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LETTER TO THE EDITOR

THE SURVIVAL OF VARIOUS INTERACTING PARTICLE SYSTEMS

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Abstract

Particles may be removed from a lattice by murder, coalescence, mutual annihilation and simple death. If the particle system is not to die out, the removed particles must be replaced by births. This letter shows that coalescence can be counteracted by arbitrarily small birth-rates and contrasts this with the situations for annihilation and pure death where there are critical phenomena. The problem is unresolved for murder.

INFINITE PARTICLE SYSTEM; ANNIHILATION; CONTACT PROCESS

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1. Introduction

In this letter we are concerned with several systems of particles on the d-dimensional lattice $\mathbb{Z}^d$. Each site is occupied by at most one particle. There are several ways in which particles can die leaving a site unoccupied. The simplest is for a particle to die at rate μ independently of all other particles. Other systems that have been considered involve the interaction of a particle with one of its $2d$ neighbours. A particle may murder one of its neighbours. It may jump to an occupied neighbouring site and the two particles coalesce (the coalescing random walk (CRW)), or, instead, both particles may be annihilated (the annihilating random walk (ARW)). It is not surprising that, if these are the only form of interactions, eventually each particle dies. The rates at which such systems thin out have been calculated in various papers (Bramson and Griffeath (1980) and Arratia (1981), for example).

If the systems are to achieve some sort of non-trivial equilibrium then there must be some mechanism for births to occur. The simplest is to allow spontaneous births at unoccupied sites. If the maximum possible death rate at a site is μ and the birth-rate is λ then as $t \to \infty$ the probability that a site is occupied must be greater than λ/μ. Even in cases where limiting measures are not known, the probabilities of individual sites and clusters of sites being occupied can be quite accurately estimated (Clifford and Sudbury (1979)).

The most fascinating problems arise when births can only occur when there are particles present on the lattice, because in these cases it is normally not clear whether the system can survive. Very often there is some critical value of the birth-rate above which survival is possible and below which extinction is certain. The purpose of this letter is to draw together some of the results in this field and to provide some new results as well. Having done this, there will emerge a sort of ordering among ways of death from the mildest to the most severe.

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2. Results for four processes

In each of these models the mechanism for a birth will be the same. An unoccupied site will become occupied at rate λ times the number of occupied neighbouring sites. We consider four models.

2.1. *The contact process* (pure death). Each particle dies at rate 1 (with exponential holding time 1). This has been one of the most studied of all interacting particle systems since it was first introduced by Harris (1974). It is known that a critical value, λ_c, exists such that the process dies out when $\lambda < \lambda_c$ and persists when $\lambda > \lambda_c$. Holley and Liggett (1978) showed that $\lambda_c < 2$ in 1-d and simulations suggest it is close to 1·66. Results for higher dimensions include $\lambda_c^{(d)} < 2/d$ and $\lambda_c^{(d)} > 1/(2d-1)$.

2.2. *The branching annihilating random walk* (BARW). As time evolves, particles execute independent random walks at rate $2d$ and annihilate each other when they meet. The BARW is the ARW with births. Results are incomplete for this process but in 1-d it is known that for $\lambda > 100$ survival is possible, and, for sufficiently small λ, extinction is certain (Bramson and Gray (1985)).

2.3. *The biased annihilating branching process* (BABP). In this process each particle murders each of its neighbours at rate 1. Since suicide is not an option the system cannot die out entirely, but it is possible that in the long run the density of particles could tend to 0. Combined results of Neuhauser and Sudbury (1993) and Mountford (1993) show that in 1-d if $\lambda > \frac{1}{3}$ and initially there are a finite non-zero number of particles then the limiting measure is the product measure $\nu_{\lambda/(1+\lambda)}$. It is conjectured that this result will be true for any $\lambda > 0$.

$\nu_{\lambda/(1+\lambda)}$ is an invariant measure in any dimension for all $\lambda > 0$.

2.4. *The branching coalescing walk* (BCRW). As time evolves, particles execute independent random walks at rate $2d$ and coalesce when they meet. Now, if a particle at x jumps to $x+1$ and coalesces with the particle there, then this looks the same as the particle at $x+1$ murdering the particle at x. Thus the BCRW is just the BABP with particles allowed to move. It might be expected that this movement allows the particles to spread and thus makes survival of the BCRW easier than for the BABP. The easiness may lie only in the proof. As will be shown in the next section, as long as the initial configuration is non-empty, the limiting measure of a BCRW is the product measure $\nu_{\lambda/(1+\lambda)}$. This constitutes the chief new result of this letter.

3. The limiting behaviour of the BCRW

In this section we exploit the well-known duality between the BCRW and the biased voter model. Let ξ_t^A be the set of occupied sites at time t in the biased voter model, then

(i) if $x \in \xi_t^A$ then x becomes vacant at a rate equal to the number of vacant neighbours;

(ii) if $x \notin \xi_t^A$ then x becomes occupied at a rate $\lambda + 1$ times the number of occupied neighbours.

In this process interactions only occur when an occupied site is adjacent to an unoccupied site. If we consider a finite configuration of occupied cells, then the probability that the next interaction will result in the appearance of a new occupied site rather than a death is $(\lambda+1)/(\lambda+2)$. Thus the probability that a finite configuration with n occupied sites will die out is the same as that a random walk starting at n, and making jumps of $+1$ and -1 with probabilities $(\lambda+1)/(\lambda+2)$ and $1/(\lambda+2)$ respectively, will ever reach 0. This is well known to be $(\lambda+1)^{-n}$. If the random walk does not reach 0 then it almost surely drifts to ∞. So, given any finite set of sites, the probability it is empty as $t \to \infty$ is $(\lambda+1)^{-\|A\|}$ where $\|A\|$ is the number of particles in A, the initial finite configuration.

The duality we need can be found in Durrett (1988), p. 43, for example. If we designate the configuration at time t of a BCRW with initial configuration B as η_t^B then

$$\boldsymbol{P}(\xi_t^A \cap B = \varnothing) = \boldsymbol{P}(\eta_t^B \cap A = \varnothing).$$

If a biased voter model does not die out then it fills all space. Thus, if A is a finite set, and B is non-empty,

$$\lim_{t \to \infty} \boldsymbol{P}(\eta_t^B \cap A = \varnothing) = \left(\frac{1}{\lambda + 1}\right)^{\|A\|}.$$

Since this is true for any finite set we have the following result.

Theorem. In the branching coalescing random walk in d dimensions with birth-rate λ, the limiting measure is $\nu_{\lambda/(1+\lambda)}$ for any non-empty initial configuration.

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