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Dunford–Pettis Properties and Spaces of Operators

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Abstract. J. Elton used an application of Ramsey theory to show that if X is an infinite dimensional Banach space, then c_0 embeds in X, ℓ_1 embeds in X, or there is a subspace of X that fails to have the Dunford–Pettis property. Bessaga and Pelczynski showed that if c_0 embeds in X^*, then ℓ_∞ embeds in X^*. Emmanuele and John showed that if c_0 embeds in $K(X,Y)$, then $K(X,Y)$ is not complemented in $L(X,Y)$. Classical results from Schauder basis theory are used in a study of Dunford–Pettis sets and strong Dunford–Pettis sets to extend each of the preceding theorems. The space $L_{w^*}(X^*,Y)$ of w^*-w continuous operators is also studied.

1 Introduction

A Banach space X has the Dunford–Pettis property (DPP) provided that every weakly compact operator T from X to any Banach space Y is completely continuous (*i.e.,* a Dunford–Pettis operator), and X is said to have the hereditary Dunford–Pettis property if every closed linear subspace of X has the DPP. Localizing these ideas, a bounded subset M of X is said to be a Dunford–Pettis (DP) subset of X if $T(M)$ is relatively compact in Y whenever $T\colon X \to Y$ is a weakly compact operator, and M is a strong (or hereditary) DP set if U is a DP subset of the closed linear span $[U]$ of U for each non-empty subset U of M. We refer the reader to Diestel [8, 9], Diestel and Uhl [10], and Andrews [1] for a guide to the extensive classical literature dealing with the DPP, equivalent formulations of the preceding definitions, and undefined notation and terminology.

Andrews [1], Bator [2], Emmanulele [15], Ghenciu and Lewis [18], and Lewis [24] provide insight into the connections between the structure of Banach spaces and properties of the (strong) DP subsets of these spaces. In particular, the following result is from [18].

Theorem 1.1 *The Banach space X does not contain a copy of c_0 if and only if every strong Dunford–Pettis subset of X is relatively compact.*

Theorem 1.1 was used in [18] to give an alternate proof of the following fundamental structure theorem due to J. Elton [11].

Theorem 1.2 [8, p. 28] *If X is an infinite dimensional Banach space, then c_0 embeds in X, ℓ_1 embeds in X, or X has a closed linear subspace which fails to have the DPP.*

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In this paper, we continue to study connections between Dunford–Pettis properties and the structure of Banach spaces. In particular, we present an improvement of Theorem 1.2, a generalization of a classical result of Bessaga and Pelczynski [5], and a strengthened version of the principal result of Emmanuele [14] and John [21] dealing with complemented spaces of operators. We also extend results in [14, 21], and Kalton [22] to the space $L_{w^*}(X^*, Y)$ of all $w^* - w$ continuous operators from the dual space X^* to Y.

2 Dunford–Pettis Sets

We begin this section with an improvement of a classical result in [10]. Proposition VI.2.4 of [10] showed that if ℓ_∞ does not embed in X^{**} and K is a compact Hausdorff space, then every operator (*i.e.,* continuous linear transformation) $T\colon C(K) \to X$ is weakly compact. Of course, it is well known [5, 9] that $\ell_\infty \not\hookrightarrow X^{**}$ if and only if ℓ_1 is not complemented in X^*. Theorem 1.1 above and [5] allow us to conclude that $\ell_\infty \not\hookrightarrow X^{**}$ if and only if every strong DP subset of X^{**} is relatively compact. Furthermore, it is easy to see that every strong DP subset of X is relatively compact if every strong DP subset of X^{**} is relatively compact. Moreover, the classical unit vector basis (e_n) of c_0 is a strong DP set which is not relatively compact. Consequently, if every strong DP subset of X is relatively compact, then every operator $T\colon Y \to X$ is unconditionally converging. (Otherwise, T would be an isomorphism on a copy of c_0 [9, p. 54], and the hypothesis on X does not allow this.) Since an unconditionally converging operator defined on any $C(K)$-space is weakly compact [6,7], we have the following result.

Theorem 2.1 *If every strong Dunford–Pettis subset of X is relatively compact, then every operator $T\colon C(K) \to X$ is weakly compact.*

Note that James [20] has provided examples of spaces X so that $c_0 \not\hookrightarrow X$ but ℓ_1 is complemented in X^*. Therefore Theorem 2.1 does indeed extend Proposition VI.2.4 of [10].

Closely associated with the notion of a Dunford–Pettis subset of X are the ideas of a limited subset of X and an L-subset of X^*. A subset S of X is limited if

$$\lim_n(\sup\{|x_n^*(x)| : x \in S\}) = 0$$

for each w^*-null sequence (x_n^*) in X^*. Note that a limited subset of X is necessarily a Dunford–Pettis subset of X [1]. The Banach space X has the Gelfand–Phillips property if each of its limited sets is relatively compact. A subset S of X^* is an L-subset of X^* if

$$\lim_n(\sup\{|x^*(x_n)| : x^* \in S\}) = 0$$

for each w-null sequence (x_n) in X. See [4] for equivalent formulations of these definitions.

The following theorem is similar in spirit to Proposition 16 in [15] and an unpublished theorem by Emmanuele.[1] Moreover, it sets the stage for corollaries which

[1] G. Emmanuele, *Limited sets and Gelfand-Phillips space.* Preprint.

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point out connections between DP properties and complementability questions involving the space $K(X, Y)$ of compact operators and the space $W(X, Y)$ of weakly compact operators in the space $L(X, Y)$ of all bounded linear transformations from X to Y.

Theorem 2.2 *Suppose that X and Y are Banach spaces.*

(i) *The following are equivalent:*

  (a) *X has the Gelfand–Phillips property.*

  (b) *If $T^*\colon X^* \to Y^*$ is a w^*-norm sequentially continuous operator, then $T\colon Y \to X$ is compact.*

  (c) *Same as* (b) *with $Y = \ell_1$.*

(ii) *The following are equivalent:*

  (a) *$|x_n^*(x_n)| \to 0$ whenever (x_n) is weakly null in X and (x_n^*) is w^*-null in X^*.*

  (b) *If B_{Y^*} is w^*-sequentially compact, then every operator $T\colon X \to Y$ is completely continuous.*

  (c) *Every operator $T\colon X \to c_0$ is completely continuous.*

Proof (i) Note first that if $T\colon Y \to X$ is an operator, then $T(B_Y)$ is limited if and only if $T^*\colon X^* \to Y^*$ is w^*-norm sequentially continuous. Thus (i)(a) implies (i)(b), and obviously (i)(b) implies (i)(c).

Now suppose that (i)(c) holds and that S is a limited subset of X which is not relatively compact. Choose $\epsilon > 0$ and subsequences (a_k) and (b_k) in S so that $\|a_k - b_k\| > \epsilon$ for all k. Using the fact that a limited subset of X is weakly precompact [1, 26], we assume that $(u_k) = (a_k - b_k) \xrightarrow{w} 0$. Define $L\colon X^* \to \ell_\infty$ by $L(x^*) = (x^*(u_k))$. Note that $L = A^*$, where $A\colon \ell_1 \to X$ is defined by $A(\lambda) = \sum \lambda_k u_k$. Thus L is an adjoint. Further, if $(x_n^*) \xrightarrow{w^*} 0$, then $\lim_n \sup_k |x_n^*(u_k)| = \lim_n ||L(x_n^*)|| = 0$ since $U = \{u_k : k \in \mathbf{N}\}$ is limited. Consequently, A is compact but U is not compact, and we have a contradiction.

(ii) Suppose that (ii)(a) holds and let Y be a Banach space so that B_{Y^*} is w^*-sequentially compact. Let $T\colon X \to Y$ be an operator, and suppose that (x_n) is weakly null and $(T(x_n)) \not\to 0$. Choose (y_n^*) in B_{Y^*} so that $y_n^*(T(x_n)) \not\to 0$, and, without loss of generality, suppose that $(y_n^*) \xrightarrow{w^*} y^*$. Thus $\langle y_n^* - y^*, T(x_n)\rangle \to 0$ by hypothesis. Consequently, $y^*(T(x_n)) \not\to 0$, and we obtain a contradiction. Thus (ii)(a) implies (ii)(b).

It is clear that (ii)(b) implies (ii)(c). Now we assume that every operator $T\colon X \to c_0$ is completely continuous. Suppose that (x_n) is w-null in X, (x_n^*) is w^*-null in X^*, and define $A\colon X \to c_0$ by $A(x) = (x_n^*(x))$. Since A is completely continuous,

$$\|A(x_n)\| = \sup_i |x_i^*(x_n)| \xrightarrow{n} 0.$$

Thus $(x_n^*(x_n)) \to 0$. ■

Corollary 2.3 *The Banach space X has the Gelfand–Phillips property if and only if every limited and weakly null sequence in X is norm null.*

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Proof If $T^*: X^* \to Y^*$ is w^*-norm sequentially continuous, $(x_n^*) \xrightarrow{w^*} 0$ in X^*, and (y_n) is a sequence from B_Y, then $|x_n^* T(y_n)| \le \|T^*(x_n^*)\| \to 0$, and $\{T(y_n) : n \in \mathbf{N}\}$ is limited. Thus without loss of generality we may assume that $(T(y_{n+1}) - T(y_n))$ is weakly null. Hence $(T(y_{n+1}) - T(y_n))$ is norm null, and T is compact. An application of the preceding theorem finishes one implication, and it is clear that $\|x_n\| \to 0$ if $(x_n) \xrightarrow{w} 0$ and $\{x_n : n \in \mathbf{N}\}$ is relatively compact. ■

Corollary 2.4 *If X has the Dunford–Pettis property, $c_0 \hookrightarrow Y$, and there exists an operator $T: X \to c_0$ which is not completely continuous, then $W(X,Y)$ is not complemented in $L(X,Y)$.*

Proof By Theorem 2.2, X contains a weakly null sequence (x_n) which is not limited. Thus X^* contains a w^*-null sequence which is not w-null. Theorem 4 in [4] gives the conclusion. ■

We remark that this corollary contains Theorem 3 in [13].

Corollary 2.5 (i) *If X is a $C(K)$-space, $c_0 \hookrightarrow Y$, and there is an operator $T: X \to c_0$ which is not weakly compact, then $W(X,Y)$ is not complemented in $L(X,Y)$.*

(ii) *If Y is an arbitrary Banach space, then $W(C[0,1],Y)$ is complemented in $L(C[0,1],Y)$ if and only if $L(C[0,1],Y) = W(C[0,1],Y)$.*

Proof (i) If $T: X \to c_0 \hookrightarrow Y$ is not weakly compact, then T is not completely continuous. Apply Corollary 2.4.

(ii) If $T: C[0,1] \to Y$ is not weakly compact, then T is not unconditionally converging, and thus T is an isomorphism on a copy of c_0 . Since c_0 is complemented in $C[0,1]$, another application of Corollary 2.4 finishes the argument. ■

Theorems 1.1, 1.2, and 2.2 cement firm connections between the Dunford–Pettis properties of X and/or X^* and the presence of ℓ_1 in X and/or X^*. Additionally, a direct combination of Theorem 2.2 and results from [12,15], and Bator [2] produces the following equivalences:

- $\ell_1 \not\hookrightarrow X$;
- X^* has the weak Radon–Nikodym property;
- Every DP subset of X^* is relatively compact;
- Every L-subset of X^* is relatively compact;
- Every completely continuous operator $T: X \to Y$ is compact.

An alternate characterization of the DPP will facilitate the extension of Theorem 1.2.

Theorem 2.6 *The following are equivalent:*

(i) *the Banach space X has the DPP;*

(ii) *every weakly null sequence in X is a DP subset of X;*

(iii) *every weakly null basic sequence in X is a DP subset of X.*

Proof Suppose that the Banach space X has the DPP and (x_n) is a weakly null sequence in X. If Y is a Banach space and $T: X \to Y$ is weakly compact, then T is

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completely continuous and $\|T(x_n)\| \to 0$. Thus (i) implies (ii), and (ii) certainly implies (iii).

Conversely, if every weakly null sequence in X is a DP subset of X, $T\colon X \to Y$ is a weakly compact operator, and $(x_n) \xrightarrow{w} 0$, then $\{T(x_n) : n \in \mathbf{N}\}$ is relatively compact. Thus $\|T(x_n)\| \to 0$, T is completely continuous, and (ii) implies (i).

Now suppose that (iii) holds, $(x_n) \xrightarrow{w} 0$, and $A = \{x_n : n \in \mathbf{N}\}$ is not a DP subset of X. Let Y be a Banach space and $T\colon X \to Y$ be a weakly compact operator so that $T(A)$ is not relatively compact. Choose $\epsilon > 0$ and a subsequence (x_{n_i}) so that $\|T(x_{n_i}) - T(x_{n_j})\| > \epsilon$ if $i \neq j$. Thus (x_{n_i}) is weakly null, and there is a $\delta > 0$ so that $\|x_{n_i} - x_{n_j}\| > \delta$ if $i \neq j$, and we may assume that (x_{n_i}) is weakly null and seminormalized. Therefore, some subsequence (s_k) of (x_{n_i}) is basic, and $\{T(s_k) : k \in \mathbf{N}\}$ is not relatively compact. Thus $\{s_k : k \in \mathbf{N}\}$ is not a DP subset of X. ■

We remark that the sequence (s_k) produced in this argument is also not a DP subset of $[s_k : k \in \mathbf{N}]$.

Corollary 2.7 *If X is an infinite dimensional Banach space, then $c_0 \hookrightarrow X$, $\ell_1 \hookrightarrow X$, or there is a seminormalized weakly null basic sequence (y_n) in X so that $\{y_n : n \in \mathbf{N}\}$ is not a DP subset of $Y = [y_n : n \in \mathbf{N}]$ and thus Y does not have the DPP.*

As a result of Theorems 1.1, 2.6, and Rosenthal's ℓ_1-theorem, Corollary 2.7 (or Theorem 1.2) can be stated in the following equivalent form. If X is an infinite-dimensional Banach space, then X contains a seminormalized weakly null basic sequence (x_n) so that $\{x_n : n \in \mathbf{N}\}$ is a strong DP set, X contains a seminormalized weakly null basic sequence (y_n) so that $\{y_n : n \in \mathbf{N}\}$ is not a DP subset of $[y_n : n \in \mathbf{N}]$, or X contains a seminormalized basic sequence (z_n) so that no seminormalized basic sequence in $[z_n]$ is weakly null.

3 Spaces of Operators

The complementability of $K(X,Y)$ and $W(X,Y)$ has long been of interest to functional analysts; (see, for example, [3,14,17,21,22] and the references therein). Building on the work of Kalton [22] and Feder [17], Emmanuele [14] and John [21] independently showed that if X and Y are infinite-dimensional and $c_0 \hookrightarrow K(X,Y)$, then $K(X,Y)$ is not complemented in $L(X,Y)$. See also [16]. The next theorem, which makes fundamental use of the DPP, allows us to use [17, 22] and obtain the Emmanuele–John theorem as an immediate corollary. The reader should compare this result with [17, Theorem 1].

Theorem 3.1 *Suppose that X and Y are Banach spaces and S is a complemented subspace of $L(X,Y)$. If there is a sequence (T_i) in S so that $\|T_i\| \not\to 0$ and $\sum_i T_i(x)$ converges unconditionally for each x, then ℓ_∞ embeds in S.*

Proof Suppose that (T_i) and S are as above and $P\colon L(X,Y) \to S$ is a projection. Define $\phi\colon \ell_\infty \to L(X,Y)$ by $\phi(b)(x) = \sum b_i T_i(x)$ for $b = (b_i) \in \ell_\infty$ and $x \in X$. Note that ϕ is a continuous linear transformation and $\phi(e_i) = T_i$ for each i.

Suppose that $\ell_\infty \not\hookrightarrow S$. Then $P\phi\colon \ell_\infty \to S$ is weakly compact [25]. Since ℓ_∞ has the DPP (all $C(K)$ spaces have the DPP), $P\phi$ is completely continuous. Thus $\|P\phi(e_i)\| = \|T_i\| \to 0$, and we have a contradiction. ■

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Corollary 3.2 *If X and Y are infinite dimensional and $c_0 \hookrightarrow K(X,Y)$, then $K(X,Y)$ is not complemented in $L(X,Y)$.*

This corollary follows directly from Theorem 3.1, Lemma 3 of [22] (*If X contains a complemented copy of ℓ_1 and Y is infinite dimensional, then $K(X,Y)$ is uncomplemented in $L(X,Y)$*), Theorem 4 of [22] (*$K(X,Y)$ contains a copy of ℓ_∞ iff $\ell_\infty \hookrightarrow X^*$ or $\ell_\infty \hookrightarrow Y$*), and Corollary 1 of [17] (*If X is infinite dimensional and $c_0 \hookrightarrow Y$, then $K(X,Y)$ is uncomplemented in $L(X,Y)$*).

We denote by $L_{w^*}(X^*,Y)$ the closed linear subspace of $L(X^*,Y)$ consisting of the w^*-w continuous operators and by $K_{w^*}(X^*,Y)$ the compact members of $L_{w^*}(X^*,Y)$. The reader is encouraged to see Ryan [27] and Emmanuele [15] for an indication of the relevance of these spaces to the study of Dunford–Pettis properties.

Our next two theorems are analogues of Theorem 4 of [22] and Corollary 1 of [17]. A series of lemmas, which closely parallels results in the opening section of [22], will be helpful in establishing these theorems. Let U denote the unit ball of X^* with the w^* topology and V denote the unit ball of Y^* with the w^* topology. For T in $L_{w^*}(X^*,Y)$, define $\chi_T \colon U \times V \to \mathbf{R}$ by $\chi_T(x^*,y^*) = y^*(Tx^*)$, $y^* \in V, x^* \in U$.

Lemma 3.3 *The mapping $T \mapsto \chi_T$ defines a linear isometry of $K_{w^*}(X^*,Y)$ onto a closed subspace of $C(U \times V)$.*

Proof Suppose $(x^*_\alpha) \xrightarrow{w^*} x^*$ in U, and $(y^*_\alpha) \xrightarrow{w^*} y^*$ in V. We have

$$\begin{aligned}|\chi_T(x^*_\alpha,y^*_\alpha) - \chi_T(x^*,y^*)| &= |y^*_\alpha(Tx^*_\alpha) - y^*(Tx^*)|\\ &\leq |y^*_\alpha(Tx^*_\alpha - Tx^*)| + |(y^*_\alpha - y^*)(Tx^*)|\\ &\leq \|Tx^*_\alpha - Tx^*\| + |(y^*_\alpha - y^*)(Tx^*)|.\end{aligned}$$

Since T is $w^* - w$ continuous and compact, T is w^*-norm continuous, and thus $\|Tx^*_\alpha - Tx^*\| \to 0$. Also, $|(y^*_\alpha - y^*)(Tx^*)|$ converges to zero because $(y^*_\alpha) \xrightarrow{w^*} y^*$ in V. Thus $(\chi_T(x^*_\alpha,y^*_\alpha))$ converges to $\chi_T(x^*,y^*)$ and $\chi_T \in C(U \times V)$. Since $\|\chi_T\| = \|T\|$ and $T \mapsto \chi_T$ is linear, the conclusion follows. ■

In the next two lemmas let (wot) denote the *weak operator topology*.

Lemma 3.4 *Let A be a subset of $K_{w^*}(X^*,Y)$. Then A is weakly compact if and only if A is (wot)-compact.*

Proof Suppose A is wot-compact and let $\chi(A)$ be $\{\chi_T : T \in A\}$. Let (T_α) be a net in A and (T_β) be a subnet convergent in the topology wot. If (T_β) converges to T in wot, then $\chi_{T_\beta}(x^*,y^*)$ converges to $\chi_T(x^*,y^*)$, for all $x^* \in X^*$ and $y^* \in Y^*$, and $\chi(A)$ is compact in the topology of pointwise convergence on $C(U \times V)$. By a result of Grothendieck [19], $\chi(A)$ is weakly compact in $C(U \times V)$. Thus A is weakly compact by the preceding result.

The other implication is clear since the topology (wot) is weaker than the weak topology on $K_{w^*}(X^*,Y)$. ■

Lemma 3.5 *Let (T_n) be a sequence of weak* $-$ weak continuous compact operators such that $(T_n) \to T$ in (wot), where T is $w^* - w$ continuous and compact. Then $(T_n) \to T$ weakly.*

Proof Let $A = \{(T_n), T\}$ and apply the previous lemma to this (wot)-compact set. ■

Since the restrictions of $w^* - w$ continuous operators to subspaces certainly need not be $w^* - w$ continuous, separability hypotheses which were not present in [17,22] are included in the statements of some of our theorems.

Theorem 3.6 *If X^* is separable, then ℓ_∞ embeds isomorphically in $K_{w^*}(X^*, Y)$ if and only if ℓ_∞ embeds isomorphically in Y.*

Proof Suppose that $\ell_\infty \hookrightarrow K_{w^*}(X^*, Y)$ and $\ell_\infty \not\hookrightarrow Y$. Let $\varphi\colon \ell_\infty \to K_{w^*}(X^*, Y)$ be an embedding, and let $T_i = \varphi(e_i)$ for each i. For $x^* \in X^*$, set $A_{x^*}(b) = \varphi(b)(x^*)$ for $b \in \ell_\infty$; for $y^* \in Y^*$, set $B_{y^*}(b) = \varphi(b)^*(y^*)$. Since $\varphi(b)$ is $w^* - w$ continuous, $\varphi(b)^*$ maps Y^* into the canonical image of X in X^{**}. Moreover, since ℓ_∞ embeds in neither X nor Y, both A_{x^*} and B_{y^*} are weakly compact and unconditionally converging. Thus $\sum b_i B_{y^*}(e_i)$ and $\sum b_i A_{x^*}(e_i)$ are unconditionally converging for each $b = (b_i) \in \ell_\infty$. Define ρ on ℓ_∞ by $\rho(b) = \sum b_i T_i$, where the series converges in the strong operator topology (sot). The unconditional convergence of the series above in X guarantees that $\rho(b)$ is $w^* - w$ continuous, and it is not difficult to see that $\rho\colon \ell_\infty \to L_{w^*}(X^*, Y)$ is a bounded linear transformation.

Now set $Y_0 = \overline{\operatorname{span}}\{\rho(b)(x^*) : b \in \ell_\infty, x^* \in X^*\}$, and note that Y_0 is separable. Let $J\colon Y_0 \to \ell_\infty$ be a linear isometric embedding, and let $A\colon Y \to \ell_\infty$ be a continuous linear extension of J. Define $D_1\colon \ell_\infty \to K_{w^*}(X^*, \ell_\infty)$ and $D_2\colon \ell_\infty \to L_{w^*}(X^*, \ell_\infty)$ by $D_1(b)(x^*) = A\varphi(b)(x^*)$ and $D_2(b)(x^*) = A\rho(b)(x^*)$. Note that $D_1(e_i)(x^*) = AT_i(x^*) = D_2(e_i)(x^*)$ for each i and each x^*.

It is not difficult to check that the $w^* - w$ analogue of Proposition 5 of [22] applies in our setting. Thus we obtain an infinite subset M of $\mathbf{N}$ so that $D_1(b) = D_2(b)$ for all $b \in \ell_\infty(M)$. It follows that $\varphi(b) = \rho(b)$, and $\varphi(b) = \sum b_i T_i(sot)$ for $b \in \ell_\infty(M)$. Lemmas 3.4 and 3.5 allow us to apply the Orlicz–Pettis Theorem and conclude that $\sum_{i \in M} T_i$ is unconditionally convergent. Of course, this is impossible since $\|T_i\| \not\to 0$. Thus $\ell_\infty \hookrightarrow Y$.

If $\tau : \ell_\infty \to Y$ is an isomorphism and $x \in X$, $\|x\| = 1$, then $b \mapsto x \otimes \tau(b)$ defines an isomorphic embedding of ℓ_∞ into $K_{w^*}(X^*, Y)$. ■

Theorem 3.7 *If X is infinite dimensional, X^* is separable, and $c_0 \hookrightarrow Y$, then $K_{w^*}(X^*, Y)$ is not complemented in $L_{w^*}(X^*, Y)$.*

Proof Since X^* is separable, X is not a Schur space. Let (x_n) be a normalized weakly null sequence in X, and consider the sequence $(x_n \otimes y_n)$ of $w^* - w$ continuous operators from X^* to Y defined by $x_n \otimes y_n(x^*) = x^*(x_n)y_n$, where (y_n) is a copy of (e_n) in Y. In fact,

$$T = \sum_{n=1}^{\infty} x_n \otimes y_n (\text{strong operator topology})$$

is a $w^* - w$ continuous operator from X^* to Y. It is important to note that the preceding series converges unconditionally in the (sot).

Now let $T_k = \sum_{n=1}^{k} x_n \otimes y_n$ for each k, and note that

$$Y_0 = \overline{\operatorname{span}}\big(\{T_k(X^*) : k \in \mathbf{N}\} \cup T(X^*)\big)$$

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is a separable subspace of of Y. Moreover, T_k and T are $w^* - w$ continuous as maps from X^* to Y. Let $J\colon Y_0 \to \ell_\infty$ be a linear isometry, and let $A\colon Y \to \ell_\infty$ be a continuous linear extension of J.

Next define $\varphi\colon \ell_\infty \to L(X^*, Y)$ by $\varphi(b) = \sum b_i x_i \otimes y_i$, (sot). As in the previous proof, $\varphi(b)$ is $w^* - w$ continuous. Thus $A\varphi(b) \in L_{w^*}(X^*, \ell_\infty)$. Now suppose that $K_{w^*}(X^*, Y)$ is complemented and let P be the projection. If b is a finitely supported member of ℓ_∞, then $AP\varphi(b) = A\varphi(b)$. By the the $w^* - w$ version of Proposition 5 of [22], we can choose an infinite subset M of $\mathbf{N}$ so that $A\varphi(b) = AP\varphi(b)$ for all $b \in \ell_\infty(M)$. Apply the Orlicz–Pettis theorem again and conclude that $\|A\varphi(e_i)\| \to 0$ for $i \in M$. This is a clear contradiction. ■

Corollary 3.8 *If X and Y are infinite dimensional Banach spaces, X^* is separable, and $c_0 \hookrightarrow K_{w^*}(X^*, Y)$, then $K_{w^*}(X^*, Y)$ is not complemented in $L_{w^*}(X^*, Y)$.*

Proof By Theorem 3.7, we may assume that $c_0 \not\hookrightarrow Y$. Suppose that $K_{w^*}(X^*, Y)$ is complemented in $L_{w^*}(X^*, Y)$. An application of (the proof of) Theorem 3.1 tells us that $\ell_\infty \hookrightarrow K_{w^*}(X^*, Y)$, and Theorem 3.6 places a copy of ℓ_∞ (and thus c_0) in Y. ■

Bessaga and Pelczynski [5] showed that if $c_0 \hookrightarrow X^*$, then $\ell_\infty \hookrightarrow X^*$. This result was generalized to spaces of operators in [23], *i.e.,* if X is infinite dimensional and $c_0 \hookrightarrow L(X, Y)$, then $\ell_\infty \hookrightarrow L(X, Y)$. In the next theorem, we use the fact that ℓ_∞ has the DPP and give a short proof of a generalization of the result in [5, 23].

Theorem 3.9 *Suppose that X and Y are Banach spaces, X is infinite dimensional, and S is a subspace of $L(X, Y)$ which is closed with respect to the strong operator topology and contains each finite rank operator. If $c_0 \hookrightarrow S$, then $\ell_\infty \hookrightarrow S$.*

Proof Suppose that $\ell_\infty \not\hookrightarrow S$. We then consider two cases.

Case 1. $c_0 \hookrightarrow Y$. Let $T\colon c_0 \to Y$ be an isomorphism. Use the Josefson–Nissenzweig Theorem and let (x_n^*) be a w^*-null sequence of norm one vectors in X^*. Define $L : \ell_\infty \to S$ by $L(b)(x) = \sum b_n x_n^*(x) T(e_n)$, $x \in X$, $b = (b_n) \in \ell_\infty$. Since $\ell_\infty \not\hookrightarrow S$, L is weakly compact and thus completely continuous. Therefore $\|L(e_i)\| = \|T(e_i)\| \to 0$, and we have a contradiction.

Case 2. $c_0 \not\hookrightarrow Y$. Let $T\colon c_0 \to S$ be an isomorphism, let $T_i = T(e_i)$, and define $L\colon \ell_\infty \to S$ by $L(b)(x) = \sum b_i T_i(x)$. Since $c_0 \not\hookrightarrow Y$ and this series is weakly unconditionally convergent, it is unconditionally convergent. Since $\ell_\infty \not\hookrightarrow S$, we obtain the same contradiction as in Case 1. Thus $\ell_\infty \hookrightarrow S$. ■

Of course, neither $K_{w^*}(X^*, Y)$ nor $L_{w^*}(X^*, Y)$ is closed with respect to the strong operator topology. For example, if $y \in Y$, $\|y\| = 1$, then $e_n \otimes y \in K_{w^*}(\ell_1, Y)$ for each n,

$$\sum_{n=1}^{k} e_n \otimes y \xrightarrow{k} (1, 1, 1, \dots, 1, \dots) \otimes y \,(sot),$$

and $(1, 1, 1, \dots, 1, \dots) \otimes y \notin L_{w^*}(\ell_1, Y)$. However, if $c_0 \hookrightarrow Y$, then ℓ_∞ does embed in $L_{w^*}(\ell_1, Y)$.

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Theorem 3.10 (i) *If X is not a Schur space and c_0 embeds in Y, then ℓ_∞ embeds in $L_{w^*}(X^*, Y)$.*

(ii) *If c_0 embeds in neither X nor Y but c_0 does embed in $L_{w^*}(X^*, Y)$, then ℓ_∞ embeds in $L_{w^*}(X^*, Y)$.*

Proof (i) Let (x_n) be a normalized and weakly null sequence in X, and let (y_n) be a copy in Y of (e_n). Define $x_n \otimes y_n \colon X^* \to Y$ by $x_n \otimes y_n(x^*) = x^*(x_n)y_n$. Then $x_n \otimes y_n \in K_{w^*}(X^*, Y)$, and $(x_n \otimes y_n) \sim (e_n)$. In fact, if c_1 and c_2 are positive constants so that

$$c_1 \max\Big\{ |a_i| : i = 1, \dots, m \Big\} \le \Big\| \sum_{i=1}^{m} a_i y_i \Big\| \le c_2 \max\Big\{ |a_i| : i = 1, \dots, m \Big\},$$

then

$$c_1 \max\Big\{ |a_i x^*(x_i)| : i = 1, \dots, m; \|x^*\| \le 1 \Big\} \le \Big\| \sum_{i=1}^{m} a_i x_i \otimes y_i \Big\| \\ \le c_2 \max\Big\{ |a_i x^*(x_i)| : i = 1, \dots, m; \|x^*\| \le 1 \Big\}.$$

Set $J(b)(x^*) = \sum b_i x^*(x_i) y_i$, $x^* \in X^*$, $b = (b_i) \in \ell_\infty$. Since $\sum |y^*(y_i)| < \infty$ for $y^* \in Y^*$, one can check that $J(b)^*(y^*) = \sum b_i y^*(y_i) x_i$ and thus $J(b) \in L_{w^*}(X^*, Y)$. Moreover, $c_1 \|b\|_\infty \le \|J(b)\| \le c_2 \|b\|_\infty$, and J is an isomorphism.

(ii) Let $B \colon c_0 \to L_{w^*}(X^*, Y)$ be an embedding, and let $T_n = B(e_n)$ for each n. Since $\sum T_n$ is weakly unconditionally convergent,

$$\sum |\langle T_n x^*, y^* \rangle| = \sum |\langle x^*, T_n^*(y^*) \rangle| < \infty$$

for $x^* \in X^*$ and $y^* \in Y^*$. Since $c_0 \not\hookrightarrow X$, and $c_0 \not\hookrightarrow Y$, $\sum T_n(x^*)$ is unconditionally convergent in Y, and $\sum T_n^*(y^*)$ is unconditionally convergent in X. Certainly $L_{w^*}(X^*, Y)$ is complemented in itself, and an application of the $w^* - w$ version of Theorem 3.1 finishes the proof. ■

The reader should see [22, p. 274 and Theorem 6] for a discussion of unconditional expansions of the identity. Note that if c_0 embeds in X or Y, then $c_0 \hookrightarrow K_{w^*}(X^*, Y)$.

Theorem 3.11 *Suppose that X^* has an unconditional compact expansion of the identity consisting of $w^* - w^*$ continuous operators. If $c_0 \not\hookrightarrow X$ and $c_0 \not\hookrightarrow Y$, then the following are equivalent:*

(i) *$K_{w^*}(X^*, Y) = L_{w^*}(X^*, Y)$;*
(ii) *$c_0 \not\hookrightarrow K_{w^*}(X^*, Y)$;*
(iii) *$\ell_\infty \not\hookrightarrow L_{w^*}(X^*, Y)$;*
(iv) *$K_{w^*}(X^*, Y)$ is complemented in $L_{w^*}(X^*, Y)$.*

Proof Since X^* has an unconditional compact expansion of the identity, X^* is separable. If $K_{w^*}(X^*, Y) = L_{w^*}(X^*, Y)$ and $\ell_\infty \hookrightarrow L_{w^*}(X^*, Y)$, then Theorem 3.6 applies and $\ell_\infty \hookrightarrow Y$. Thus (i) implies (iii).

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The preceding theorem immediately shows that (iii) implies (ii). Now suppose that $T\colon X^* \to Y$ is a non-compact $w^* - w$ continuous operator. Let (B_n) be a $w^* - w^*$ continuous unconditional compact expansion of the identity. Thus $\sum_{n=1}^{\infty} TB_n x^*$ converges unconditionally to Tx^* for each $x^* \in X^*$. Let $T_n = TB_n$. An application of the uniform boundedness principle shows that $\sum T_n$ is weakly unconditionally convergent. The non-compactness of T certainly shows that $\sum T_n$ is not unconditionally convergent. Thus $c_0 \hookrightarrow K_{w^*}(X^*, Y)$, and (ii) yields (i).

Clearly (i) implies (iv). To see that (iv) implies (i), suppose not, argue as in the preceding paragraph, and apply Theorem 3.1 to obtain (the contradiction) that ℓ_∞ embeds in Y. ∎

Emmanuele [15] said that a Banach space X has the DPrcP whenever each Dunford–Pettis subset of X is relatively compact. As noted in the previous section, Emmanuele [15] and Bator [2] independently showed that if $\ell_1 \not\hookrightarrow X$, then X^* has the DPrcP. Note that if $\ell_1 \not\hookrightarrow X^*$, then X and X^* have the DPrcP. Emmanuele [15, p. 482] also asked if every Dunford–Pettis subset of $K_{w^*}(X^*, Y)$ must be relatively compact when X and Y have the DPrcP and $K_{w^*}(X^*, Y) = L_{w^*}(X^*, Y)$. Note that if X and Y have the DPrcP, then every DP subset of either X or Y is a strong DP set which is relatively compact.

Theorem 3.12 *If every strong DP set in both X and Y is relatively compact, X^* is separable, and $K_{w^*}(X^*, Y) = L_{w^*}(X^*, Y)$, then every strong DP set in $L_{w^*}(X^*, Y)$ is relatively compact.*

Proof Suppose there is a strong DP set in $L_{w^*}(X^*, Y)$ which is not relatively compact. By Theorem 1.1, $c_0 \hookrightarrow L_{w^*}(X^*, Y)$, and c_0 embeds in neither X nor Y. Thus, by Theorem 3.10, $\ell_\infty \hookrightarrow L_{w^*}(X^*, Y)$. Therefore $\ell_\infty \hookrightarrow Y$, and we have a contradiction. ∎

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