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LOWER BOUNDS FOR INDUCED FORESTS IN CUBIC GRAPHS

BY

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ABSTRACT. If G is a connected cubic graph with p vertices, $p > 4$, then G has a vertex-induced forest containing at least $(5p - 2)/8$ vertices. In case G is triangle-free, the lower bound is improved to $(2p - 1)/3$. Examples are given to show that no such lower bound is possible for vertex-induced trees.

In [2] and [4], it was shown that in a cubic graph with p vertices, a largest vertex-induced forest contains no more than $[3/4p - 2]$ vertices. If G is cubic and cyclically 4-connected, this upper bound is achieved, as was shown by Payan and Sakarovitch [3]. In the present article, making no assumptions about high connectivity, we provide lower bounds for the size of largest vertex-induced forests in connected graphs with maximum degree three.

Throughout what follows, we use the following definitions. A vertex set is independent if no two of its vertices are adjacent. An independent vertex set I in a graph G is strongly independent if the subgraph $G - I$ is connected. A subgraph H of G is vertex-induced or simply induced if, whenever v and w are vertices of H adjacent in G, it follows that v and w are adjacent in H. A forest is a graph with no cycles. A tree is a connected forest.

Our first proposition is taken from [4].

PROPOSITION 1. *Let G be a connected graph with maximum degree 3. Let I be a maximum strongly independent vertex set in G. Then no two cycles in $G - I$ have a vertex in common.*

PROOF. Suppose that C_1 and C_2 are distinct intersecting cycles of $G - I$. Their union is connected and bridgeless, and contains a vertex v whose degree in $C_1 \cup C_2$, and hence in $G - I$, is 3. Therefore v is adjacent to no vertex of I, so $I \cup \{v\}$ is strongly independent, which is a contradiction. □

PROPOSITION 2. *If G is a connected cubic graph with p vertices where $p \geqq 6$, then G has a strongly independent vertex set I with $|I| \geq (p + 6)/8$.*

PROOF. Let I be a maximum strongly independent vertex set in G. Consider the subgraph $F = G - I$. By proposition 1, no two cycles of F intersect. Denote by A, B,

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C, and D the sets of vertices of F of degree one, of degree two and in no cycle of F, of degree two and in some cycle of F, and of degree three, respectively. If F is a cycle, then

$$|I| = p/4 \geqq (p + 6)/8 \quad \text{for} \quad p \geqq 6.$$

Thus, since F is connected, we may assume that each cycle of F includes at least one vertex of D. Let H be the bipartite subgraph of G induced by the bipartition $(I, A \cup B \cup C)$. In H, the vertices of I have degree three, those of A have degree two, and those of B and C have degree one. For $X = A, B, C$, we shall refer to an edge of H with one end in I and the other end in X as an (I, X)-edge.

Let c be the number of cycles in F. We shall show that

(1) $$c \geqq (p - 2|I|)/3.$$

To establish this inequality, we assign a weight $w(e)$ to each edge e of H, as follows:

$$w(e) = \begin{cases} 1/2 & \text{if } e \text{ is an } (I, A)\text{-edge;} \\ 1 & \text{if } e \text{ is an } (I, B)\text{-edge;} \\ (k-3)/(k-1) & \text{if } e \text{ is an } (I, C)\text{-edge and the} \\ \text{end of } e \text{ in } C \text{ belongs to a } k\text{-cycle of } F. \end{cases}$$

We define the weight of a subgraph of H to be the sum of the weights of its edges. In particular, denoting by c_k the number of cycles in F of length k, and noting that each cycle of F includes at least one vertex of D,

$$\begin{aligned} w(H) &\leqq |A| + |B| + \underset{k \geqq 3}{\Sigma}(k-3)c_k \\ &= |A| + |B| + \sum_{k \geqq 3} kc_k - 3 \sum_{k \geqq 3} c_k \\ &\leqq p - |I| - 3c \end{aligned}$$

Thus, to establish (1), it suffices to prove that

(2) $$w(H) \geqq |I|.$$

We achieve this by considering each connected component of H individually. Let H have components $H_1, H_2, \ldots, H_m$. For $X = I, A, B, C$, and any component H_i, set

$$X_i = X \cap V(H_i).$$

We shall show that

(3) $$w(H_i) \geqq |I_i|, \qquad 1 \leqq i \leqq m.$$

Counting the edges of H_i in two ways yields

(4) $$2|A_i| + |B_i| + |C_i| = 3|I_i|.$$

Next, we consider the subgraph $H_i - (B_i \cup C_i)$. Since it has $|I_i| + |A_i|$ vertices and $2|A_i|$ edges and is connected,

$$2|A_i| \geqq |I_i| + |A_i| - 1.$$

Thus

$$|A_i| \geqq |I_i| - 1. \tag{5}$$

Two cases arise.

Case 1: $|A_i| \geqq |I_i|$. In this case

$$w(H_i) \geqq |A_i| \geqq |I_i|$$

Case 2: $|A_i| = |I_i| - 1$. By (4)

$$|B_i| + |C_i| = |I_i| + 2 \geqq 3.$$

If $|B_i| \geqq 1$, then

$$w(H_i) \geqq |A_i| + |B_i| \geqq |I_i| - 1 + 1 = |I_i|.$$

If $|B_i| = 0$, then $|C_i| \geqq 3$. We now observe that the vertices of $|C_i|$ must lie on a common cycle of F.

Suppose, to the contrary, that vertices x, y of $|C_i|$ belong to distinct cycles of F. Let P be an (x, y)-path in H_i, and, for $X = 1, A$, set

$$X \cap V(P) = X_p.$$

Then

$$(I \backslash I_p) \cup A_p \cup \{x, y\}$$

is a strongly independent vertex set of cardinality $|I| + 1$. But this contradicts our choice of I. Therefore the vertices of C_i do indeed lie on a common cycle of F.

Let the length of this common cycle be k. Then $k \geqq |C_i| + 1 \geqq 4$, and so

$$w(H_i) = |A_i| + |B_i| + ((k-3)/(k-1))|C_i| \geqq |I_i| - 1 + 0 + 1 = |I_i|.$$

Therefore (3) holds in both cases. It follows that (2), and hence (1) also hold.

Finally, we count the edges of F in two ways, and deduce from (1) that

$$3p/2 - 3|I| = p - |I| + c - 1 \leqq p - |I| + (p - 2|I|)/3 - 1,$$

whence

$$|I| \geqq (p + 6)/8.$$

We now determine when equality holds in Proposition 2.

PROPOSITION 3. *(Corollary to the proof of Proposition 2). Equality holds in the statement of Proposition 2 if and only if G is derived from a cubic tree (all vertices degree three or one) by blowing up each degree three vertex to a triangle and attaching* K_4 *with one subdivided edge at each degree one vertex.*

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PROOF. In order that equality hold, equation (3) must hold for each component. If case 1 of the proof of Proposition 2 holds, then $|A_i| = |I_i|$ and by (4) $|A_i| = |C_i|$. By the argument in case 2, the vertices of C_i all lie on the same cycle of F. Equality in (3) forces this cycle to be a triangle. Thus in case 1, we have

a) $|I_i| = 2, \quad |A_i| = 2, \quad |B_i| = 0, \quad |C_i| = 2.$

Similar reasoning in case 2 yields possibilities:

b) $|I_i| = 1, \quad |A_i| = 0, \quad |B_i| = 1, \quad |C_i| = 2, \quad k = 3;$

c) $|I_i| = 1, \quad |A_i| = 0, \quad |B_i| = 0, \quad |C_i| = 3, \quad k = 4.$

If a) holds, we have the situation in figure 1, where $I_i = \{i_1, i_2\}$, $A_i = \{a_1, a_2\}$, and

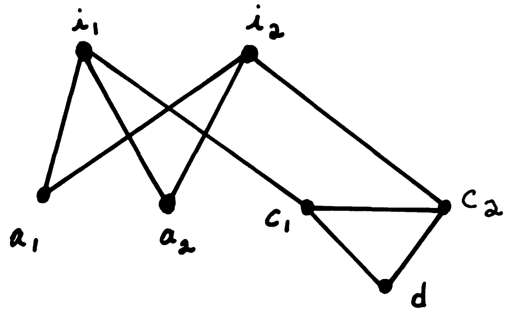

FIGURE 1

$C_i = \{c_1, c_2\}$. Now $I' = (I - I_i) \cup A_i$ is a strongly independent set satisfying $|I'| = (p + 6)/8$. Letting F', H', A', B', C', D' be defined by analogy with the proof of Proposition 2, we observe that every component of H' must fall into one of the cases a), b), or c). Since a_1 and a_2 are clearly in the same component of H', only case a) is possible. We have $I_i' = A_i$, $A_i' = I_i$, $B_i' = \phi$ and $|C_i'| = 2$. Let $C_i' = \{c_1', c_2'\}$. Then we have the situation pictured in figure 2.

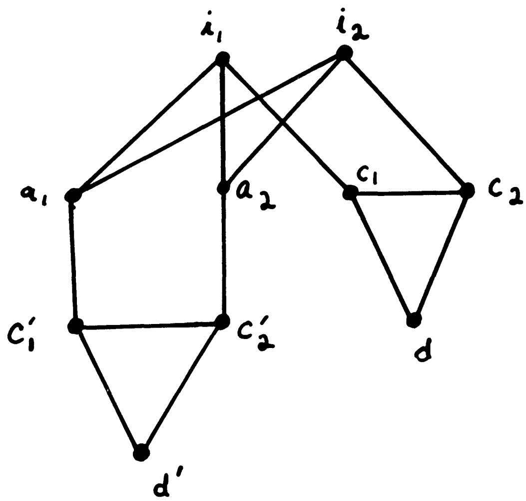

FIGURE 2

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Since F is connected, there is a $d-d'$ path in F. Hence $I'' = (I\backslash\{i_1\}) \cup \{c_1, c_1'\}$ is strongly independent. This contradicts the maximality of I, and equality in Proposition 2 may not hold in case a).

If b) holds, we have the situation of Figure 3, where $I_i = \{v\}$, $A_i = \phi$, $B_i = \{b\}$, $C_i = \{c_1, c_2\}$.

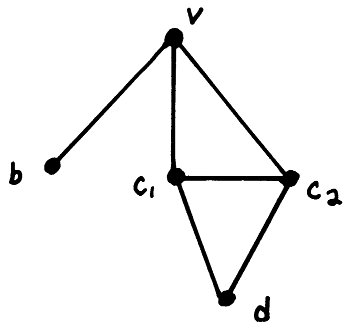

FIGURE 3

Letting $I' = (I\backslash\{v\} \cup \{c_1\}$, we observe that I' is strongly independent and we are in case c). In fact, we may assume that every component of H' is of type c), as shown in Figure 4.

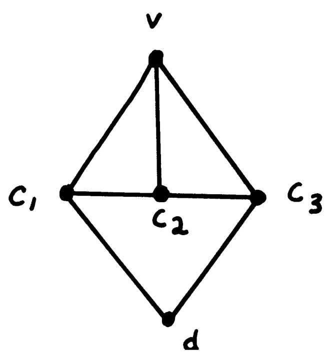

FIGURE 4

Let F' be the subgraph of F induced by D. Since, in the equality case, $\Sigma k c_k = p - |I|$, every vertex of F', except those of degree one, is on a cycle of F'. If c_k' is the number of cycles of length k in F', then

$$\Sigma k c_k' = p - 5|I|.$$

Contracting each cycle to a vertex, we obtain a tree T with c_k' vertices of degree k and $|I|$ vertices of degree one. Therefore, summing degrees of vertices of T, we get

$$\Sigma k c_k' + |I| = 2(\Sigma c_k' + |I| - 1),$$

$$\text{or } \Sigma(k-2)c_k' = |I| - 2.$$

Hence $2\Sigma(k - 3)c'_k = 3(\Sigma(k - 2)c'_k) - \Sigma kc'_k = 3|I| - 6 + 5|I| - p = 0$, since $|I| = (p + 6)/8$. It follows that every cycle in F' is a triangle, and so G has the indicated structure.□

THEOREM 4. *If G is a connected graph with $\Delta \leqq 3$ and $p > 4$, then G has an induced forest with at least $(5p - 2)/8$ vertices.*

PROOF. First assume G is cubic. Let I be a maximum strongly independent vertex set with $|I| \geqq (p + 6)/8$. $G - I$ has cycles disjoint, and may be reduced to a forest by removal of one vertex from each cycle. Suppose $G - I$ has c cycles. Then it has $p - |I| + c - 1$ edges. Hence $3/2p - 3|I| = p - |I| + c - 1$, or $c = 1/2p - 2|I| + 1$. Hence, there is a forest in G with $p - |I| - c$ vertices. This is $p - |I| - 1/2p + 2|I| - 1$, or $1/2p + |I| - 1$. But $|I| \geqq (p + 6)/8$; so there is a forest with at least $(5p - 2)/8$ vertices. In case G is not cubic, we embed G as an induced subgraph of a cubic graph H in the standard way [1]. H consists of several disjoint copies of G joined by some new edges. Since H has an induced forest with at least $(5p - 2)/8$ vertices, certainly the restriction of such a forest to at least one of the copies of G attains this ratio.□

The bound of Theorem 4 is achieved in the examples cited following proposition 2. To see this, it suffices to note that to reduce such a graph to a forest, one must remove at least one vertex from each triangle and at least two from each subdivided K_4.

We now turn our attention to triangle-free graphs with maximum degree three.

PROPOSITION 5. *If G is a connected cubic triangle-free graph with p vertices, then G has a strongly independent vertex set I with $|I| \geqq (p + 4)/6$.*

PROOF. We proceed, following the notation of proposition 2, to show that

$$c \leqq \frac{p - 2|I|}{4}.$$

Our assignment of weights is as in the proof of proposition 2, except that $w(e) = (k - 4)/(k - 1)$ if e has one end in a k-cycle of F. It follows that $w(H) \leqq p - |I| - 4c$, and again, we show that $w(H) \geqq |I|$ by considering each component H_i. The only troublesome case occurs when $|A_i| = |I_i| - 1$ and $|B_i| = 0$. Again, all vertices of C_i lie on a single cycle of F. If $|C_i| = 3$, then $|I_i| = 1$ and the k-cycle containing C_i must have $k \geqq 6$ to avoid triangles. In this case $((k - 4)/(k - 1))|C_i| \geqq 1$. On the other hand, if $|C_i| \geqq 4$, then certainly $k \geqq 5$, and again $((k - 4)/(k - 1))|C_i| \geqq 1$. It follows that $w(H) \geqq |I|$. Hence, as in the proof of proposition 2, $c \leqq (p - 2|I|)/4$, and $|I| \geqq (p + 4)/6$.□

Let H be the graph in figure 5. By joining n copies of H cyclically, one obtains a cubic triangle-free graph in which the size of a largest strongly independent set is $(p + 6)/6$. Hence, if proposition 5 could be improved to strict inequality, it would be sharp.

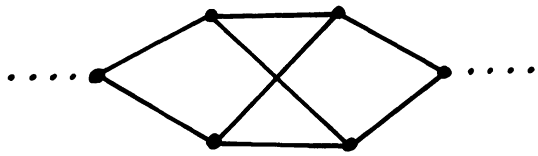

FIGURE 5

THEOREM 5. *If G is a connected triangle-free graph with* $\Delta \leqq 3$ *then G has an induced forest with at least* $(2p - 1)/3$ *vertices.*

PROOF. Similar to proof of theorem 4.□

That this bound is nearly sharp follows from considering the examples above.

In what remains of this article, we will consider a modification of the question we have addressed above. In particular, we inquire about the size of a largest induced tree in a cubic graph. We will construct a sequence of examples in which the proportion of vertices in a largest induced tree approaches zero. Begin with three complete binary trees of n levels (each with $1 + 2 + 2^2 + \ldots 2^{n-1}$ vertices) and join their three roots to a new vertex. In the resulting graph, each vertex has degree one or three. Blow up each vertex of degree three to a triangle and attach a copy of K_4 with a subdivided edge to each vertex of degree one. Call the result G_n.

It is not difficult to check that G_n has $6(2^{n+1} - 1)$ vertices, and that a largest induced tree in G_n has $4n + 4$ vertices. Therefore, the proportion of vertices in a largest induced tree approaches zero quite rapidly.

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