorem.

To prove part (ii) of the Theorem, recall from § 2.4 that G is acted upon by an involution $\tau : G \to G$. We claim that τ lifts to extension (3.12),

$$\hat{\tau} : \widehat{G} \to \widehat{G}, \quad \hat{\tau}^2 = \mathrm{Id},$$

such that the restriction of $\hat{\tau}$ to $L^+\mathbb{G}_m$ is given by the formula

$$\hat{\tau}(f(h)) = f(-h)^{-1}, \quad f(h) \in R[[h]]^*. \tag{3.14}$$

Consider the homomorphism $L^+\mathbb{G}_m \to \mathbb{G}_m$ sending $f(h) \in L^+\mathbb{G}_m(R) = R[[h]]^*$ to $f(0) \in R^*$. This fits into the following diagram of group scheme extensions.

$$\begin{array}{ccccccccc} 1 & \to & L^+\mathbb{G}_m & \to & \widehat{G} & \to & G & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & \mathbb{G}_m & \to & \widehat{G/G^{\geq 2}} & \to & G/G^{\geq 2} & \to & 1 \end{array}$$

The action of $\hat{\tau}$ on $\widehat{G}$ descends to $\widehat{G/G^{\geq 2}}$. For the $\mathbb{Z}/2\mathbb{Z}$-action on R^* given by formula $c \mapsto c^{-1}$, we have that

$$H^1(\mathbb{Z}/2\mathbb{Z}, R^*) \xrightarrow{\sim} R^*/R^{*2}. \tag{3.15}$$

In particular, every cohomology class gets killed after a finite étale extension of R. It follows that the sequence of $\mathbb{Z}/2\mathbb{Z}$-invariants

$$1 \to \mathbb{G}_m^{\hat{\tau}=1} \to (\widehat{G/G^{\geq 2}})^{\hat{\tau}=1} \to (G/G^{\geq 2})^{\tau=1} \to 1 \tag{3.16}$$

is exact. Note that $\mathbb{G}_m^{\hat{\tau}=1} = \mu_2 = \{1, -1\}$. We claim that (3.16) is a split extension:

$$(\widehat{G/G^{\geq 2}})^{\hat{\tau}=1} \xrightarrow{\sim} (G/G^{\geq 2})^{\tau=1} \times \mu_2. \tag{3.17}$$

Indeed, the determinant homomorphism

$$\widehat{G} \hookrightarrow L^+\,\mathrm{GL}(p^{3n}) \xrightarrow{\det} L^+\mathbb{G}_m \tag{3.18}$$

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composed with the map $L^+G_m \to \mathbb{G}_m$ factors through $\widehat{G/G^{\geq 2}}$ and commutes with τ. Hence, it defines a homomorphism

$$(\widehat{G/G^{\geq 2}})^{\hat{\tau}=1} \to \mu_2 \tag{3.19}$$

whose restriction to μ_2 is the identity. This gives a splitting of extension (3.16). We derive from (3.17) that the extension $\widehat{G/G^{\geq 2}}$ has the form

$$1 \to \mathbb{G}_m \to G_0 \ltimes \underline{A}_0^* \to G_0 \ltimes \underline{A}_0^*/\mathbb{G}_m \to 1. \tag{3.20}$$

Now we can prove that

$$i^*[\mathcal{O}_h^\sharp] = i^*(\delta(\gamma)). \tag{3.21}$$

To see this, consider the gerbe $\mathcal{S}$ of splittings of the Azumaya algebra $\mathcal{O}_h^\sharp$. By definition, this is a sheaf of groupoids on $(X')_{fl}$ whose sections $\mathcal{S}(Z)$ over $Z \to X'$ is the groupoid of splittings of the pullback of $\mathcal{O}_h^\sharp$ to Z. This is a gerbe naturally banded by the sheaf $L^+\mathbb{G}_m$ meaning that the automorphism group of any object of $\mathcal{S}(Z)$ is canonically identified with $L^+\mathbb{G}_m(Z)$. By construction of $\mathcal{O}_h^\sharp$ and the uniqueness statement in Proposition 3.3 this gerbe is equivalent to the gerbe of liftings of G-torsor $\mathcal{M}_{X,\mathcal{O}_h,s}$ to a $\widehat{G}$-torsor. It follows that the gerbe of splittings $\overline{\mathcal{S}}$ of the Azumaya algebra $i^*\mathcal{O}_h^\sharp$ is equivalent to the $\mathbb{G}_m$-gerbe of liftings of $G/G^{\geq 2}$-torsor $\mathcal{L} := \mathcal{M}_{X,\mathcal{O}_h,s} \times_G G/G^{\geq 2}$ to a $\widehat{G/G^{\geq 2}}$-torsor. The set of isomorphism classes of torsors over $G/G^{\geq 2} \xrightarrow{\sim} G_0 \ltimes \underline{A}_0^*/\mathbb{G}_m$ lifting a given G_0-torsor $\mathcal{M}_{X,[\eta]}$ is in bijection ρ with the set $H^1_{\text{et}}(X', \mathcal{O}_{X'}^*/\mathcal{O}_{X'}^{p^*})$ of isomorphism classes of torsors over the group scheme $\underline{A}_0^*/\mathbb{G}_m \times_{G_0} \mathcal{M}_{X,[\eta]}$. It follows from (3.20) that given a $G/G^{\geq 2}$-torsor $\mathcal{L}$ the $\mathbb{G}_m$-gerbe of liftings of $\mathcal{L}$ to a torsor over $\widehat{G/G^{\geq 2}}$ is equivalent to the gerbe of liftings of $\underline{A}_0^*/\mathbb{G}_m \times_{G_0} \mathcal{M}_{X,[\eta]}$-torsor $\rho(\mathcal{L})$ to a torsor over $\underline{A}_0^* \times_{G_0} \mathcal{M}_{X,[\eta]}$. This proves (3.21).

To prove the last assertion of Theorem 1 observe that the kernel of the restriction $i^*\colon \mathrm{Br}(X'[[h]]) \to \mathrm{Br}(X')$ is a subgroup of the group $H^2_{\mathrm{Zar}}(X', \mathbb{W}(\mathcal{O}_{X'}))$, where $\mathbb{W}(\mathcal{O}_{X'})$ is the additive group of the ring of big Witt vectors, that is,

$$\mathbb{W}(\mathcal{O}_{X'}) = (1 + h\mathcal{O}_{X'}[[h]])^*.$$

We claim that vanishing of $H^2(X', \mathcal{O}_{X'})$ implies vanishing of $H^2(X', \mathbb{W}(\mathcal{O}_{X'}))$. Indeed, $\mathbb{W}(\mathcal{O}_{X'})$ is the inverse limit of the groups of truncated Witt vectors

$$\mathbb{W}(\mathcal{O}_{X'}) \cong \lim_{\leftarrow} \mathbb{W}_m(\mathcal{O}_{X'}).$$

Using the exact sequence

$$0 \to \mathbb{W}_l(\mathcal{O}_{X'}) \xrightarrow{V^m} \mathbb{W}_{l+m}(\mathcal{O}_{X'}) \to \mathbb{W}_m(\mathcal{O}_{X'}) \to 0$$

it follows that, for every positive integer m, the group $H^2(X, \mathbb{W}_m(\mathcal{O}_{X'}))$ is trivial and consequently the restriction homomorphism

$$H^1(X, \mathbb{W}_{l+m}(\mathcal{O}_{X'})) \to H^1(X, \mathbb{W}_m(\mathcal{O}_{X'}))$$

is surjective, for every l and m. Hence, by Proposition 13.3.1 from [EGA III, Chapter 0], we have

$$H^2(X', \mathbb{W}(\mathcal{O}_{X'})) \xrightarrow{\sim} \lim_{\leftarrow} H^2(X', \mathbb{W}_m(\mathcal{O}_{X'})) = 0$$

as desired.

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Remark 3.6. Observe that under the assumptions of Theorem 1, we have that

$$p^{3n}([\mathcal{O}_h^\sharp] - \delta(\gamma)) = 0.$$

Indeed, $\mathcal{O}_h^\sharp$ is an Azumaya algebra of rank p^{6n} and, hence, its class in the Brauer group is killed by p^{3n}. On the other hand, the class $\delta(\gamma))$ is killed by p.

Remark 3.7. The proof of Theorem 1 shows that vanishing of $H^2(X', \mathcal{O}_{X'})$ implies surjectivity of the map ρ (see formula (1.4)), which does not follow directly from the Bezrukavnikov–Kaledin theorem. Indeed, from (3.12) we derive an extension

$$1 \to \widehat{G^{\geq 2}} \to \widehat{G}/\mathbb{G}_m \to G_0 \ltimes \underline{A}_0^*/\mathbb{G}_m \to 1.$$

Consider the $G_0 \ltimes \underline{A}_0^*/\mathbb{G}_m$-torsor $\mathcal{M}$ corresponding to a restricted structure $[\eta]$ and a class $\gamma \in H^1_{\text{et}}(X', \mathcal{O}^*_{X'}/\mathcal{O}^{*p}_{X'})$. Using that $H^2(X', \mathcal{O}_{X'}) = 0$ we infer that $\mathcal{M}$ can be lifted to a $\widehat{G}/\mathbb{G}_m$-torsor $\tilde{\mathcal{M}}$. Pushing forward the latter under the homomorphism $\widehat{G}/\mathbb{G}_m \to G$ we get a quantization with ρ-invariant γ.

Remark 3.8. In [BK08, Proposition 1.24] the authors erroneously assert the subgroup $G \hookrightarrow L\,\mathrm{PGL}(p^n)$ from (3.5) preserves a lattice, that is, possibly after conjugation by an element of $\mathrm{PGL}(p^n, k((h)))$, factors through $L^+\,\mathrm{PGL}(p^n) \subset L\,\mathrm{PGL}(p^n)$. This claim led the authors to a mistake in the statement of Proposition 1.24. In fact, even the subgroup of translations $\operatorname{Spec} A_0 = \alpha_p^{2n} \subset G$ does not admit an invariant lattice. This follows from the fact the commutator map

$$\operatorname{Lie} \alpha_p^{2n} \otimes \operatorname{Lie} \alpha_p^{2n} \to \operatorname{Lie} L\mathbb{G}_m = k((h))$$

arising from extension (3.6) is given by the formula $(1/h)\sum_i dy_i \wedge dx_i$, i.e. does not factor through $\operatorname{Lie} L^+\mathbb{G}_m = k[[h]]$.

4. Central extensions of the group of Poisson automorphisms

In this section we prove Basic Lemma 4 and derive from it Proposition 3.3. For the duration of this section we fix a symplectic vector space (V, ω_V) of dimension $2n$ and denote by A_h the corresponding restricted Weyl algebra, G its group of automorphisms and G_0 the quotient of G by the subgroup of automorphisms identical modulo h viewed as a group scheme of automorphisms of A_0 preserving the class $[\eta_V]$.

4.1 Properties of G_0

Recall from [BK08, Proposition 3.4] that the reduced subgroup $G_0^0 = (G_0)_{\text{red}} \subset G_0$ is equal to the stabilizer of the point $\operatorname{Spec} k \hookrightarrow \operatorname{Spec} A_0$: for every k-algebra R, $G_0^0(R)$ is the subgroup of $G_0(R)$ that consists of R-linear automorphisms of $A_0 \otimes R$ that preserve the kernel of the homomorphism $A_0 \otimes R \to R$ induced by $A_0 \to k$. According to [BK08, Lemma 3.3] the Lie algebra of G_0 (respectively, G_0^0) is the algebra of all Hamiltonian vector fields[10] on $\operatorname{Spec} A_0$ (respectively, the algebra of all Hamiltonian vector fields vanishing at $\operatorname{Spec} k \hookrightarrow \operatorname{Spec} A_0$). In particular, we have

$$\dim G_0 = \dim G_0^0 = \dim_k \operatorname{Lie} G_0^0 = \dim_k m^2 = p^{2n} - 2n - 1, \tag{4.1}$$

where m is the maximal ideal in A_0.

[10] Recall that a vector field is said to be Hamiltonian if it has the form H_f, for some $f \in A_0$.

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Denote by α_p the Frobenius kernel on $\mathbb{G}_a$. The finite group scheme $\alpha := \alpha_p^{2n}$ acts on $\operatorname{Spec} A_0 = \alpha$ by translations inducing the inclusion $\alpha \hookrightarrow G_0$. Observe that the product morphism

$$\alpha \times G_0^0 \to G_0$$

induces an isomorphism of the underlying schemes.

Lemma 4.1. *The group schemes G_0 and G_0^0 are connected. Moreover, there is a surjective homomorphism*

$$G_0^0 \twoheadrightarrow \mathrm{Sp}(2n)$$

whose kernel is a unipotent algebraic group.

Proof. It suffices to prove the assertions for the reduced group G_0^0. To show that G_0^0 is connected we consider the filtration

$$\cdots \subset F^2G_0^0 \subset F^1G_0^0 \subset G_0^0$$

by normal group subschemes of G_0^0 and prove that all the associated quotients are connected. Namely, for any k-algebra R, we set

$$F^iG_0^0(R) = \{\phi \in G_0^0(R) \,|\, \phi = \mathrm{Id} \mod m^{i+1} \otimes R\}.$$

It is easy to see that this functor is representable by a normal group subscheme of G_0^0.

The action of G_0^0 on the tangent space $(m/m^2)^*$ preserves ω. Thus, it gives rise to a monomorphism

$$G_0^0/F^1G_0^0 \to \mathrm{Sp}(2n),$$

which is, in fact, an isomorphism because it has a section. In particular, we have that

$$\dim G_0^0/F^1G_0^0 = \dim \mathrm{Sp}(2n) = \dim_k m^2/m^3. \tag{4.2}$$

To check that the other quotients are connected we construct injective homomorphisms

$$\alpha_i : F^iG_0^0/F^{i+1}G_0^0 \hookrightarrow \underline{m^{i+2}/m^{i+3}}, \quad i \geq 1, \tag{4.3}$$

where $\underline{m^{i+2}/m^{i+3}}$ is the vector group associated to the space m^{i+2}/m^{i+3} and then using (4.1) conclude that α_i are isomorphisms. For the sake of brevity we only define α_i on k-points. Take $\phi \in F^iG_0^0(k)$ and consider $\phi^* : A_0 \to A_0$. By definition, $\phi^* = \mathrm{Id} \bmod m^{i+1}$, so $\phi^* - \mathrm{Id}$ maps m^r to m^{i+r}, for every $r \geq 0$. Hence $\phi^* - \mathrm{Id}$ defines a homogeneous degree i map $\theta_\phi : \oplus m^r/m^{r+1} = A \to A$ which is, in fact, a derivation. Let us show that θ_ϕ lies in a Lie algebra of G_0^0, i.e. $L_{\theta_\phi}\eta$ is exact. Indeed, since $\phi \in G_0^0$ we have $\phi^*\eta = \eta + dK$ for some $K \in A_0$. But then $L_{\theta_\phi}\eta = dK_{i+2}$, where K_{i+2} is the homogeneous component of K of degree $i+2$. It follows that θ_ϕ is Hamiltonian: $\iota_{\theta_\phi}\omega = \iota_{\theta_\phi}d\eta = d(K_{i+2} - \iota_{\theta_\phi}\eta)$. Set

$$\alpha_i(\phi) = K_{i+2} - \iota_{\theta_\phi}\eta \in m^{i+2}/m^{i+3}.$$

Using the identity $\theta_{\phi\circ\psi} = \theta_\phi + \theta_\psi$, for every $\phi, \psi \in F^iG_0^0(k)$, one checks that α is a group homomorphism and that it factors through $F^iG_0^0/F^{i+1}G_0^0$. For the injectivity of (4.3) observe that $\theta_\phi = 0$ if and only if $\phi \in F^{i+1}G_0^0(k)$.

From (4.3) we have that, for every $i \geq 1$,

$$\dim F^iG_0^0/F^{i+1}G_0^0 \leq \dim_k m^{i+2}/m^{i+3}.$$

If for some i the inequality is strict, then using (4.2) we would have that $\dim G_0^0 < \dim_k m^2$ contradicting to (4.1). It follows that all α_i are isomorphisms as desired. □

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Recall from Lemma A.1[11] a decomposition

$$\mathrm{Gr}_{\mathbb{G}_m} \xrightarrow{\sim} \underline{\mathbb{Z}} \times \hat{\mathbb{W}}.$$

COROLLARY 4.2. *The extension (1.15) admits a unique reduction to* $\hat{\mathbb{W}} \subset \mathrm{Gr}_{\mathbb{G}_m}$. *Notation:*

$$1 \to \hat{\mathbb{W}} \to \widetilde{G}_0^e \to G_0 \to 1.$$

Proof. By Lemma 4.1 we have that $Hom(G_0, \underline{\mathbb{Z}}) = 0$. The uniqueness part follows. For the existence, note that the composition

$$\widetilde{G} \hookrightarrow L\,\mathrm{GL}(p^n) \xrightarrow{\det} L\mathbb{G}_m \twoheadrightarrow \underline{\mathbb{Z}}$$

factors through $\widetilde{G}_0$. We claim that setting $\widetilde{G}_0^e := \ker(\tilde{G}_0 \to \underline{\mathbb{Z}})$ does the job. Indeed, the only assertion that requires a proof is the surjectivity of the projection $\widetilde{G}_0^e \to G_0$. By construction, $\widetilde{G}_0^e$ projects onto the kernel of the homomorphism $G_0 \to \underline{\mathbb{Z}}/p^n$ induced by $\widetilde{G}_0 \to \underline{\mathbb{Z}}$. But by Lemma 4.1 every such homomorphism is trivial. □

Consider $m^2 \subset A_0$ as a Lie subalgebra of A_0 equipped with Poisson bracket. Recall that

$$m^2 \xrightarrow{f \mapsto H_f} \mathrm{Lie}\, G_0^0 \tag{4.4}$$

is an isomorphism of Lie algebras. The grading on A_0 induces a grading on the Lie algebra m^2:

$$m^2 \cong \bigoplus_{2 \le i \le 2n(p-1)} m^i/m^{i+1}$$

such that the Lie bracket has degree -2.

LEMMA 4.3. *Isomorphism (4.4) induces*

$$[\mathrm{Lie}\, G_0^0, \mathrm{Lie}\, G_0^0] \cong \bigoplus_{2 \le i < 2n(p-1)} m^i/m^{i+1}.$$

In particular, $[\mathrm{Lie}\, G_0^0, \mathrm{Lie}\, G_0^0]$ *has codimension* 1 *in* $\mathrm{Lie}\, G_0^0$. *Moreover,* $\mathfrak{sp}(2n) = m^2/m^3$ *together with any nonzero element* $z \in m^3/m^4$ *generate the Lie algebra* $[\mathrm{Lie}\, G_0^0, \mathrm{Lie}\, G_0^0]$.

Proof. By a direct computation the Poisson bracket of any two monomials of total degree $2n(p-1)+2$ is 0 i.e. $[\mathrm{Lie}\, G_0^0, \mathrm{Lie}\, G_0^0]$ does not contain nonzero homogeneous elements of degree $2n(p-1)$. Hence, $[\mathrm{Lie}\, G_0^0, \mathrm{Lie}\, G_0^0] \subset \bigoplus_{2 \le i < 2n(p-1)} m^i/m^{i+1}$. Since $[\mathfrak{sp}(2n), \mathfrak{sp}(2n)] = \mathfrak{sp}(2n)$, we have that $m^2/m^3 \subset [\mathrm{Lie}\, G_0^0, \mathrm{Lie}\, G_0^0]$. Also it is clear that $[\mathrm{Lie}\, G_0^0, \mathrm{Lie}\, G_0^0]$ contains at least one nonzero element of degree 3 (e.g. $\{x_1^2, x_1 y_1^2\} = 2x_1^2 y_1$). To complete the proof of the lemma it suffices to verify that the Lie subalgebra $\mathfrak{g}$ generated by m^2/m^3 and a nonzero element of degree 3 coincides with $\bigoplus_{2 \le i < 2n(p-1)} m^i/m^{i+1}$. We check by induction on d that $m^d/m^{d+1} \subset \mathfrak{g}$ provided that $2 \le d < 2n(p-1)$. The base of induction, $d = 3$, can be easily checked directly follows from Lemma A.6.

Choose a symplectic basis (x_i, y_j) for V^* and let $E = x_1^{a_1} y_1^{b_1} \dots x_n^{a_n} y_n^{b_n} \in m^d/m^{d+1}$ with $d > 3$. Note that:

- $3a_i x_1^{a_i} y_i^{b_i} = \{x_i^{a_i+1} y_1^{b_i-2}, y_i^3\}$ (and $-3b_i x_1^{a_i} y_1^{b_i} = \{x_i^{a_i-2} y_1^{b_i+1}, y_1^3\}$);
- $(a-1-2b) x_i^a y_i^b = \{x_i^{a-1} y_i^b, x_i^2 y_i\}$ (and $(2a-b+1) x_i^a y_i^b = \{x_i^a y_i^{b-1}, x_i y_i^2\}$);
- $-x_i x_j = \{x_i y_i, x_i x_j\}$;
- $x_i^{p-1} y_i^{p-1} + 2(p-1) x_i^{p-2} y_i^{p-2} x_j y_j = \{x_i^{p-1} y_i^{p-3} x_j, y_i^2 y_j\}$;
- $2(p-1) x_i^{p-2} y_i^{p-2} x_j^2 y_j + 2 x_i^{p-1} y_i^{p-1} x_j = \{x_i^{p-1} y_i^{p-3} x_j^2, y_i^2 y_j\}$.

[11] We remark that all the results of the Appendix, in particular, Lemma A.1, do not depend on anything from the main body of the paper.

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Assume first that $p > 3$. Then if for some i we have $a_i < p-1$ and $b_i \geq 2$ (or $a_i \geq 2$ and $b_i < p-1$), then by the first formula above E is generated by elements of degree $2 \leq d' < d$, which are in $\mathfrak{g}$ by the induction assumption. Otherwise, for all i the pair (a_i, b_i) equals $(p-1, p-1)$, $(1,1)$, $(1,0)$, or $(0,1)$.

If for some i the pair is $(1,1)$ we get that from the second formula that $E \in \mathfrak{g}$. If there are at least two pairs of the type $(1,0)$ or $(0,1)$ we are done by the third formula.

Otherwise we may assume that $E = x_1^{p-1}y_1^{p-1}\dots x_{n-r}^{p-1}y_{n-r}^{p-1}$ or $E = x_1^{p-1}y_1^{p-1}\dots x_{n-r}^{p-1}y_{n-r}^{p-1}x_{n-r+1}$ for some $0 < r < n$. In these cases we are done by the last two formulas.

Now assume $p = 3$. For $d > 3$ note that if for some i the number $a_i + b_i - 1$ is not divisible by 3 we are done using the second formula. Thus, assume that for all i the pair (a_i, b_i) equals either $(p-1, p-1)$, $(1,1)$, $(1,0)$ or $(0,1)$. In these cases we proceed as above. □

Lemma 4.4. *We have the following commutative diagram.*

$$\begin{array}{ccc} G_0^0/[G_0^0, G_0^0] & \xrightarrow{\cong} & \mathbb{G}_a \\ \downarrow \cong & \nearrow & \\ G_0/[G_0, G_0] & & \end{array}$$

Moreover, the projection $G_0 \to G_0/[G_0, G_0]$ admits a section yielding to a decomposition $G_0 \cong [G_0, G_0] \rtimes \mathbb{G}_a$. Lastly, we have that

$$\mathrm{Lie}[G_0^0, G_0^0] = [\mathrm{Lie}\, G_0^0, \mathrm{Lie}\, G_0^0]. \tag{4.5}$$

Proof. Let us construct a group scheme homomorphism ϕ from G_0 to $\mathbb{G}_a$. For a k-algebra R and $g \in G_0(R)$, we have that $g(\eta) = \eta + df \in \Omega^1_{A_0 \otimes R/R}$, for some $f \in \mathrm{coker}(R \to A_0 \otimes R)$. Consider the element

$$\phi(g) = [f \cdot \omega^n] \in H^{2n}_{\mathrm{DR}}(A_0 \otimes R/R) \xrightarrow{\sim} R,$$

where the isomorphism above is induced by $k \xrightarrow{\sim} H^{2n}_{\mathrm{DR}}(A_0)$ that takes $1 \in k$ to the inverse Cartier operator applied to ω^n.

To show that ϕ is a homomorphism consider two elements $g_1, g_2 \in G_0(R)$. Write $g_1(\eta) = \eta + df_1, g_2(\eta) = \eta + df_2$. Since $g_2 \circ g_1(\eta) = \eta + df_2 + d(g_2(f_1))$ the image of $g_2 \circ g_1$ equals $[(f_2 + g_2(f_1)) \cdot (\omega)^n]$. On the other hand, $\phi(g_1) + \phi(g_2) = [(f_2 + f_1) \cdot (\omega)^n]$. Thus, it suffices to prove that G_0 acts trivially on $H^{2n}_{\mathrm{DR}}(A_0)$. We claim that, in fact, every 1-dimensional representation of G_0 is trivial. Indeed, G_0 is generated by two subgroups $\alpha = \alpha_p^{2n}$ and G_0^0. Since α has no nontrivial homomorphisms to $\mathbb{G}_m$ it suffices to prove the assertion for G_0^0. By Lemma 4.1 G_0^0 is an extension of $\mathrm{Sp}(2n)$ by a unipotent group and neither of the two groups has nontrivial 1-dimensional representations. This proves that ϕ is homomorphism.

The restriction of ϕ to G_0^0 yields a homomorphism

$$G_0^0/[G_0^0, G_0^0] \to \mathbb{G}_a. \tag{4.6}$$

Next, we shall construct a homomorphism $s : \mathbb{G}_a = \mathrm{Spec}\, k[t] \to G_0^0$ whose composition with the projection $G_0^0 \to G_0^0/[G_0^0, G_0^0]$ followed by (4.6) is Id. Set $u = \prod x_i^{p-1} \prod y_i^{p-1} \in A_0$. Define $s(t) \in Aut_{k[t]}(A_0[t])$ sending $f \in A_0$ to $f - (t/2)\{f, u\}$. One verifies directly that s is group homomorphism and a section of (4.6). Let us check that (4.6) is an isomorphism. First, from Lemma 4.3 we know that $[\mathrm{Lie}\, G_0^0, \mathrm{Lie}\, G_0^0]$ has codimension 1 in $\mathrm{Lie}\, G_0^0$. Second, since $G_0^0 = (G_0)_{\mathrm{red}}$ is smooth, both groups $[G_0^0, G_0^0]$ and $G_0^0/[G_0^0, G_0^0]$ are also smooth. Moreover, we have that $[\mathrm{Lie}\, G_0^0, \mathrm{Lie}\, G_0^0] \subset \mathrm{Lie}[G_0^0, G_0^0]$ (see e.g. [Bor91, Proposition 3.17]). It follows that the dimension of $G_0^0/[G_0^0, G_0^0]$ is at most 1. Thus, (4.6) is a homomorphism from a smooth connected algebraic

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group of dimension ≤ 1 to $\mathbb{G}_a$ and as we have already seen this homomorphism admits a section. It follows that (4.6) is an isomorphism. This also proves formula (4.5).

To complete the proof of lemma it suffices to check that the homomorphism $G_0^0/[G_0^0, G_0^0] \to G_0/[G_0, G_0]$ is surjective. Since G_0 is generated by α and G_0^0 it is enough to show that $\alpha \in [G_0, G_0]$. Consider the subgroup $P = \alpha \rtimes \mathrm{Sp}(2n) \subset G_0$. We claim that $[P, P] = P$. Indeed, there is a surjection $[P, P] \twoheadrightarrow [\mathrm{Sp}(2n), \mathrm{Sp}(2n)] = \mathrm{Sp}(2n)$. The kernel of the surjection is a subgroup of α whose Lie algebra is a $\mathrm{Sp}(2n)$-invariant subspace of $\operatorname{Lie}\alpha$. It follows that the kernel is either trivial which clearly not the case or equal to α as desired. □

Next, we shall show that $[G_0, G_0]$ is generated by $\mathrm{Sp}(2n)$, α, and a certain one-parameter subgroup $\mathbb{G}_a \subset G_0^0$. We start with the following observation.

LEMMA 4.5. *Let $f \in A_h$ be an element of the restricted Weyl algebra such that $f^{(p+1)/2} = 0$. Consider the homomorphism*

$$\tilde{\lambda}_f : \mathbb{G}_a = \operatorname{Spec} k[\tau] \to \underline{A_h(h^{-1})}^*$$

given by the formula

$$e^{\tau f/h} = \sum_{i=0}^{(p-1)/2} \frac{(\tau f)^i}{h^i i!}.$$

Then the pth power of the operator $\mathrm{ad}_{\tau f/h} : A_h \hat{\otimes} k[\tau] \to A_h \hat{\otimes} k[\tau]$ is zero and

$$\mathrm{Ad}_{e^{\tau f/h}} = \sum_{i=0}^{p-1} \frac{\mathrm{ad}_{\tau f}^i}{h^i i!} =: e^{\mathrm{ad}_{\tau f/h}}. \tag{4.7}$$

In particular, $e^{\tau f/h}$ normalizes the lattice $\underline{A_h} \subset \underline{A_h(h^{-1})}$ and, thus, defines a homomorphism

$$\tilde{\lambda}_f : \mathbb{G}_a \to \tilde{G}. \tag{4.8}$$

Proof. The only assertion that requires a proof is formula (4.7). Both sides of the equation can be thought as homomorphisms from $\mathbb{G}_a$ to the loop group of $k((h))$-linear automorphisms of $A_h(h^{-1})$. One readily sees that the differentials of these homomorphisms at $\tau = 0$ are equal. It follows that the homomorphisms are equal on the subscheme $\alpha_p \subset \mathbb{G}_a$. Also, by the assumption on f, both homomorphisms are given by matrices in $\mathrm{End}_{k[\tau]((h))}(A_h \hat{\otimes} k[\tau](h^{-1}))$ whose entries are polynomials in τ of degree less than p. Therefore, the homomorphisms are equal on $\mathbb{G}_a$. □

Let (x_i, y_j) be a symplectic basis for V^*. For $p > 3$, define a homomorphism

$$\lambda : \mathbb{G}_a = \operatorname{Spec} k[\tau] \hookrightarrow G_0^0 \tag{4.9}$$

by the equations

$$\lambda(\tau, x_i) = x_i, \quad \text{for all } i$$

$$\lambda(\tau, y_1) = y_1 + 3\tau x_1^2, \quad \lambda(\tau, y_i) = y_i, \quad \text{for all } i \neq 1.$$

The differential of λ is the Hamiltonian vector field $H_{-x_1^3}$. The construction from Lemma 4.5 gives a lifting $\tilde{\lambda}_{x_1^3} = e^{\tau x_1^3/h} : \mathbb{G}_a \to \widetilde{G}$ of λ.

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For $p = 3$ define λ by

$$\lambda(\tau, x_i) = x_i, \quad \lambda(\tau, y_i) = y_i \quad \text{for all } i \neq 1$$

$$\lambda(\tau, x_1) = x_1 + \tau x_1^2, \quad \lambda(\tau, y_1) = y_1 - 2\tau x_1 y_1 + 2\tau^2 x_1^2 y_1.$$

The differential of λ in this case is $H_{-x_1^2 y_1}$. The homomorphism $\tilde{\lambda}_{x_1^3} = e^{\tau x_1^2 y_1/h} : \mathbb{G}_a \to \tilde{G}$ lifts λ.

Lemma 4.6. *The group scheme $H := [G_0, G_0]$ is generated by $\mathrm{Sp}(2n)$, α, and the image of λ.*

Proof. First, we show that α and $[G_0^0, G_0^0]$ generate H. Indeed, since $\alpha \subset H$, we have that, for any k-algebra R

$$H(R) = \alpha(R)(G_0^0(R) \cap H(R)).$$

Thus, it suffices to prove that $G_0^0(R) \cap H(R) = [G_0^0, G_0^0](R)$. By Lemma 4.4 $G_0^0/[G_0^0, G_0^0] \cong G_0/H$, so the assertion holds.

Thus, it remains to prove that $[G_0^0, G_0^0]$ is generated by $\mathrm{Sp}(2n)$ and the image of λ. Since the groups in question are smooth it suffices to verify that $\mathrm{Lie}[G_0^0, G_0^0]$ is generated by $\mathfrak{sp}(2n)$ and $\mathrm{Lie}\,\lambda(\mathbb{G}_a)$. But this is immediate from Lemmas 4.3 and 4.4. □

Consider the extension

$$1 \to \hat{\mathbb{W}} \to \widetilde{G}_0^e \to G_0 \to 1 \tag{4.10}$$

from Lemma 4.2.

Lemma 4.7. *The restriction of the extension (4.10) to G_0^0 admits a unique splitting, that is, there exists a unique homomorphism $G_0^0 \to \widetilde{G}_0^e$ whose composition with the projection $\widetilde{G}_0^e \to G_0$ is the identity.*

Proof. Recall from (3.6) the extension

$$1 \to L\mathbb{G}_m \to \widetilde{G} \to G \to 1.$$

The kernel $\underline{A}_h^*/\mathbb{G}_m$ of surjection $\pi : G \to G_0$ is a pro-unipotent group scheme. Thus, by part (iii) of Proposition A.5 the restriction of the above extension to $\pi^{-1}G_0^0 \hookrightarrow G$ admits a unique reduction to $L^+\mathbb{G}_m$. Equivalently, the extension

$$1 \to \mathrm{Gr}_{\mathbb{G}_m} \to \widetilde{G}_0 \to G_0 \to 1$$

admits a unique splitting $\upsilon : G_0^0 \to \tilde{G}_0$ over $G_0^0 \subset G_0$. It remains to show that υ lands in $\widetilde{G}_0^e$. From the proof of Lemma 4.2 $\widetilde{G}_0^e$ is the kernel of a homomorphism $\widetilde{G}_0 \to \underline{\mathbb{Z}}$. Since G_0^0 is connected its composition with υ is identically 0 as desired. □

Recall from Remark 3.2 an isomorphism of Lie algebras $\mathrm{Lie}\,\widetilde{G}_0^e = \mathrm{Lie}\,\widetilde{G}_0 \xrightarrow{\sim} A_0 \oplus h^{-2}k[h^{-1}]$. Also recall an identification $\mathrm{Lie}\,G_0^0 \xrightarrow{\sim} m^2 \subset A_0$.

Lemma 4.8. *The morphism $\mathrm{Lie}\,G_0^0 \to \mathrm{Lie}\,\widetilde{G}_0^e$ induced by the splitting from Lemma 4.7 equals the composition $m^2 \hookrightarrow A_0 \hookrightarrow \mathrm{Lie}\,\widetilde{G}_0^e$.*

Proof. The difference of the two morphisms of Lie algebras is a homomorphism from $\mathrm{Lie}\,G_0^0$ to the abelian Lie algebra $\mathrm{Lie}\,\hat{\mathbb{W}}$. Thus, it suffices to check that the morphisms coincide on the one-dimensional Lie algebra $\mathrm{Lie}\,G_0^0/[\mathrm{Lie}\,G_0^0, \mathrm{Lie}\,G_0^0]$, which is immediate from Lemma 4.4. □

4.2 Extensions of α by $\hat{\mathbb{W}}$

In this subsection we shall apply the theory of restricted Lie algebras to study the category of central extensions of the group scheme α by the group ind-scheme $\hat{\mathbb{W}}$. Recall (see

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e.g. [DG80, Chapter II, § 7]) that the Lie algebra of a group scheme H over a field of characteristic $p > 0$ is equipped with the pth power operation giving $\operatorname{Lie} H$ a restricted Lie algebra structure. We are not aware of a written account of such theory for group ind-schemes. Therefore, we shall use the following trick to reduce our problem to the well documented setup.

For an affine scheme S and an affine group scheme H over k denote by $\underline{\mathrm{Mor}}(S, H)$ the *fpqc* sheaf of groups assigning to a scheme T over k the group $\mathrm{Mor}(S \times T, H)$.

LEMMA 4.9. *Let G be a finite connected group scheme over k, H a smooth commutative group scheme, and let S be an affine scheme. Then the groupoid of central extensions of G by $\underline{\mathrm{Mor}}(S, H)$ in the category of fpqc sheaves of groups is equivalent to the category of central extensions of $G_S = G \times S$ by $H_S = H \times S$ in the category of group schemes over S.*

Proof. Assume we are given a central extension

$$1 \to H_S \to K \xrightarrow{\pi} G_S \to 1, \tag{4.11}$$

and let us construct a central extension F of G by $\underline{\mathrm{Mor}}(S, H)$. For any scheme T over $\mathrm{Spec}(k)$ define $F(T) = \{g \in K(T \times S) | T \times S \xrightarrow{\pi(g)} G \times S \to G \text{ factors through the projection to } T\}$. It is easy to see that the resulting F is a sheaf and that $\underline{\mathrm{Mor}}(S, H)$ injects into it, so it is left to prove that $F \to G$ is a surjection. Indeed, since the morphism π is flat and H_S is smooth we get that $K \to G_S$ is formally smooth [Sta, Lemma 29.33.3]. Then since the map $\mathrm{Spec}(k) \times S \to G \times S$ is a nilpotent thickening we get that π has a section. Thus, $F \to G$ is surjective.

Conversely, if we have a central extension

$$1 \to \underline{\mathrm{Mor}}(S, H) \xrightarrow{i} F \to G \to 1, \tag{4.12}$$

define for any S-scheme T the group $K(T)$ to be $F(T) \times^{\mathrm{Mor}(T \times S, H)} \mathrm{Mor}(T, H)$. Here the map $\mathrm{Mor}(T \times S, H) \to F(T)$ is induced by i and $Mor(T \times S, H) \to \mathrm{Mor}(T, H)$ is defined to be the restriction to the graph of the structure morphism $T \to S$. Let $H_S(T) \to F(T) \times \mathrm{Mor}(T, H)$ be the homomorphism whose composition with the first projection takes $H_S(T)$ to the neutral element and whose composition with the second projection is the identity map. This defines an injection of sheaves $H_S \hookrightarrow K$, making K into an H_S-torsor over G_S representable by a scheme [Mil80, Chapter III, Theorem 4.3]. That is enough. □

COROLLARY 4.10. *The groupoid of central extensions of α by $\underline{\mathrm{Mor}}(\mathbb{A}^1, \mathbb{G}_m)$ in the category of fpqc sheaves is equivalent to the groupoid of central extensions of $\alpha \times \mathbb{A}^1$ by $\mathbb{G}_m \times \mathbb{A}^1$ in the category of group schemes over $\mathbb{A}^1$.*

Remark 4.11. The above groupoids are discrete, i.e. objects do not have nontrivial automorphisms. Indeed, the Cartier dual group to α is isomorphic to itself. In particular, it has no nontrivial $\mathbb{A}^1$ points. Hence, every homomorphism $\alpha \times \mathbb{A}^1 \to \mathbb{G}_m \times \mathbb{A}^1$ in the category of group schemes over $\mathbb{A}^1$ is trivial.

Observe that the evaluation at 0 defines a split surjection $\underline{\mathrm{Mor}}(\mathbb{A}^1, \mathbb{G}_m) \to \mathbb{G}_m$, whose kernel is identified with $\hat{\mathbb{W}}$. Hence, we have a decomposition $\underline{\mathrm{Mor}}(\mathbb{A}^1, \mathbb{G}_m) = \hat{\mathbb{W}} \times \mathbb{G}_m$.

Declare the restricted Lie algebra of Gr_G to be $\mathrm{Lie}(\mathrm{Gr}_G^0) = h^{-1}k[h^{-1}]$ with the trivial Lie bracket and the restricted power operation given by the absolute Frobenius.

THEOREM 3. *The groupoid of central extensions of the group scheme α by $\hat{\mathbb{W}}$ is equivalent to the groupoid of central extensions of the restricted Lie algebra $\mathrm{Lie}(\alpha)$ by $\mathrm{Lie}(\hat{\mathbb{W}})$.*

Proof. Using that the multiplication by p is 0 on α and surjective on $\hat{\mathbb{W}}$ it follows that every extension of α by $\hat{\mathbb{W}}$ admits a reduction to $\mathrm{Gr}_{\mu_p}^0$. Moreover, since $\mathrm{Hom}(\alpha, \hat{\mathbb{W}}) = 0$ (by Corollary A.3),

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such a reduction is unique. Thus, the groupoid of central extensions of α by $\hat{\mathbb{W}}$ is equivalent to the groupoid of central extensions of α by $\mathrm{Gr}^0_{\mu_p}$. The groupoid of central extensions of α by $\mathrm{Gr}^0_{\mu_p}$ is equivalent to a full subcategory of the groupoid of central extensions of $\alpha \times \mathbb{A}^1$ by $\mu_p \times \mathbb{A}^1$. This subcategory classifies families of central extensions whose fiber over $0 \in \mathbb{A}^1$ is a trivial extension.

Next we claim that for an extension

$$\mu_p \times \mathbb{A}^1 \to K \to \alpha \times \mathbb{A}^1 \tag{4.13}$$

the $\mathbb{A}^1$-group scheme is of height 1. Indeed, the Frobenius map $K \to K$ factors as $K \to \alpha \times \mathbb{A}^1 \to \mu_p \times \mathbb{A}^1 \to K$. From Remark 4.11 we conclude that this map is trivial.

Thus, by [DG80, Chapter II, § 7, Theorem 3.5] the category of central extensions of $\alpha \times \mathbb{A}^1$ by $\mu_p \times \mathbb{A}^1$ is equivalent to the category of extensions of corresponding restricted Lie algebras over $\mathbb{A}^1$. The latter is equivalent to the category of extensions of $\mathrm{Lie}(\alpha)$ by $\mathrm{Lie}(\mu_p \times \mathbb{A}^1)$ in the category of restricted Lie algebras over k. The above equivalence induces an equivalence between the subcategories of central extensions of $\alpha \times \mathbb{A}^1$ by $\mu_p \times \mathbb{A}^1$ trivial over $0 \in \mathbb{A}^1$ and the category of restricted Lie algebra extensions of $\mathrm{Lie}(\alpha)$ by $\mathrm{Lie}(\hat{\mathbb{W}})$. □

Corollary 4.12. *The groupoid of central extensions of the group scheme α by $\hat{\mathbb{W}}$ which split over any factor $\alpha_p \subset \alpha$ is equivalent to the set of* $\mathrm{Lie}(\hat{\mathbb{W}})$*-valued skew-symmetric bilinear forms on* $\mathrm{Lie}(\alpha)$ *viewed as a groupoid with no nontrivial morphisms.*

Proof. Given a central extension of α by $\hat{\mathbb{W}}$ let

$$0 \to \mathrm{Lie}(\hat{\mathbb{W}}) \to \mathfrak{L} \to \mathrm{Lie}(\alpha) \to 0 \tag{4.14}$$

be the corresponding extension of Lie algebras. The commutator on $\mathfrak{L}$ defines a $\mathrm{Lie}(\hat{\mathbb{W}})$-valued skew-symmetric bilinear form on $\mathrm{Lie}(\alpha)$. To construct the functor in the other direction set $\mathfrak{L} = \mathrm{Lie}(\hat{\mathbb{W}}) \oplus \mathrm{Lie}(\alpha)$ as a vector space. The skew-symmetric form defines a Lie bracket on $\mathfrak{L}$ making $\mathfrak{L}$ a central extension of Lie algebras. Define the restricted power operation on $\mathfrak{L}$ by the formula $(f, g)^{[p]} = f^{[p]}$. Let us check that $\mathfrak{L}$ is a restricted Lie algebra. We have to check that the restricted power operation satisfies

$$(X + Y)^{[p]} = X^{[p]} + Y^{[p]} + \sum_{i=1}^{p-1} \frac{s_i(X, Y)}{i}$$

for X and Y in the Lie algebra, and $s_i(X, Y)$ being the coefficient of t^{i-1} in the formal expression $\mathrm{ad}(tX + Y)^{p-1}(X)$. Since $p > 2$ the polynomial $s_i(X, Y) = 0$ for every i as desired. It remains to check that every extension (4.14) of restricted Lie algebras that splits over every factor $\mathrm{Lie}(\alpha_p) \subset \mathrm{Lie}(\alpha)$ arises this way. As observed above, the restricted power operation on $\mathfrak{L}$ is additive. Now consider the subspace V of $\mathfrak{L}$ consisting of elements annulated by $[p]$-power operation. The projection defines an embedding $V \hookrightarrow \mathrm{Lie}(\alpha)$. Since the extension has a section over each $\mathrm{Lie}(\alpha_p)$ the embedding is an isomorphism and we win. □

4.3 Geometric description of $\widetilde{G}^e_0$

Let

$$1 \to \hat{\mathbb{W}} \to \widetilde{G}^e_0 \to G_0 \to 1 \tag{4.15}$$

be the extension from Lemma 4.2, and let $\widetilde{\alpha}$ be its restriction to α.

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Let us check that $\widetilde{\alpha}$ satisfies the assumptions of Corollary 4.12, that is splits over every subgroup $\alpha_p \subset \alpha$. For any $v \in V$ set $f = \omega_V(v, \cdot) \in V^*$ and define a homomorphism

$$\alpha_p = \operatorname{Spec} k[\epsilon]/(\epsilon^p) \to \widetilde{G} \subset \underline{A_h[h^{-1}]^*}, \quad \epsilon \mapsto e^{\epsilon f/h}.$$

Here $e^{\epsilon f/h}$ denotes the restricted exponent, i.e. $e^{\epsilon f/h} = 1 + \epsilon f/h + \cdots + 1/(p-1)!(\epsilon f/h)^{p-1}$. It is easy to see that $e^{\epsilon f/h} \in \underline{A_h[h^{-1}]^*}(\alpha_p)$ normalizes the lattice $A_h \otimes k[\epsilon]/(\epsilon^p)$; therefore, it lies in $\widetilde{G}(\alpha_p)$. Then the composition of this homomorphism with the projection to $\widetilde{G}_0$ is a lift of the embedding $\alpha_p \subset \alpha$ corresponding to v as desired.

The natural action of $\mathrm{Sp}_{2n} \subset G_0^0 \subset G_0$ on α by conjugation lifts to an action on $\widetilde{\alpha}$. Indeed, by Lemma 4.7 the extension (4.15) splits uniquely over $\mathrm{Sp}_{2n} \subset G_0^0$.

For future purposes note that the symplectic basis (x_i, y_i) for V^* gives rise to a scheme-theoretic section of $\pi : \widetilde{\alpha} \to \alpha$. Namely, $\alpha \cong \alpha_p^{2n}$, and we define

$$t : \alpha = \operatorname{Spec}(k[\epsilon_1, \dots \epsilon_n, \delta_1, \dots \delta_n,]/(\epsilon_i^p = \delta_i^p = 0)) \to \tilde{\alpha} \tag{4.16}$$

by $e^{\epsilon_1 x_1/h} \dots e^{\epsilon_n x_n/h} e^{\delta_1 y_1/h} \dots e^{\delta_n y_n/h}$. The section t is not a group homomorphism and it does depend on the choice of symplectic basis. However, its differential

$$dt(e) : \mathrm{Lie}(\alpha) \to \mathrm{Lie}(\widetilde{\alpha})$$

is the unique linear map compatible with the restricted power operation.

Recall that a connection on a (trivial) $\hat{\mathbb{W}}$-torsor $\widetilde{\alpha}$ is a function

$$\nabla : \{\text{sections } s : \alpha \to \widetilde{\alpha}\} \to \Omega^1_\alpha \otimes \mathrm{Lie}(\hat{\mathbb{W}}) \tag{4.17}$$

such that, for any $c \in \hat{\mathbb{W}}(\alpha)$, one has $\nabla(cs) = \nabla(s) + c^{-1}\, dc$. Denote by $\mathrm{Conn}(\widetilde{\alpha}, \hat{\mathbb{W}})$ the set of $\hat{\mathbb{W}}$-connections on $\widetilde{\alpha}$. More generally, we define the space $\underline{\mathrm{Conn}}(\widetilde{\alpha}, \hat{\mathbb{W}})$ of connections on $\widetilde{\alpha}$ to be the functor $(k - \text{algebras})^{\mathrm{op}} \to \mathrm{Sets}$ sending an algebra R to the set of functions

$$\nabla : \{\text{sections } s : \alpha \times \operatorname{Spec} R \to \widetilde{\alpha}\} \to \Omega^1_\alpha \otimes \mathrm{Lie}(\hat{\mathbb{W}}) \otimes R \tag{4.18}$$

with $\nabla(cs) = \nabla(s) + c^{-1}\, dc$, for every $c \in \mathbb{W}(\alpha \times \operatorname{Spec} R)$. The group scheme S_α of automorphisms of the scheme α acts on the space $\underline{\mathrm{Conn}}(\widetilde{\alpha}, \hat{\mathbb{W}})$. In particular, for any subgroup $H \subset S_\alpha$, we have a subset $\mathrm{Conn}(\widetilde{\alpha}, \hat{\mathbb{W}})^H \subset \mathrm{Conn}(\widetilde{\alpha}, \hat{\mathbb{W}})$ of H-invariant connections.

LEMMA 4.13. *There exists a unique* $\mathrm{Sp}_{2n} \ltimes \alpha$*-invariant* $\hat{\mathbb{W}}$*-connection* ∇ *on* $\widetilde{\alpha}$.

Proof. Denote by $\mathrm{Conn}(\widetilde{\alpha}, \hat{\mathbb{W}})^\alpha$ the set of α-invariant connections. We have that

$$\mathrm{Conn}(\widetilde{\alpha}, \hat{\mathbb{W}})^\alpha = \{\text{linear maps } f : \mathrm{Lie}(\widetilde{\alpha}) \to \mathrm{Lie}(\hat{\mathbb{W}}) \text{ such that } f|_{\mathrm{Lie}(\hat{\mathbb{W}})} = \mathrm{Id}\}.$$

For an α-invariant connection ∇, the corresponding f is given by the formula

$$(\nabla(s) - ds) \circ d\pi(e) + \mathrm{Id},$$

for any section s.

Now since Sp_{2n} normalizes α in G_0^0 we get that Sp_{2n} acts on $\mathrm{Conn}(\widetilde{\alpha}, \hat{\mathbb{W}})^\alpha$. Suppose we are given two Sp_{2n}-invariant connections in $\mathrm{Conn}(\widetilde{\alpha}, \hat{\mathbb{W}})^\alpha$. Then their difference gives a morphism $\mathrm{Lie}(\alpha) \to \mathrm{Lie}(\hat{\mathbb{W}})$ of representation of Sp_{2n}, which has to be trivial since $\mathrm{Lie}(\alpha)$ is a nontrivial irreducible representation whereas the action of Sp_{2n} on $\mathrm{Lie}(\hat{\mathbb{W}})$ is trivial. Thus, we get the uniqueness.

To prove the existence take (a unique) $f : \mathrm{Lie}(\tilde{\alpha}) \to \mathrm{Lie}(\hat{\mathbb{W}})$ that commutes with the restricted power operation (see the proof of Corollary 4.12). This morphism is Sp_{2n}-invariant since the action of Sp_{2n} respects the restricted structure. □

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We will need an explicit formula for the $\mathrm{Sp}_{2n} \ltimes \alpha$-invariant connection ∇. Let t be the section defined in (4.16). We claim that

$$\nabla(t) = \frac{\eta}{h} = \Sigma \frac{\delta_i \, d\epsilon_i}{h}. \tag{4.19}$$

To see this let us show that the connection given by (4.19) is $\mathrm{Sp}_{2n} \ltimes \alpha$-invariant. Pick a k-algebra R and a point $a \in \alpha(R)$ given by $\epsilon_i \mapsto \epsilon_i' \in R$. Then $t(a)$ is an R-point of $\widetilde{\alpha}$ that acts on $\widetilde{\alpha}$ by translation γ. The composition $\gamma \circ t : \alpha \times \mathrm{Spec}(R) \to \widetilde{\alpha} \times \mathrm{Spec}(R)$ is given by the formula

$$t(a) \Pi e^{\epsilon_i x_i/h} \Pi e^{\delta_i y_i/h} = e^{-(\Sigma \epsilon_i \delta_i'/h)} \Pi e^{(\epsilon_i + \epsilon_i')x_i/h} \Pi e^{(\delta_i + \delta_i')y_i/h}. \tag{4.20}$$

Let $\bar{\gamma}$ be the translation by a acting on α. Then $t_\gamma = \gamma^{-1} t \bar{\gamma}$ defines another section of π. The invariance of ∇ under the action of γ reads as

$$\bar{\gamma}^* \nabla(t) = \nabla(t_\gamma).$$

Since $\bar{\gamma}(\epsilon_j) = \epsilon_j + \epsilon_j'$ and $\bar{\gamma}(\delta_j) = \delta_j + \delta_j'$, we have that

$$\bar{\gamma}^* \nabla(t) = \bar{\gamma}^* \frac{\eta}{h} = \frac{\eta}{h} + \Sigma \frac{\delta_i' \, d\epsilon_i}{h}.$$

On the other hand, from (4.20) we have $t_\gamma = e^{\Sigma \epsilon_i \delta_i'/h} t$ and, therefore,

$$\nabla(t_\gamma) = \nabla(t) + e^{-(\Sigma \epsilon_i \delta_i'/h)} \, de^{\Sigma \epsilon_i \delta_i'/h} = \frac{\eta}{h} + \Sigma \frac{\delta_i' \, d\epsilon_i}{h}.$$

Thus, ∇ is α-invariant. Let us show that ∇ is also Sp_{2n}-invariant. Indeed, the morphism $\{f : \mathrm{Lie}(\widetilde{\alpha}) \to \mathrm{Lie}(\hat{\mathbb{W}})\}$ coincides with the differential of t, which is, as we observed above, a unique linear map compatible with the restricted power operation. Therefore, it is Sp_{2n}-invariant.

Define S_α to be the group scheme of automorphisms of the scheme α, that is,

$$S_\alpha(T) = \mathrm{Aut}_T(\alpha \times T).$$

Define also $S_{\widetilde{\alpha}}^{\hat{\mathbb{W}}}$ to be the *fpqc* sheaf of automorphisms of the torsor $\widetilde{\alpha}$, that is,

$$S_{\widetilde{\alpha}}^{\hat{\mathbb{W}}}(T) = \{\phi \in S_\alpha(T), \tilde{\phi} : \widetilde{\alpha} \times T \to \phi^*(\widetilde{\alpha} \times T)\}.$$

Finally, define $S_{\widetilde{\alpha}}^{\nabla}$ to be subsheaf of $S_{\widetilde{\alpha}}^{\hat{\mathbb{W}}}$ of endomorphisms preserving the connection ∇ on $\widetilde{\alpha}$.

Lemma 4.14. *The morphism $S_{\widetilde{\alpha}}^{\hat{\mathbb{W}}} \to S_\alpha$ fits into a short exact sequence*

$$1 \to \mathrm{Mor}(\alpha, \hat{\mathbb{W}}) \to S_{\widetilde{\alpha}}^{\hat{\mathbb{W}}} \to S_\alpha \to 1.$$

Moreover, the kernel of the composition $S_{\widetilde{\alpha}}^{\nabla} \hookrightarrow S_{\widetilde{\alpha}}^{\hat{\mathbb{W}}} \to S_\alpha$ is the sheaf $\hat{\mathbb{W}} \subset \mathrm{Mor}(\alpha, \hat{\mathbb{W}})$ of constant maps. Finally, the image of $S_{\widetilde{\alpha}}^{\nabla}$ in S_α belongs to G_0.

Proof. The map $\mathrm{Mor}(\alpha, \hat{\mathbb{W}}) \to S_{\widetilde{\alpha}}$ takes f to the translation by $f \circ \pi$, and exactness is immediate because $\widetilde{\alpha}$ is a trivial torsor. If the translation by $f \circ \pi$ preserves the connection, then $df = 0$. This implies that f is constant, that is, $f \in \hat{\mathbb{W}}$.

To see that $\mathrm{Im}(S_{\widetilde{\alpha}}^{\nabla}) \subset G_0$ pick $\gamma \in S_{\widetilde{\alpha}}^{\nabla}$ and let $\bar{\gamma}$ be its image in S_α. We have that

$$\bar{\gamma}^* \nabla(t) = \nabla(t_\gamma) = \nabla(t) + c^{-1} \, dc \in \Omega_\alpha^1 \otimes \mathrm{Lie}(\hat{\mathbb{W}}) \tag{4.21}$$

for some $c = (1 + \Sigma_{i>0} a_i h^{-i}) \in \hat{\mathbb{W}}(\alpha)$. Consider the group homomorphism $\hat{\mathbb{W}} \to \mathbb{G}_a$ that takes a series $(1 + \Sigma a_i h^{-i}) \in \hat{\mathbb{W}}(\alpha)$ to a_1. This defines a morphism $\Omega_\alpha^1 \otimes \mathrm{Lie}(\hat{\mathbb{W}}) \to \Omega_\alpha^1$, and the image

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of $\nabla(t) \in \Omega^1_\alpha \otimes \mathrm{Lie}(\hat{\mathbb{W}})$ is precisely η. Hence, we have from (4.21)

$$\bar{\gamma}^*\eta = \eta + da_1$$

as desired. □

Recall from Lemma 4.7 that the extension $\widetilde{G}_0^e \to G_0$ admits a unique splitting $G_0^0 \hookrightarrow \widetilde{G}_0^e$ over G_0^0. The left action of $\widetilde{G}_0^e$ on $\widetilde{G}_0^e/G_0^0 \xrightarrow{\sim} \widetilde{\alpha}$ defines a homomorphism

$$\widetilde{G}_0^e \to S^{\hat{\mathbb{W}}}_{\widetilde{\alpha}}. \tag{4.22}$$

THEOREM 4. *Let ∇ be a $\mathrm{Sp}_{2n} \ltimes \alpha$-invariant connection on $\tilde{\alpha}$. Then homomorphism (4.22) induces an isomorphism $\widetilde{G}_0^e \xrightarrow{\sim} S^{\nabla}_{\widetilde{\alpha}}$.*

Proof. Let us show that $\widetilde{G}_0^e$ is a subsheaf of $S^{\nabla}_{\widetilde{\alpha}}$, that is, ∇ is G_0-invariant. Denote by H the commutator subgroup of G_0. We shall first show that ∇ is H-invariant.

By Lemma 4.6 the group H is generated by α, $\mathrm{Sp}(2n)$, and a certain one-parameter subgroup $\lambda(\tau) : \mathbb{G}_a \to G_0^0$. By assumption, ∇ is $\mathrm{Sp}_{2n} \ltimes \alpha$-invariant. It remains to check that ∇ is $\mathbb{G}_a$-invariant. We use formula (4.19) describing ∇ in coordinates corresponding to the trivialization t of the torsor $\tilde{\alpha}$. First, assume that $p > 3$. Homomorphism λ has a unique lifting $\tilde{\lambda}$ to $\widetilde{G}_0^{0,e}$ that can be explicitly computed using the construction from Lemma 4.5

$$\tilde{\lambda} = e^{\tau x_1^3/h} : \mathbb{G}_a \to \widetilde{G}_0^{0,e}.$$

The invariance of ∇ under the action of $\mathbb{G}_a$ reads as

$$\gamma^*\nabla(t) = \nabla(t'), \tag{4.23}$$

where $t' : \mathbb{G}_a \times \alpha \to \widetilde{\alpha}$ is the composition

$$\mathbb{G}_a \times \alpha \xrightarrow{\mathrm{Id}\times\gamma} \mathbb{G}_a \times \alpha \xrightarrow{\mathrm{Id}\times t} \mathbb{G}_a \times \widetilde{\alpha} \xrightarrow{\mathrm{Id}\times\tilde{\gamma}^{-1}} \mathbb{G}_a \times \widetilde{\alpha} \xrightarrow{\mathrm{pr}_{\widetilde{\alpha}}} \widetilde{\alpha}.$$

We have to compute t'. The following equality of morphisms $\mathbb{G}_a \times \alpha \to \widetilde{G}_0^e$ holds:

$$e^{\tau(x_1^3/h)}\Pi e^{\epsilon_i x_i/h} e^{\delta_i y_i/h} = e^{-\tau(2\delta_1^3/h)}(e^{(\epsilon_1+3\tau\delta_1^2)x_1/h} e^{\delta_1 y_1/h} \cdots e^{\epsilon_n x_n/h} e^{\delta_n y_n/h}) e^{\tau((x_1+\delta_1)^3 - 3\delta_1^2 x_1 - \delta_1^3)/h}. \tag{4.24}$$

We claim that the last factor $e^{\tau((x_1+\delta_1)^3-3\delta_1^2x_1-\delta_1^3)/h}$ maps $\mathbb{G}_a \times \alpha$ to $G_0^0 \subset \widetilde{G}_0^e$ as follows.

$$\begin{array}{ccc} G_0^0 & \longrightarrow & \widetilde{G}_0^{0,e} \\ \downarrow{\scriptstyle \mathrm{Id}} & & \downarrow \\ G_0^0 & \xrightarrow{\mathrm{Id}} & G_0^0 \end{array}$$

Indeed, the same formula defines an extension of the morphism $\mathbb{G}_a \times \alpha \to \widetilde{G}_0^e$ to a morphism from a *reduced* scheme $\mathbb{G}_a \times \mathrm{Spec}\, k[[\epsilon_i, \delta_j]]$ to $\widetilde{G}_0^e$.[12] The composition of the latter with the projection $\widetilde{G}_0^e \to G_0$ lands in $G_0^0 \subset G_0$. But the projection $\widetilde{G}_0^{0,e} \to G_0^0$ induces an isomorphism on points with values in any reduced k-algebra. Thus, the morphism $e^{\tau((x_1+\delta_1)^3-3\delta_1^2x_1-\delta_1^3)/h} : \mathbb{G}_a \times \mathrm{Spec}\, k[[\epsilon_i, \delta_j]] \to \widetilde{G}_0^e$ factors through G_0^0 and the claim follows.

$$\gamma^*\nabla(t) = \gamma^*\frac{\eta}{h} = \frac{\eta}{h} + \frac{\delta_1 d(3\tau\delta_1^2)}{h} = \frac{\eta}{h} + \frac{2\tau\, d\delta_1^3}{h},$$

$$\nabla(t') = \nabla(t) + e^{-\tau(2\delta_1^3/h)}\, de^{\tau(2\delta_1^3/h)} = \frac{\eta}{h} + \frac{2\tau\, d\delta_1^3}{h}.$$

[12] Indeed, the pth power of $(x_1+\delta_1)^3 - 3\delta_1^2x_1 - \delta_1^3 = x_1^3 + 3x_1^2\delta_1 \in A_h[[\delta_1]]$ is zero.

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For $p=3$, the lift $\tilde{\gamma}$ is given by $e^{\tau(x_1^2y_1/h)}$. Write

$$e^{\tau(x_1^2y_1/h)}\Pi e^{\epsilon_i x_i/h}e^{\delta_i y_i/h} = f(\tau,\epsilon_1,\delta_1)(e^{(\epsilon_1+\tau\delta_1\epsilon_1)x_1/h}e^{(\delta_1+\tau\delta_1^2)y_1/h}e^{\epsilon_2x_2/h}e^{\delta_2y_2/h}\cdots e^{\epsilon_nx_n/h}e^{\delta_ny_n/h})s \tag{4.25}$$

for some uniquely determined $s\in G_0^0(\mathbb{G}_a\times\alpha)\subset\widetilde{G}_0^{0,e}(\mathbb{G}_a\times\alpha)$ and $f(\tau,\epsilon_1,\delta_1)\in\hat{\mathbb{W}}(\mathbb{G}_a\times\alpha)$. We claim that

$$f(\tau,\epsilon_1,\delta_1)=e^{\tau(\delta_1^2\epsilon_1/h)}. \tag{4.26}$$

A direct verification of this formula is unpleasant; instead we deduce it from the following facts. Using a computation in Lie algebras from Lemma 4.8 one verifies (4.26) modulo τ^2.

Also, it is easy to see that the left-hand side is invariant under the action of the multiplicative group given by $\tau\to\tau/a$, $\epsilon_1\to\epsilon_1/a$, $\delta_1\to a\delta_1$, $x_1\to ax_1$, $y_1\to y/a$. Thus, the element $f(\tau,\epsilon_1,\delta_1)$ must be also invariant under this transformation. Finally, $f(\tau,\epsilon_1,\delta_1)$ satisfies the following cocycle condition:

$$f(\tau_1+\tau_2,\epsilon_1,\delta_1)=f(\tau_1,\epsilon_1,\delta_1)f(\tau_2,\epsilon_1+\tau_1\delta_1\epsilon_1,\delta_1+\tau_1\delta_1^2). \tag{4.27}$$

There exists a unique f satisfying the above properties and it is given by (4.26). It follows that $\tilde{\gamma}$ carries the section t to $t'=e^{\tau(2\delta_i^2\epsilon_i/h)}t$ and (4.23) follows.

We have proved that ∇ is H-invariant. Note that since $\mathrm{Sp}_{2n}\subset H$ and $\alpha\subset H$ we can see that ∇ is a unique H-invariant connection.

The group scheme G_0 acts on $\underline{\mathrm{Conn}}(\widetilde{\alpha},\hat{\mathbb{W}})$, the subgroup H is normal in G_0, hence G_0/H acts on the space of H-invariant connections. But since the latter consists of one element this action must by trivial. Hence, ∇ is G_0-invariant.

It follows that homomorphism (4.22) factors through $S_{\widetilde{\alpha}}^{\nabla}$. Thus, by Lemma 4.14 we have a commutative diagram

$$\begin{array}{ccc}\widetilde{G}_0^e & \xrightarrow{\beta} & S_{\widetilde{\alpha}}^{\nabla}\\ \downarrow & & \downarrow\\ G_0 & \xrightarrow{Id} & G_0\end{array}$$

where β induces an isomorphism on the kernels of the vertical arrows. Hence, β is an isomorphism as desired. □

4.4 Proof of the Basic Lemma

We will prove the assertion for the extensions by $\hat{\mathbb{W}}$ (as opposed to $\mathrm{Gr}_{\mathbb{G}_m}$) which is *a priori* stronger than that stated in § 1.8.

Recall the setup. Let $i:V\hookrightarrow V^\flat$ be a morphism of symplectic vector spaces such that the restriction to V of the symplectic form on $V^\flat$ is ω_V. Let $\widetilde{G}_0^e\to G_0$ and $\widetilde{G}_0^{\flat,e}\to G_0^\flat$ be the corresponding extensions. Denote by $G_0^\sharp\subset G_0^\flat$ the group subscheme that consists of automorphisms preserving kernel of the homomorphism $i^*:A_0^\flat\to A_0$. We have a natural homomorphism $G_0^\sharp\to G_0$.

Theorem 5 (Basic Lemma). *The homomorphism $G_0^\sharp\to G_0$ lifts uniquely to a homomorphism of central extensions*

$$\widetilde{G}_0^{\flat,e}\times_{G_0^\flat}G_0^\sharp\to\widetilde{G}_0^e.$$

Proof. The uniqueness follows from Corollary A.3. Let us prove the existence. By Corollary 4.12 the morphism $i:\alpha\to\alpha^\flat$ lifts uniquely to a morphism $\tilde{i}:\widetilde{\alpha}\to\widetilde{\alpha}^\flat$ of extensions. Moreover, the

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pullback of the (unique) $\mathrm{Sp}(V^\flat) \ltimes \alpha^\flat$-invariant connection $\nabla^\flat$ on $\widetilde{\alpha}^\flat$ is the (unique) $\mathrm{Sp}(V) \ltimes \alpha$-invariant connection ∇ on $\widetilde{\alpha}$. Thus, we have a homomorphism

$$S^{\nabla^\flat}_{\widetilde{\alpha}^\flat} \times_{G_0^\flat} G_0^\sharp \to S^{\nabla}_{\widetilde{\alpha}}$$

lifting $G_0^\sharp \to G_0$. It remains to apply Theorem 3. □

COROLLARY 4.15. *There exists a unique isomorphism of central extensions of $G_0^\sharp$:*

$$\widetilde{G}_0^{\flat,e} \times_{G_0^\flat} G_0^\sharp \xrightarrow{\sim} \widetilde{G}_0^e \times_{G_0} G_0^\sharp.$$

4.5 Proof of the Proposition 3.3

We will prove a stronger assertion for the extensions by $\hat{\mathbb{W}}$ (as opposed to $\mathrm{Gr}_{\mathbb{G}_m}$). Let (V, ω) be a symplectic vector space and let η be a homogeneous 1-form on the scheme $\mathbf{V}$ whose differential equals ω, that is, a vector $\eta \in V^* \otimes V^*$ whose skew-symmetrization is ω. Denote by

$$i : V \hookrightarrow V^\flat := V \oplus V^*$$

the linear morphism corresponding to the graph $\Gamma_\eta : \mathbf{V} \hookrightarrow \mathbb{T}^*_{\mathbf{V}}$ of η. Explicitly, the composition of i with the first projection is Id and its composition $V \to V^*$ with the second projection is given by $\eta \in V^* \otimes V^*$. In § 3.2 we defined a homomorphism $\psi_0 : G_0 \to G_0^\flat$. We have to prove that ψ_0 lifts uniquely to a homomorphism $\tilde{\psi}_0 : \widetilde{G}_0^e \to \widetilde{G}_0^{\flat,e}$ of extensions. The uniqueness follows from Corollary A.3. To prove the existence we observe that by construction of ψ_0 it factors through the subgroup $G_0^\sharp \subset G_0^\flat$ that consists of automorphisms preserving kernel of the homomorphism $i^* : A_0^\flat \to A_0$ and its composition

$$G_0 \xrightarrow{\psi_0} G_0^\sharp \longrightarrow G_0$$

with restriction morphism is the identity.[13] Consider the homomorphism

$$\widetilde{G}_0^e \xrightarrow{(\mathrm{Id},\psi_0)} \widetilde{G}_0^e \times_{G_0} G_0^\sharp.$$

Using Corollary 4.15 we get a morphism

$$\widetilde{G}_0^e \times_{G_0} G_0^\sharp \xrightarrow{\sim} \widetilde{G}_0^{\flat,e} \times_{G_0^\flat} G_0^\sharp \xrightarrow{\mathrm{pr}} \widetilde{G}_0^{\flat,e}.$$

Its composition with (Id, ψ_0) is the desired lift $\tilde{\psi}_0 : \widetilde{G}_0^e \to \widetilde{G}_0^{\flat,e}$.

5. $\mathbb{G}_m$-equivariant quantizations

In this section we consider quantizations of symplectic varieties (X, ω) equipped with an action of the multiplicative group $\mathbb{G}_m$ such that the form ω has a positive weight m with respect to this action and m is invertible in k. We recall the notion of a $\mathbb{G}_m$-equivariant Frobenius constant quantization O_h of such (X, ω). By definition, O_h is a $\mathbb{G}_m$-equivariant sheaf of $\mathcal{O}_{X'[h]}$-algebras on $X' \times \mathrm{Spec}\, k[h]$. In particular, specializing $h = 1$ we have a sheaf $O_{h=1}$ of $\mathcal{O}_{X'}$-algebras over X'.

[13] Homomorphism ψ_0 can be described in a coordinate-free way as follows. Consider the subgroup $G_0^{\sharp,f}$ of $G_0^\sharp$ that consists of scheme-theoretic automorphisms g of $\alpha^\flat$ fitting in the commutative diagram

$$\begin{array}{ccc} \alpha^\flat & \xrightarrow{g} & \alpha^\flat \\ \downarrow \pi & & \downarrow \pi \\ \alpha & \xrightarrow{\bar{g}} & \alpha, \end{array}$$

for some $\bar{g} \in G_0$. The restriction of the projection $G_0^\sharp \to G_0$ to $G_0^{\sharp,f}$ is an isomorphism and ψ_0 is its inverse.

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We show that if the action of $\mathbb{G}_m$ on X is contracting, then $O_{h=1}$ is an Azumaya algebra over X' and using Theorem 1 compute its class in the Brauer group $\mathrm{Br}(X')$ proving a conjecture of Kubrak and Travkin [KT19].

5.1 Definition of $\mathbb{G}_m$-equivariant quantizations

Let X be a smooth variety over k equipped with a symplectic 2-form ω and a $\mathbb{G}_m$-action

$$\lambda : \mathbb{G}_m \times X \to X. \tag{5.1}$$

We shall say that ω is of weight m with respect to the $\mathbb{G}_m$-action if the following identity holds in $\Gamma(\mathbb{G}_m \times X, \Omega^2_{\mathbb{G}_m \times X/\mathbb{G}_m})$:

$$\lambda^*\omega = z^m \,\mathrm{pr}_X^*\,\omega. \tag{5.2}$$

Here z denotes the coordinate on $\mathbb{G}_m$ and $\mathrm{pr}_X : \mathbb{G}_m \times X \to X$ the projection. For the duration of this section we shall assume that ω is of weight m with m invertible in k.

The $\mathbb{G}_m$-action on X defines a homomorphism from the Lie algebra of $\mathbb{G}_m$ to the Lie algebra of vector fields on X. Denote by θ the image of the generator of $\mathrm{Lie}\,\mathbb{G}_m$. Formula (5.2) together with the identity $d\omega = 0$ imply that

$$d\iota_\theta\omega = m\omega.$$

Hence, setting $\eta = (1/m)\iota_\theta\omega$, defines a restricted Poisson structure on X. Endow $X'[h] := X' \times \mathrm{Spec}\, k[h]$ with the $\mathbb{G}_m$-action given by the composition

$$\mathbb{G}_m \times X' \xrightarrow{F\times \mathrm{Id}} \mathbb{G}_m \times X' \xrightarrow{\lambda} X'$$

(where $F : \mathbb{G}_m \to \mathbb{G}_m$ is given by $F^*(z) = z^p$) on the first factor and by $h \mapsto z^m h$ on the second factor.

A $\mathbb{G}_m$-equivariant Frobenius-constant quantization of X consists of a $\mathbb{G}_m$-equivariant sheaf O_h of associative $\mathcal{O}_{X'[h]}$-algebras on $X'[h]$ together with an isomorphism of $\mathbb{G}_m$-equivariant $\mathcal{O}_{X'}$-algebras

$$O_h/(h) \xrightarrow{\sim} \mathcal{O}_X \tag{5.3}$$

such that O_h is locally free as an $\mathcal{O}_{X'[h]}$-module and the restriction $\mathcal{O}_h := \lim O_h/(h^n)$ of O_h to the formal completion of $X'[h]$ along the divisor $h = 0$ (equipped with the central homomorphism $s : \mathcal{O}_{X'}[[h]] \to \mathcal{O}_h$ and (5.3)) is a Frobenius-constant quantization of X compatible with the restricted Poisson structure given by the 1-form $\eta = (1/m)\iota_\theta\omega$.

For example, if X is affine, then a $\mathbb{G}_m$-equivariant Frobenius-constant quantization of X is determined by a graded $\mathcal{O}(X')[h]$-algebra $O_h(X'[h])$ (with $\deg h = m$) together with $O_h(X'[h])/(h) \xrightarrow{\sim} \mathcal{O}(X)$.

5.2 $\mathbb{A}^1$-action

In the following, we shall consider $\mathbb{G}_m$-actions on a scheme X satisfying the property that: morphism (5.2) extends to a morphism

$$\tilde{\lambda} : \mathbb{A}^1 \times X \to X. \tag{5.4}$$

If X is reduced and separated, which we shall assume to be the case for rest of this section, then $\tilde{\lambda}$ defines an action of the monoid $\mathbb{A}^1$ on X. In particular, the restriction of $\tilde{\lambda}$ to the closed subscheme $X \hookrightarrow \mathbb{A}^1 \times X$ given by the equation $z = 0$ factors through the subscheme $X^{\mathbb{G}_m} \hookrightarrow X$ of fixed points:

$$\tilde{\lambda}_0 : X \to X_m^{\mathbb{G}} \hookrightarrow X.$$

Moreover, $\tilde{\lambda}$ exhibits $X^{\mathbb{G}_m}$ as a $\mathbb{A}^1$-homotopy retract of X.

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Also note if X is a proper scheme with a nontrivial action of $\mathbb{G}_m$, then $\tilde{\lambda}$ does not exist. This can be seen by looking at the closure of a 1-dimensional $\mathbb{G}_m$-orbit in X.

5.3 Main result

By definition, a $\mathbb{G}_m$-equivariant Frobenius-constant quantization O_h gives rise to a Frobenius-constant quantization $\mathcal{O}_h$ and, thus, a class $\rho(O_h) \in H^1_{\text{et}}(X', \mathcal{O}^*_{X'}/\mathcal{O}^{*p}_{X'})$ (see § 1.4). Denote by $[\rho(O_h)] \in \text{Br}(X')$ the image of $\rho(O_h)$ under the homomorphism $H^1_{\text{et}}(X', \mathcal{O}^*_{X'}/\mathcal{O}^{*p}_{X'}) \to H^2_{\text{et}}(X', \mathcal{O}^*_{X'})$.

Recall from [OV07, § 4.2] a homomorphism

$$\Omega^1(X') \to \text{Br}(X'), \quad \eta \mapsto [\eta] \tag{5.5}$$

that carries a 1-form η to the class of the Azumaya algebra D_X restricted to the graph of the embedding $\Gamma_\eta : X' \to \mathbb{T}^*_{X'}$ given by η. Finally, denote by $O_{h=1}$ the sheaf of $\mathcal{O}_{X'}$-algebras $O_h/(h-1)$. The following result has been conjectured in [KT19, § 0.3, Question 2].

Theorem 6. *Let X be a smooth variety over k equipped with a $\mathbb{G}_m$-action (5.2) and a symplectic form ω of weight $m > 0$ relatively prime to the characteristic of k. Assume that the morphism (5.1) extends to a morphism (5.4). Then, for every $\mathbb{G}_m$-equivariant Frobenius-constant quantization O_h of X, the restriction of O_h to $X'[h, h^{-1}] = X' \times \text{Spec}\, k[h, h^{-1}] \subset X'[h]$ is an Azumaya algebra. Moreover, the following equality in $\text{Br}(X')$ holds:*

$$[O_{h=1}] = \left[\frac{1}{m}\iota_\theta\omega\right] + \tilde{\lambda}_0^*[\rho(O_h)].$$

Proof. To prove the Azumaya property of $O_h(h^{-1})$ consider the morphism

$$\psi : O_h \otimes_{\mathcal{O}_{X'[h]}} O_h^{\text{op}} \to \text{End}_{\mathcal{O}_{X'[h]}}(O_h).$$

This is a morphism of vector bundles over $X'[h]$ of the same rank. We have to prove that ψ is an isomorphism away from the divisor $h = 0$. Denote by

$$\det\psi : \wedge^{\text{top}}(O_h \otimes_{\mathcal{O}_{X'[h]}} O_h^{\text{op}}) \to \wedge^{\text{top}}(\text{End}_{\mathcal{O}_{X'[h]}}(O_h))$$

the determinant of ψ and by $Z \hookrightarrow X'[h]$. Let $(x_0, h_0) \in Z$ be a k-point of Z. We shall check that $h_0 = 0$. Using (5.4) and the positivity of m the $\mathbb{G}_m$-action on $X'[h]$ extends to a morphism

$$\tilde{\lambda}(h) : \mathbb{A}^1 \times X'[h] \to X'[h]. \tag{5.6}$$

It follows that the closure T of the $\mathbb{G}_m$-orbit of (x_0, h_0) intersects the divisor $h = 0$ at some point $(x_0', 0)$. Since Z is closed and $\mathbb{G}_m$-invariant we have that $T \subset Z$ i.e. $\det\psi$ is identically 0 on T. On the other hand, using the Azumaya property of the formal quantization $\mathcal{O}_h(h^{-1})$ we see that the restriction of ψ to the formal punctured neighborhood of $(x_0', 0) \in T$ is an isomorphism. This contradiction proves the first assertion of the theorem.

For the second assertion, consider algebra $D_{X,h}$ obtained from the filtered algebra of differential D_Y operators via the Rees construction (see § 1.2). The p-curvature homomorphism makes $D_{X,h}$ into an algebra over $S^\cdot T_{X'}[h]$. The graph $\Gamma_\eta : X' \to \mathbb{T}^*_{X'}$ of the differential form $\eta = (1/m)\iota_\theta\omega$ defines a sheaf of ideals $I_{\Gamma_\eta} \subset S^\cdot T_{X'}$. The quotient $D_{X,[\eta],h} = D_{X,h}/I_{\Gamma_\eta}$ can be viewed as a $\mathbb{G}_m$-equivariant sheaf of $\mathcal{O}_{X'[h]}$-algebras over $X'[h]$. By construction, the restriction of $D_{X,[\eta],h}$ to the formal completion of $X'[h]$ along the divisor $h = 0$ is isomorphic to the algebra $\mathcal{D}_{X,[\eta],h}$ constructed in § 1.5. Now given a $\mathbb{G}_m$-equivariant Frobenius-constant quantization O_h we consider the tensor product $O_h \otimes_{\mathcal{O}_{X'[h]}} D^{\text{op}}_{X,[\eta],h}$. Using Theorem 1 and the Beauville–Laszlo theorem [BL95] there exists a sheaf $O_h^\sharp$ of $\mathcal{O}_{X'[h]}$-algebras over $X'[h]$ whose restriction to

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$X'[h,h^{-1}]$ is $(O_h \otimes_{\mathcal{O}_{X'[h]}} D^{\mathrm{op}}_{X,[\eta],h})(h^{-1})$ and whose restriction to the formal completion of $X'[h]$ along the divisor $h=0$ is an Azumaya algebra. It follows that $O_h^\sharp$ is an Azumaya algebra over $X'[h]$. We claim that the following equality holds in $\mathrm{Br}(\mathbb{A}^1 \times X'[h])$:

$$\tilde{\lambda}(h)^*([O_h^\sharp]) = \mathrm{pr}^*_{X'[h]}([O_h^\sharp]). \tag{5.7}$$

Indeed, since $(O_h \otimes_{\mathcal{O}_{X'[h]}} D^{\mathrm{op}}_{X,[\eta],h})(h^{-1})$ is $\mathbb{G}_m$-equivariant the equality holds after the restriction to $\mathbb{G}_m \times X'[h,h^{-1}]$. Now the claim follows from the injectivity of the restriction morphism $\mathrm{Br}(\mathbb{A}^1 \times X'[h]) \to \mathrm{Br}(\mathbb{G}_m \times X'[h,h^{-1}])$. Restricting the classes in (5.7) to the divisor $X'[h] \xrightarrow{z=0} \mathbb{A}^1 \times X'[h]$ we find that

$$\tilde{\lambda}(h)_0^*([O_h^\sharp]) = [O_h^\sharp]. \tag{5.8}$$

Morphism $\tilde{\lambda}(h)_0^* : X'[h] \to X'[h]$ factors as follows:

$$X'[h] \xrightarrow{\mathrm{pr}_{X'}} X' \xrightarrow{\tilde{\lambda}(h)_0^*} X' \xrightarrow{h=0} X'[h].$$

By Theorem 1 the restriction of $[O_h^\sharp]$ to the divisor $X' \xrightarrow{h=0} X'[h]$ is equal to $[\rho(O_h)]$. Using (5.8) and restricting to the divisor $h=1$ we find that

$$[O_{h=1}^\sharp] = [\rho(O_h)]$$

as desired. □

Acknowledgements

This paper has grown out of our attempt to correct an error in [BK08, Proposition 1.24]. This proposition asserts, in particular, that the algebra $\mathcal{O}_h(h^{-1})$ extends to an Azumaya algebra over the formal scheme $X'[[h]]$. In fact, the result stated in [BK08, Proposition 1.24] is similar to our formula (1.9) with the exception that the class $[\mathcal{O}_h^\sharp]$ at the left-hand side of (1.9) is replaced by the class of an extension of $\mathcal{O}_h(h^{-1})$ to $X'[[h]]$. In particular, [BK08, Proposition 1.24] asserts that the Azumaya algebra $\mathcal{O}_h(h^{-1})$ corresponding to the $\mathbb{Z}/2\mathbb{Z}$-equivariant Frobenius-constant quantization splits which is definitely not the case since the algebra $\mathcal{O}_h$ has no zero divisors. In § 3, Remark 3.8, we indicate where the error in the proof of [BK08, Proposition 1.24] is. In particular, we will see that $\mathcal{O}_h(h^{-1})$ never extends to an Azumaya algebra over $X'[[h]]$.

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Conflicts of Interest

None.

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Appendix A

A.1 The affine grassmannian for $\mathbb{G}_m$

Let G be an algebraic group over a field k. Denote by

$$LG : \mathrm{Aff}_k^{\mathrm{op}} \to \mathrm{Groups}$$

the corresponding loop group, that is a sheaf of groups on the category of affine schemes over k equipped with the *fpqc* topology sending k-scheme $\operatorname{Spec} R$ to $G(R((h)))$. Also, let

$$L^+G : \mathrm{Aff}_k^{\mathrm{op}} \to \mathrm{Groups}$$

be the sheaf of groups sending k-scheme $\operatorname{Spec} R$ to $G(R[[h]])$. It is known (see e.g. [Zhu17, Proposition 1.3.2]) that L^+G is represented by a group scheme over k and that LG is an ind-affine scheme. Denote by Gr_G the affine grassmannian for G. By definition, Gr_G is the *fpqc* sheaf associated to the presheaf $R \mapsto LG(R)/L^+G(R)$.

Recall the structure of the affine grassmannian for $\mathbb{G}_m$. The following result is well-known (see, for example, [Con94]); for the reader's convenience we include its proof.

LEMMA A.1. *For a commutative ring R such that $\operatorname{Spec} R$ is connected there is a decomposition*

$$R((h))^* = R^* \times \mathbb{W}(R) \times \mathbb{Z} \times \hat{\mathbb{W}}(R),$$

where $\mathbb{W}(R)$ is the subgroup of $R[[h]]^*$ formed by formal power series with constant term 1, $\hat{\mathbb{W}}(R)$ is the group of polynomials of the form $1 + \Sigma a_i h^{-i}$ with nilpotent coefficients $a_i \in R$. In addition, we have that

$$\hat{\mathbb{W}}(R) = \ker(R[h^{-1}]^* \xrightarrow{f \mapsto f(0)} R^*).$$

Proof. The claim follows from the fact that under the assumptions of the lemma

$$R((h))^* = \left\{ \sum_i a_i h^i \in R((h)); \exists i_0 : a_{i_0} \in R^*, a_j \text{ nilpotent for all } j < i_0 \right\}.$$

To show this replace R by $R/\mathfrak{N}_R$, where $\mathfrak{N}_R$ stands for the nilradical. We need to show that a Laurent polynomial is invertible if and only if its first nonzero coefficient is invertible in R. Suppose that

$$A(h)B(h) = 1 \tag{A.1}$$

for $A(h), B(h) \in R((h))$ such that

$$\begin{aligned} A(h) &= a_{-N}h^{-N} + a_{-N+1}h^{-N+1} + \cdots, \\ B(h) &= b_{-N}h^{-M} + b_{-M+1}h^{-M+1} + \cdots, \end{aligned} \tag{A.2}$$

where $b_{-M} \neq 0$ and $a_{-N} \neq 0$. From (A.1) we have that $N + M \geq 0$. If $N + M = 0$, then $a_{-N}b_{-M} = 1$ and we are done. Otherwise, we have from (A.1)

$$a_{-N}b_{-M} = 0, \quad a_{-N}b_{-M+1} + a_{-N+1}b_{-M} = 0, \ldots \tag{A.3}$$

$$a_{-N}b_{-M+(N+M)} + \cdots + a_{-N+(N+M)}b_{-M} = 1. \tag{A.4}$$

Using (A.3) we get $a_{-N}^2 b_{-M+1} = 0$ and similarly $a_{-N}^i b_{-M+i} = 0$ for every $i \leq N + M$. Multiplying both sides of (A.4) by a_{-N}^{N+M} we infer

$$a_{-N}^{N+M}(1 - a_{-N}b_{-M+(N+M)}) = 0.$$

Since a_{-N} and $1 - a_{-N}b_{-M+(N+M)}$ are coprime,

$$R \xrightarrow{\sim} R/(a_{-N}) \times R/(1 - a_{-N}b_{-M+(N+M)}),$$

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Spec R is connected, and $a_{-N} \neq 0$ we conclude that $1 - a_{-N} b_{-M+(N+M)} = 0$. Hence, a_{-N} is invertible as desired. □

Using the lemma, we have decompositions

$$L\mathbb{G}_m \xrightarrow{\sim} \mathbb{G}_m \times \mathbb{W} \times \underline{\mathbb{Z}} \times \hat{\mathbb{W}},$$

$$\mathrm{Gr}_{\mathbb{G}_m} \xrightarrow{\sim} \underline{\mathbb{Z}} \times \hat{\mathbb{W}},$$

where $\mathbb{W}$ is the group scheme of big Witt vectors and $\hat{\mathbb{W}}$ is a group ind-scheme whose group of R-points is defined in the lemma.

A.2 Subgroups of $L\,\mathrm{GL}(n)$

Proposition A.2. *Let G be an affine group scheme over a field k, and let $\phi : G \to L\,\mathrm{GL}(n)$ be a homomorphism. Then there exists an element $g \in \mathrm{GL}(n, k((h)))$ such that ϕ factors through $gL^+\,\mathrm{GL}(n)g^{-1}$:*

$$G \xrightarrow{\phi} g(L^+\,\mathrm{GL}(n))g^{-1} \hookrightarrow L\,\mathrm{GL}(n).$$

Proof. Set $V = k^n$. We have to show that there exists a $\phi(G)$-invariant $k[[h]]$-lattice

$$\Lambda \subset V((h)).$$

Informally, our Λ will be constructed starting with the lattice $\Lambda_0 = V[[h]]$ as the intersection $\bigcap_g g\Lambda_0$. Since we make no assumptions on k and G one has give a meaning to the latter. We shall do it as follows.

The morphism ϕ is given by a matrix $A \in \mathrm{GL}(n, \mathcal{O}(G)((h)))$ such that

$$A \otimes A = \Delta(A) \in \mathrm{GL}(n, (\mathcal{O}(G) \otimes \mathcal{O}(G))((h))), \tag{A.5}$$

where $\Delta : \mathcal{O}(G) \to \mathcal{O}(G) \otimes \mathcal{O}(G)$ is the comultiplication on $\mathcal{O}(G)$ given by the product morphism $G \times G \to G$ and such that the image of A under the evaluation at $1 \in G(k)$ homomorphism $\mathrm{GL}(n, \mathcal{O}(G)((h))) \to \mathrm{GL}(n, k((h)))$ is the identity matrix.

Set

$$\Lambda = \{v \in V((h)) \text{ such that } Av \in V \otimes_k \mathcal{O}(G)[[h]]\}.$$

Then Λ is a $k[[h]]$-submodule of $V[[h]]$ contains $h^N V[[h]]$, for sufficiently large N. Hence, Λ is a lattice. It remains to show that Λ is $\phi(G)$-invariant, that is

$$A(\Lambda) \subset \Lambda \otimes_{k[[h]]} \mathcal{O}(G)[[h]].$$

The matrix A defines $\mathcal{O}(G)((h))$-linear maps

$$\begin{aligned} V((h)) \otimes_{k[[h]]} \mathcal{O}(G)[[h]] \xrightarrow{A\otimes \mathrm{Id}} (V((h)) \otimes_{k[[h]]} \mathcal{O}(G)[[h]]) \otimes_{k[[h]]} \mathcal{O}(G)[[h]] \to \\ \to V((h)) \otimes_{k[[h]]} (\mathcal{O}(G) \otimes_k \mathcal{O}(G))[[h]], \end{aligned} \tag{A.6}$$

where the second map in (A.6) is induced by the embedding

$$\mathcal{O}(G)[[h]] \otimes_{k[[h]]} \mathcal{O}(G)[[h]] \to (\mathcal{O}(G) \otimes_k \mathcal{O}(G))[[h]]. \tag{A.7}$$

Since the cokernel of (A.7) and $\mathcal{O}(G)[[h]]$ are both flat $k[[h]]$-modules it follows that $\Lambda \otimes_{k[[h]]} \mathcal{O}(G)[[h]]$ is precisely the preimage of $V[[h]] \otimes_{k[[h]]} (\mathcal{O}(G) \otimes_k \mathcal{O}(G))[[h]]$ under the composition (A.6).

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Hence, it suffices to check that (A.6) carries $A(\Lambda)$ to $V[[h]] \otimes_{k[[h]]} (\mathcal{O}(G) \otimes_k \mathcal{O}(G))[[h]]$. But the composition

$$V((h)) \xrightarrow{A} V((h)) \otimes_{k[[h]]} \mathcal{O}(G)[[h]] \xrightarrow{(A.6)} V((h)) \otimes_{k[[h]]} (\mathcal{O}(G) \otimes_k \mathcal{O}(G))[[h]] \tag{A.8}$$

is equal to

$$V((h)) \xrightarrow{A} V((h)) \otimes_{k[[h]]} \mathcal{O}(G)[[h]] \xrightarrow{Id \otimes \Delta} V((h)) \otimes_{k[[h]]} (\mathcal{O}(G) \otimes_k \mathcal{O}(G))[[h]]$$

by (A.5). Hence, it carries Λ to $V[[h]] \otimes_{k[[h]]} (\mathcal{O}(G) \otimes_k \mathcal{O}(G))[[h]]$ and we win. □

COROLLARY A.3. *There are no nontrivial homomorphisms from an affine group scheme to* $L\mathbb{G}_m/L^+\mathbb{G}_m$.

A.3 Subgroups of $L\,\mathrm{PGL}(n)$

Remark A.4. The analogous assertion for $L\,\mathrm{PGL}(n)$ does not hold.

Consider the homomorphism of loop groups *fpqc* sheaves

$$L\,\mathrm{GL}(n) \to L\,\mathrm{PGL}(n) \tag{A.9}$$

induced by the projection $\mathrm{GL}(n) \to \mathrm{PGL}(n)$. We do not know if (A.9) is surjective as a morphism of *fpqc* sheaves. However, we shall see below that (A.9) is surjective over any affine group subscheme of $L\,\mathrm{PGL}(n)$ of finite type over k. For our applications we need a bit more general statement.

Recall that an affine group scheme H over a perfect field k is said to be pro-unipotent if there exists a filtration

$$\cdots \subset H^{\geq i} \subset \cdots \subset H^{\geq 1} = H$$

by normal group subschemes such that

$$H \xrightarrow{\sim} \varprojlim H/H^{\geq i}$$

and every quotient $H/H^{\geq i}$ is unipotent (i.e. has a finite composition series with all quotient groups isomorphic to the additive group $\mathbb{G}_a$).

PROPOSITION A.5. *Let G be an affine group scheme over a perfect field k, and let $\phi : G \to L\,\mathrm{PGL}(n)$ be a homomorphism. Assume that G has a normal pro-unipotent group subscheme $G^{\geq 1} \subset G$ such that $\phi(G^{\geq 1}) \subset L^+\,\mathrm{PGL}(n)$ and the quotient $G_0 = G/G^{\geq 1}$ has finite type over k. Then the following assertions hold.*

(i) *The morphism of fpqc sheaves $\tilde{G} := G \times_{L\,\mathrm{PGL}(n)} L\,\mathrm{GL}(n) \to G$ given by the projection to the first coordinate is surjective for the Zariski topology on G (and, consequently, for the fpqc topology).*

(ii) *The following two conditions are equivalent.*

(1) *There exists an element $g \in \mathrm{PGL}(n, k((h)))$ such that ϕ factors through $gL^+\,\mathrm{PGL}(n)g^{-1}$:*

$$G \xrightarrow{\phi} g(L^+\,\mathrm{PGL}(n))g^{-1} \hookrightarrow L\,\mathrm{PGL}(n).$$

(2) *The extension*

$$1 \to L\mathbb{G}_m \to \tilde{G} \to G \to 1 \tag{A.10}$$

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admits a reduction $\tilde{G}^+$ *to* $L^+\mathbb{G}_m$ *as follows.*

$$\begin{array}{ccccccccc} 1 & \to & L^+\mathbb{G}_m & \to & \tilde{G}^+ & \to & G & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{Id}} & & \\ 1 & \to & L\mathbb{G}_m & \to & \tilde{G} & \to & G & \to & 1 \end{array}$$

(iii) *Assume that* G_0 *is smooth and connected. Then extension (A.10) admits a unique reduction* $\tilde{G}^+$ *to* $L^+\mathbb{G}_m$.

Proof. For part (i) observe that the morphism of schemes

$$L^+\,\mathrm{GL}(n) \to L^+\,\mathrm{PGL}(n)$$

admits a section locally for the Zariski topology on $L^+\,\mathrm{PGL}(n)$. Also since every $G^{\geq 1}$-torsor over an affine scheme is trivial the projection

$$G \twoheadrightarrow G_0$$

admits a scheme-theoretic section $s : G_0 \to G$. Hence, it suffices to check that the composition $G_0 \xrightarrow{s} G \xrightarrow{\phi} L\,\mathrm{PGL}(n)$ lifts locally for the Zariski topology on G_0 to scheme-theoretic morphism $G_0 \to L\,\mathrm{GL}(n)$. Set $G_0 = \operatorname{Spec} R$. Then $\phi \circ s$ defines a morphism

$$\operatorname{Spec} R((h)) \to \mathrm{PGL}(n). \tag{A.11}$$

The pullback of the $\mathbb{G}_m$-torsor $\mathrm{GL}(n) \to \mathrm{PGL}(n)$ defines a $\mathbb{G}_m$-torsor L over $\operatorname{Spec} R((h))$. Observe that $\phi \circ s$ admits a lifting to $L\,\mathrm{GL}(n)$ if and only if L is trivial. Thus, to complete the proof of part (i) we have to show that there exists an affine open covering $\operatorname{Spec} R = \cup U_i$ such that the pullback of L to $\operatorname{Spec} \mathcal{O}(U_i)((h))$ is trivial for every i. We shall prove a stronger assertion: the morphism $\operatorname{Spec} R((h)) \to \operatorname{Spec} R$ induces an isomorphism

$$\mathrm{Pic}(R) \xrightarrow{\sim} \mathrm{Pic}(R((h))). \tag{A.12}$$

Since G_0 is a group scheme and k is perfect, the reduction R_{red} is smooth over k. Since R is a finitely generated k-algebra, the kernel of the projection $R \to R_{\mathrm{red}}$ is a nilpotent ideal. It follows that $(R((h))_{\mathrm{red}} \xrightarrow{\sim} R_{\mathrm{red}}((h))$. Consequently, we have that

$$\mathrm{Pic}(R) \xrightarrow{\sim} \mathrm{Pic}(R_{\mathrm{red}}), \quad \mathrm{Pic}(R((h))) \xrightarrow{\sim} \mathrm{Pic}(R_{\mathrm{red}}((h))).$$

Next, using regularity of $R_{\mathrm{red}}((h))$ we conclude that

$$\mathrm{Pic}(R_{\mathrm{red}}((h))) \cong \mathrm{Cl}(R_{\mathrm{red}}((h))) \cong \mathrm{Cl}(R_{\mathrm{red}}[[h]]) \cong \mathrm{Pic}(R_{\mathrm{red}}[[h]]) \cong \mathrm{Pic}(R_{\mathrm{red}}).$$

This proves part (i).

For part (ii), let $\tilde{G}^+$ be a reduction of $\tilde{G}$ to $L^+\mathbb{G}_m$. Since $L^+\mathbb{G}_m$ is an affine group scheme (as opposed to merely a group ind-scheme) $\tilde{G}^+$ is also an affine group scheme. Applying Proposition A.2 we conclude that the homomorphism $\tilde{G}^+ \to L\,\mathrm{GL}(n)$ factors through $gL^+\,\mathrm{GL}(n)g^{-1}$, for some $g \in \mathrm{GL}(n, k((h)))$. Hence, $G \to L\,\mathrm{PGL}(n)$ factors through $gL^+\,\mathrm{PGL}(n)g^{-1}$. The inverse implication is clear.

Finally, for part (iii), set $\bar{G} := \tilde{G}/L^+\mathbb{G}_m$. We have to show that the central extension

$$\mathrm{Gr}_{\mathbb{G}_m} \to \bar{G} \to G$$

admits a unique splitting. We shall first construct a scheme-theoretic section of the projection $\bar{G} \to G$. Using part (i) there exists an open cover $G = \cup U_i$ and sections $s_i : U_i \to \tilde{G}$ of the projection $\tilde{G} \to G$. Let $\bar{s}_i : U_i \to \bar{G}$ be the composition of s_i with the quotient map $\tilde{G} \to \bar{G}$.

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Since G is reduced the morphisms

$$\bar{s}_i\bar{s}_j^{-1} : U_i \cap U_j \to \mathrm{Gr}_{\mathbb{G}_m} = \hat{\mathbb{W}} \times \mathbb{Z}$$

lands at the second factor. Hence the collection $\{\bar{s}_i\bar{s}_j^{-1}\}$ defines a Čech 1-cocycle for the constant sheaf $\mathbb{Z}$ on G. Since G is irreducible, we have that $\check{H}^1(G,\mathbb{Z}) = 0$. Thus, we have a global scheme-theoretic section $\bar{s} : G \to \bar{G}$ of the projection $\bar{G} \to G$. We claim that every such section satisfying $\bar{s}(1) = 1$ is a group homomorphism. To see this it suffices to show that the following diagram is commutative.

$$\begin{array}{ccc} G \times G & \xrightarrow{m} & G \\ \downarrow{\scriptstyle \bar{s}\times\bar{s}} & & \downarrow{\scriptstyle \bar{s}} \\ \bar{G} \times \bar{G} & \xrightarrow{\bar{m}} & \bar{G} \end{array}$$

In turn, this follows from the fact that every scheme-theoretic morphism from a connected reduced scheme to $\mathrm{Gr}_{\mathbb{G}_m}$ is constant. □

A.4 A representation of $\mathfrak{sp}(2n)$

In this subsection we prove irreducibility of a certain representation of the Lie algebra $\mathfrak{sp}(2n)$ that we used in the proof of Lemma 4.3. We use notation from § 4.1.

LEMMA A.6. *For every integer l with $0 \le l < 2(p-1)$, the adjoint representation of the Lie algebra $\mathfrak{sp}(2n) = m^2/m^3$ on m^l/m^{l+1} is irreducible.*

Proof. Write m_n^l/m_n^{l+1} for m^k/m^{k+1}. It is easy to verify the assertion of the lemma for $n = 1$: in fact, the representation of $\mathfrak{sp}(2) = m_1/m_1^2$ on m_1^l/m_1^{l+1} is irreducible for every $l \ge 0$. Moreover, the representations m_1^l/m_1^{l+1} and $m_1^{l'}/m_1^{l'+1}$ are isomorphic if and only if $l + l' = 2p - 2$.

To prove the lemma in general, consider the restriction of the representation of $\mathfrak{sp}(2n)$ on m_n^l/m_n^{l+1} to the Lie subalgebra

$$\mathfrak{sp}(2)^{\oplus n} \hookrightarrow \mathfrak{sp}(2n)$$

of the block diagonal matrices. The latter representation decomposes as follows:

$$m_n^l/m_n^{l+1} = \bigoplus_{i_1+\cdots+i_n=l} m_1^{i_1}/m_1^{i_1+1} \otimes \cdots \otimes m_1^{i_n}/m_1^{i_n+1}. \tag{A.13}$$

By the Jacobson density theorem the representation of $\mathfrak{sp}(2)^{\oplus n}$ on each summand is irreducible. Moreover, if $l < 2(p-1)$, then these direct summands are pairwise nonisomorphic. It follows that any subspace $V \subset m_n^l/m_n^{l+1}$ invariant under the $\mathfrak{sp}(2)^{\oplus n}$-action is the sum of *some* of the summands appearing in (A.13). Hence, it suffices to prove that if a $\mathfrak{sp}(2n)$-subrepresentation $W \subset m_n^l/m_n^{l+1}$ contains $m_1^{i_1}/m_1^{i_1+1} \otimes \cdots \otimes m_1^{i_n}/m_1^{i_n+1}$, for some partition $(i_1,\dots,i_n)$ of l with $i_1 > 0$, the projection of W to $m_1^{i_1-1}/m_1^{i_1} \otimes m_1^{i_2+1}/m_1^{i_2+2} \otimes \cdots \otimes m_1^{i_n}/m_1^{i_n+1}$ is nonzero. This reduces the proof to the case $n = 2$ which is shown by direct inspection. □

REFERENCES

BL95 A. Beauville and Y. Laszlo, *Un lemme de descente*, C. R. Acad. Sci. **320** (1995), 335–340.

BB07 R. Bezrukavnikov and A. Braverman, *Geometric Langlands correspondence for D-modules in prime characteristic: the* GL(n) *case*, Pure Appl. Math. Q. **3** (2007), 153–179; Special Issue: In honor of Robert D. MacPherson. Part 3.

BF14 R. Bezrukavnikov and M. Finkelberg, *Wreath Macdonald polynomials and categorical McKay correspondence*, Camb. J. Math. **2** (2014), 163–190.

https://doi.org/10.1112/S0010437X23007601 Published online by Cambridge University Press

BK04a R. Bezrukavnikov and D. Kaledin, *Fedosov quantization in algebraic context*, Mosc. Math. J. **4** (2004), 559–592.

BK04b R. Bezrukavnikov and D. Kaledin, *McKay equivalence for symplectic resolutions of singularities*, Proc. Steklov Inst. Math. **246** (2004), 13–33.

BK08 R. Bezrukavnikov and D. Kaledin, *Fedosov quantization in positive characteristic*, J. Amer. Math. Soc. **21** (2008), 409–438.

BL21 R. Bezrukavnikov and I. Losev, *Etingof conjecture for quantized quiver varieties*, Invent. Math. **223** (2021), 1097–1226.

BMR02 R. Bezrukavnikov, I. Mirkovic and D. Rumynin, *Localization of modules for a semisimple Lie algebra in prime characteristic*, Ann. of Math. (2) **167** (2002), 945–991.

BKTV22 E. Bogdanova, D. Kubrak, R. Travkin and V. Vologodsky, *The canonical global quantization of symplectic varieties in characteristic p*, Preprint (2022), arXiv:2211.17261.

Bor91 A. Borel, *Linear Algebraic groups*, Graduate Texts in Mathematics, vol. 126 (Springer, New York, 1991).

Con94 C. Contou-Carrere, *Jacobienne locale, groupe de bivecteurs de Witt universel, et symbole modere*, C. R. Math. Acad. Sci. Paris **318** (1994), 743–746.

DG80 M. Demazure and P. Gabriel, *Introduction to algebraic geometry and algebraic groups*, North-Holland Mathematics Studies, vol. 39 (North-Holland, 1980).

EGA III A. Grothendieck and J. Dieudonne, *Éléments de géométrie algébrique. III. Étude cohomologique des faisceaux cohérents, Première partie. Étude globale élémentaire de quelques classes de morphismes*, Publ. Math. Inst. Hautes Études Sci. **8** (1961), 222.

Kal08 D. Kaledin, *Derived equivalences by quantization*, Geom. Funct. Anal. **17** (2008), 1968–2004.

Kat72 N. Katz, *Algebraic solutions of differential equations (p-curvature and the Hodge filtration)*, Invent. Math. **18** (1972), 1–118.

KT19 D. Kubrak and R. Travkin, *Resolutions with conical slices and descent for the Brauer group classes of certain central reductions of differential operators in characteristic p*, Int. Math. Res. Not. IMRN **2019** (2019), rnz169.

Mil80 J. Milne, *Étale cohomology*, Princeton Mathematical Series, vol. 33 (Princeton University Press, 1980).

Mol17 S. Molokov, *Quantizations in characteristic p*, Bachelor's thesis, HSE University (2017).

Mun22 J. Mundinger, *Quantization of restricted Lagrangian subvarieties in positive characteristic*, Adv. Math. **410** (2022), Paper No. 108760.

OV07 A. Ogus and V. Vologodsky, *Nonabelian Hodge theory in characteristic p*, Publ. Math. Inst. Hautes Études Sci. **106** (2007), 1–138.

Sta The Stacks Project Authors. *Stacks Project*, https://stacks.math.columbia.edu.

Zhu17 X. Zhu, *An introduction to affine Grassmannians and the geometric Satake equivalence*, in *Geometry of moduli spaces and representation theory*, IAS/Park City Mathematics Series, vol. 24 (American Mathematical Society, Providence, RI, 2017), 59–154.

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