by this formula, will be in the same ratio to the theoretical amount that the net premium is to the premium charged; but this will not be so when, as is often the case, the premiums are formed in some other way.

I remain, Sir,

Your obedient servant,

T. B. SPRAGUE.

Liverpool and London Insurance Company,
20, *Poultry, London,*
4*th December,* 1858.

FORMULÆ FOR THE PREMIUM FOR A TERM ASSURANCE ON TWO JOINT LIVES.

To the Editor of the Assurance Magazine.

Sir,—In the last Number of your *Journal*, your talented correspondent, Mr. T. B. Sprague, remarks, that the formulæ given by him for the single and annual premium of a temporary assurance on the joint lives of two persons may probably be *new* to the readers of the *Assurance Magazine*, as they are not given in David Jones' *Treatise on Annuities*, or in any other work with which he is acquainted. It is true that the *commutation* formulæ for joint lives do not appear in the first volume of Jones, but your readers will find them capitulated on pages 5 and 6 of the second volume of that work, also in a paper on "Life Contingencies," by Professor De Morgan, in the *Companion to the British Almanack*, for the year 1842, where the subject is treated in a very elegant and masterly style.

I know from experience that the formulæ in question do not frequently occur in practice; but it is well to know how to investigate them, inasmuch that the exact expressions are always preferable to the approximate methods, when, as in the present case, the results are not too complicated for practical use.

The mode of investigation adopted by Mr. Sprague, in your last Number, is very neat and concise, but as that method is based upon Jones' formulæ,

$$(\mathrm{A})_{\overline{t}|}=r\left\{1+(a)_{\overline{t-1}|}\right\}-(a)_{\overline{t}|},$$

the demonstration of which may be considered the most difficult part of the question, it has occurred to me that the following solution, investigated from first principles, may be acceptable to your readers.

I regret that my time has been so much occupied during the last quarter as to prevent me from continuing my paper on "Finite Differences" in the present Number; but I will, if possible, resume the subject in the *Journal* for April next.

I am, Sir,

Your obedient servant,

WM. CURTIS OTTER, F.R.A.S.

Problem.

Required the single and annual premium for the assurance of £1 *payable on the death of the first of two lives,* A *and* B, *aged x and y, provided such death takes place before the expiration of t years.*

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SOLUTION.

The payment of the £1 will evidently depend upon the probability of one or other or both of the given lives failing within t years.

Now, the probability that one or other or both of the lives are dead at the end of the nth year from the present time is equal to the probability of both the lives being in existence at the beginning of the nth year, *minus* the probability of their both being alive at the end of the nth year, the expression for which is

$$(p_{x.y,\,n-1} - p_{x.y,\,n}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Discounting this probability, and integrating the result between the limits assigned by the tables of mortality used, we get

$$\sum_{n=z-x.}^{n=1}(r.p_{x.y,\,n-1}) - \sum_{n=z-x.}^{n=1}(r.p_{x.y,\,n})$$

$$= \frac{a_{x-1,\,y-1}}{p_{x-1,\,y-1,\,1}} - a_{x\,y} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and, similarly,

$$\sum_{n=z-x.}^{n=t}(r.p_{x\,y,\,n-1}) - \sum_{n=z-x.}^{n=t}(r.p_{x.y,\,n})$$

$$= r^t.p_{x.y,\,t}\left\{\frac{a_{x+t,\,y+t}}{p_{x+t,\,y+t,\,1}} - a_{x+t,\,y+t}\right\} \quad . \quad . \quad (3)$$

but, from the construction of the annuity tables, we have

$$a_{x-1,\,y-1} = r.p_{x-1,\,y-1,\,1}\,(1 + a_{x.y}),$$

therefore, by substitution in (2), we get

$$r(1 + a_{x.y}) - a_{x.y} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

or, $$r\left(1 + \frac{N_{x.y}}{D_{x.y}}\right) - \frac{N_{x.y}}{D_{x.y}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

or, $$\frac{r.N_{x-1,\,y-1} - N_{x,\,y}}{D_{x.y}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

similarly (3) becomes

$$r^t.p_{x.y,\,t}\left\{r(1 + a_{x+t,\,y+t}) - a_{x+t,\,y+t}\right\} \quad . \quad . \quad . \quad . \quad . \quad (7)$$

or, $$\frac{D_{x+t,\,y+t}}{D_{x.y}}\left\{r\left(1 + \frac{N_{x+t,\,y+t}}{D_{x+t,\,y+t}}\right) - \frac{N_{x+t,\,y+t}}{D_{x+t,\,y+t}}\right\}. \quad . \quad . \quad . \quad (8)$$

or, $$\frac{r.N_{x+t-1,\,y+t-1} - N_{x+t,\,y+t}}{D_{x.y}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Formulæ (2), (4), (5), and (6), represent the present value of £1 payable on the death of the first of two lives aged x and y.

Also, (3), (7), (8), and (9), represent the present value of £1 payable on the death of the first of two lives, provided that such death takes place after the expiration of t years.

Hence the difference between any corresponding pairs of these expressions is the present value of £1 *payable on the death of the first of the two lives, provided such death occurs before the end of t years.*

$$=\frac{r(N_{x-1,\,y-1}-N_{x+t-1,\,y+t-1})-N_{x.y}-N_{x+t,\,y+t}}{D_{x\,y}} \quad . \quad (10)$$

or, $$1-\frac{D_{x+t,\,y+t}-(1-r).(N_{x-1,\,y-1}-N_{x+t-1,\,y+t-1})}{D_{x\,y}} \quad . \quad (11)$$

or, $$\frac{M_{x\,y}-M_{x+t,\,y+t}}{D_{x.y}} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (12)$$

Since, by the construction of the commutation tables,

$$\begin{aligned} M_x &= r.N_{x-1}-N_x \\ &= D_x-(1-r)N_{x-1}, \end{aligned}$$

the annual premium is obtained by dividing the preceding expressions by $(N_{x-1,\,y-1}-N_{x+t-1,\,y+t-1})$, in lieu of $D_{x.y}$.

Thus, from (10) we get

$$r-\frac{N_{x.y}-N_{x+t,\,y+t}}{N_{x-1,\,y-1}-N_{x+t-1,\,y+t-1}}$$

for the annual premium sought—being the same as that obtained by Mr. Sprague in the October Number of the *Journal*.

Obs.—All questions in which the joint duration of two lives occurs can always be solved most expeditiously by simply substituting $D_{x.y}$, $N_{x.y}$, &c., for D_x, N_x, &c., in the single life formulæ, and using the joint commutation tables, in lieu of the single ones, in their practical application.

Ex. gr.—The single premium for an assurance of £1 for t years, on a life aged x, is

$$A_{(x)\overline{t|}}=\frac{M_x-M_{x+t}}{D_x}$$

$$=\frac{r(N_{x-1}-N_{x+t-1})-(N_x-N_{x+t})}{D_x}=1-\frac{D_{x+t}-(1-r)(N_{x-1}-N_{x+t-1})}{D_x}$$

∴ the annual premium for the same is

$$\begin{aligned} P_{(x)\overline{t|}} &= \frac{M_x-M_{x+t}}{N_{x-1}-N_{x+t-1}} \\ &= r-\frac{N_x-N_{x+t}}{N_{x-1}-N_{x+t-1}} \\ &= \frac{D_x-D_{x+t}-(1-r)(N_{x-1}-N_{x+t-1})}{N_{x-1}-N_{x+t-1}}; \end{aligned}$$

hence the single and annual premiums for a similar temporary assurance on two joint lives aged x and y are—

$$\begin{aligned} A_{(x.y)\,\overline{t|}} &= \frac{M_{x\,y}-M_{x+t,\,y+t}}{D_{x.y}} \\ &= \frac{r(N_{x-1\;y-1}-N_{x+t-1,\,y+t-1})-(N_{x.y}-N_{x+t,\,y+t})}{D_{x.y}} \end{aligned}$$

and $$\begin{aligned} P_{(x.y)\,\overline{t|}} &= \frac{M_{x.y}-M_{x+t,\,y+t}}{N_{x-1,\,y-1}-N_{x+t-1,\,y+t-1}} \\ &= r-\frac{N_{x.y}-N_{x+t,\,y+t}}{N_{x-1,\,y-1}-N_{x+t-1,\,y+t-1}}. \end{aligned}$$

In a similar manner may all the joint life formulæ be derived from the single, without the trouble of investigating each particular case independently of the single life formulæ.

Obs.—*Formulæ* (2), (3), (4), and (7), are, I believe, *new*, as I have never seen them in any work that has come under my notice; (2) and (4) will be found very serviceable in computing a table of reversions by means of a reciprocal probability table, such as Table XXIV. given in the first volume of Jones, page 550, and the ordinary life annuity tables.

W. C. O.

ON CERTAIN COMMUTATION FORMULÆ.

To the Editor of the Assurance Magazine.

SIR,—The communications of Messrs. Laundy and Sprague,* in the last Number of the *Assurance Magazine*, have induced me to send you a few "Commutation Table" formulæ which I have had occasion to make use of in practice, and which, as not lying quite on the surface of the subject, may prove a useful contribution to those interested in them.

The notation used is that employed by Professor De Morgan, where "l_x" represents the number living at age x, "$p_{x.n}$" the probability of a life aged x living n years, "v"$=\dfrac{1}{1+r}$, the present value of £1 due a year hence, &c.

1. Endowment of £1 payable to (x) if alive n years hence; premium to be returned in the event of death; p per £1 Office commission added.

Single premium, $\dfrac{(1+p)\,D_{x+n}}{D_x-(1+p)\,(M_x-M_{x+n})}$.

Annual premium, $\dfrac{(1+p)\,D_{x+n}}{(N_{x-1}-N_{x+n-1})-(1+p)\,(R_x-R_{x+n}-n\,M_{x+n})}$.

2. *Life assurance.*—Annual premium payable until death, on which the whole of the premiums paid (without interest) are to be returned. Commission of p per £ on net value.

$$\pi=\frac{(1+p)\,M_x}{N_{x-1}-(1+p)\,R_x}.$$

3. Assurance for n years on (x), with endowment payable should he survive that term, but the benefit payable only provided (y) be alive.

Single premium, $\dfrac{D_{x+n,\,y+n}}{D_{xy}}\,.\,(1-A_{\overline{x+n,\,y+n}}^{1})+A_{\overline{x.y}}^{1}$.

Annual premium, $\left\{D_{x+n,\,y+n}(1-A_{\overline{x+n,\,y+n}}^{1})+D_{x.y}A_{\overline{x.y}}^{1}\right\}\div$
$(N_{x-1,\,y-1}-N_{x+n-1,\,y+n-1})$.

For the Carlisle 3 per cent. rates, the values $A_{\overline{x.y}}^{1}$, &c., are got by inspection in Gray, Smith, and Orchard's tables.

4. Endowment of £1 payable to (y) should he survive n years, and provided (x) die before.

* Mr. Sprague has overlooked the circumstance that his (joint life) formula previously appeared in Professor De Morgan's able paper in the *Companion to the Almanac* for 1842.